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CAUCHY TYPE PROBLEM WITH RIGHT FRACTIONAL h-SUM

Sharip Beksultan[1], Tokmagambetov Nariman[2]
[1]Sharip.beksultan@gmail.com , [2]nariman.tokmagambetov@gmail.com
L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan

Before stating our results, we introduce some definitions and notations. Let $h>0$ and put $T=\{a,a+h,a+2h,...,b\}$ with $a\in R$ and $b=a+kh$ for $k\in\{2,3,...\}$. Let us denote by F_T the set of real valued functions defined on T, $\sigma_h(t)=t+h$ and $\rho_h=\{t-h\}$ [1-2].

Definition 1. For a function $f\in F_T$ the forward h-difference operator is defined as

$$(\Delta_h f)(t)=\frac{f(\sigma_h(t))-f(t)}{\sigma_h(t)-t},\ t\in\{a,a+h,a+2h,...,\rho_h(b)\},$$

while the h-difference sum is given by

$$(\Delta_{a,h}^{-1} f)(t)=\sum_{k=\frac{a}{h}}^{\frac{t}{h}} f(kh)h,\ \ t\in\{a,a+h,a+2h,...,\rho_h(b)\}$$

Definition 2. For arbitrary $x,y\in R$ the h-factorial function is defined by

$$x_h^{(y)}=h^y\frac{\Gamma\left(\frac{x}{h}+1\right)}{\Gamma\left(\frac{x}{h}+1-y\right)}$$

where Γ is the Euler gamma function.

Definition 3. Let $f\in F_T$. The right fractional h-sum of order $\alpha>0$ is the operators $(\Delta_{a,h}^{-\alpha} f)(t):F_T\to F_{T_a}$, $T=\{a+\alpha h,a+\alpha h+h,a+\alpha h+2h,...,b+\alpha h\}$, given by

$$\left(\Delta_{a,h}^{-\alpha} f\right)(t) = \sum_{k=\frac{a}{h}}^{\frac{t}{h}-\alpha} \left(t - \sigma_h(kh)\right)_h^{\alpha-1} f(kh)h\,,$$

we define $\left(\Delta_{a,h}^{0} f\right)(t) = f(t)$.

Definition 4. Let $\alpha > 0$ and set $\gamma = [\alpha]$ (*integer part*). The right fractional h-difference $\Delta_{a,h}^{\alpha} f$ of order α of a function $f \in F_T$ is defined

$$\left(\Delta_{a,h}^{\alpha} f\right)(t) = \left(\Delta_{a,h}^{\gamma} \Delta_{a,h}^{\alpha-\gamma} f\right)(t)$$

We denote

$$L_h^p[a,b] := \left\{ f \in F_T : \left(\left(\Delta_{a,h}^{-1} \left|f\right|^p\right)(b)\right)^{\frac{1}{p}} < \infty \right\}.$$

Our first main result reads:

Theorem. Let $\alpha > 0$, $\gamma = [\alpha]$, G be an open set in R and $f(.,.) : (a,b] \times G \to R$ be a function such that $f\left(x, y(x)\right) \in L_h^1[a,b]$ for any $y \in G$. If $y(x) \in L_h^1[a,b]$, then $y(x)$ satisfies a.e. in the following relations:

$$\left(\Delta_{a,h}^{\alpha} y\right)(t) = f(x,y),\ \alpha > 0$$
$$\left(\Delta_{a,h}^{\alpha-k} y\right)(t) = b_k,\ b_k \in R,\ \left(k = 0,1,2,\ldots,[\alpha]\right)$$

if and only if, $y(x)$ satisfies a.e. the integral equation

$$y(x) := \sum_{k=0}^{n} \frac{b_k}{\Gamma(\alpha - k + 1)} (x-a)_h^{\alpha-\gamma} + \left(\Delta_{a,h}^{-\alpha} f\right)(x)$$

Literature

1. N.R.O. Bastos, R.A.C. Ferreira, and D.F.M. Torres, Necessary optimality conditions for fractional difference problems of the calculus of variations, Discrete Contin. Dyn. Syst. 29, 2 (2011), 417–437.