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The solution to the Cauchy type problem for the homogenous q-fractional differential equation

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In this article we present the method for finding solutions to boundary value problems for the homogenous q-fractional differential equation based on the reduction to Volterra integral equations.

In this work we will use the following definitions of the Riemann-Liouville q-fractional integrals and q-fractional derivatives on a finite interval. For more information see e.g. the books [1], [2] and [3].

For $q \in (0,1)$, define

$$[m]_q = \frac{1-q^m}{1-q}, \qquad (m \in R)$$

The q-analog of the power function $(n-m)^k$ with $k \in N_0 = \{0,1,2,...\}$ is

$$(n-m)_q^0 = 1, \qquad (n-m)_q^k = \prod_{i=0}^{k-1}\left(n - q^i m\right), \qquad k \in N, \quad n,m \in R$$

More generally, if $\alpha \in R$, then

$$(n-m)_q^\alpha = \prod_{i=1}^{\infty}\frac{n-q^i m}{n-q^{\alpha+i}m}, \qquad (n \neq 0)$$

Note if $m=0$, then $n_q^\alpha = n^\alpha$. We also use the notation $0_q^\alpha = 0^\alpha$ for $\alpha > 0$. The q-gamma function is defined by

$$\Gamma_q(\alpha) = \frac{(1-q)_q^{\alpha-1}}{(1-q)^{\alpha-1}}, \qquad (\alpha \in R \setminus \{0,-1,-2,...\})$$

Obviously, $\Gamma_q(\alpha+1) = [\alpha]_q \Gamma_q(\alpha)$.

The q-derivative of a function f is defined by

$$(D_q f)(t) = \frac{f(qt) - f(t)}{qt - t} \text{ for } t \neq 0 \text{ and } (D_q f)(0) = \lim_{t\to 0}(D_q f)(t),$$

and the q-derivatives of higher order are given by

$$\left(D_q^0 f\right)(t) = f(t) \quad \text{and} \quad \left(D_q^k f\right)(t) = D_q\left(D_q^{k-1} f\right)(t), \qquad k \in N$$

The q-integral of a function f defined on the interval $[0, b]$ is given by

$$\left(I_q f\right)(t) = \int_0^t f(s) d_q s = t(1-q)\sum_{i=0}^{\infty} f\left(q^i t\right) q^i, \qquad t \in [0, b]$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b f(s) d_q s = \int_0^b f(s) d_q s - \int_0^a f(s) d_q s$$

Similar to derivatives, an operator I_q^k is given by

$$\left(I_q^0 f\right)(t) = f(t) \text{ and } \left(I_q^k f\right)(t) = I_q\left(I_q^{k-1} f\right)(t), \qquad k \in N$$

For any $s, t > 0$, the q-beta function is defined by

$$B_q(s,t) = \int_0^1 t^{s-1}(1-qt)_q^{t-1} d_q t,$$

The expression of q-beta function in terms of the q-gamma function can be written as

$$B_q(s,t) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}$$

The q- Mittag–Leffler function is defined by

$$\mathrm{E}_{q,\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k$$

with

$$c_0 = 1 \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma_q[\alpha(jm+l)+1]}{\Gamma_q[\alpha(jm+l+1)+1]} \qquad (k \in N)$$

Definition 1. Let $\alpha \geq 0$ and f be a function defined on $[0, T]$. The fractional q-integral of Riemann-Liouville type is given by $\left(I_q^0 f\right)(t) = f(t)$ and

$$\left(I_q^\alpha f\right)(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)_q^{\alpha-1} f(s) d_q s, \quad \alpha > 0, \; t \in [0, T]$$

Definition 2. The fractional q-derivative of Riemann-Liouville type of order $\alpha \geq 0$ is defined by $\left(D_q^0 f\right)(t) = f(t)$ and

$$\left(D_q^\alpha f\right)(t) = \left(D_q^l I_q^{l-\alpha} f\right)(t), \qquad \alpha > 0$$

where l is the smallest integer greater than or equal to α.

The q-integral is defined by

$$\left(I_{q,0} f\right)(x)=\int_0^x f(t)d_q(t)=x(1-q)\sum_{k=0}^{\infty} f\left(xq^k\right)q^k \qquad \left(0<|q|<1\right)$$

The fractional q-integral is defined by

$$\left(I_{q,a}^{\alpha} f\right)(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x (qt/x;q)_{\alpha-1} f(t)d_q(t) \qquad \left(x>a;\alpha\in R^{+}\right)$$

The fractional q–derivative of Riemann–Liouville type is

$$\left(D_{q,a}^{\alpha} f\right)(x)=\begin{cases}\left(I_{q,a}^{-\alpha}\right)(x) & \alpha\leq 0\\ \left(D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f\right)(x) & \alpha>0\end{cases}$$

where $[\alpha]$ denotes the smallest integer greater or equal to α .

We consider the Cauchy type problem for the homogenous q-fractional differential equation of order $\alpha>0$ with the following initial conditions:

$$\left(D_{q,0+}^{\alpha} y\right)(x)-\lambda x^{\beta} y(x)=0 \quad (\alpha>0,\lambda\in R) \qquad (1)$$

$$\left(D_{q,0+}^{\alpha-k} y\right)(0+)=b_k, \quad \left(b_k\in R; k=1,2,...,n=-[-\alpha]\right) \qquad (2)$$

with $\beta>-\{\alpha\}$.

Theorem 1. *Let* $\alpha>0$, $n=-[-\alpha]$ $\lambda\in R$ *and* $\beta\geq 0$. *Then the Cauchy type problem* (1)-(2) *has a unique solution* $y(x)\in C_{q,n-\alpha}^{\alpha}[a,b]$ *and this solution is given by*

$$y(x)=\sum_{j=1}^{n}\frac{b_j}{\Gamma_q(\alpha-j+1)}x^{\alpha-j}\mathrm{E}_{q,\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-j}{\alpha}}\left[\left(\lambda x^{\alpha+\beta}\right)\right]$$

Literature

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