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THE CAUCHY TYPE PROBLEM WITH CAPUTO FRACTIONAL q-DERIVATIVE

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Our paper is devoted to fractional q-difference equation based on Caputo fractional derivative. We investigated question concerning the solvability of this equation in a certain space of functions.

In this paper, we assume that $0 < q < 1$ and $0 \leq a < b < \infty$, also some of needed q-notations are given as follows. The q-shifted factorial is defined by $(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i), n \in N$, $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$ and for $\alpha \in R$ define $(a;q)_\alpha := \frac{(a;q)_\infty}{(q^\alpha a;q)_\infty}$.

For more details we refer the reader to the book [1]. The q-gamma function was introduced by Jackson [2] $\Gamma_q(z)=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}$ $(0<|q|<1)$ for any $z>0$; the q-beta function is $B_q(x,y):=\int_0^1 t^{x-1}(qt;q)_{y-1}d_qt$ $(x,y>0)$; and Askey proved that [3]: $B_q(x,y)=\frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}$.

The q-difference operator, which was introduced by Jackson, is defined by [4]:

$$D_q f(z):=\frac{f(z)-f(qz)}{z-zq} \quad \text{for } z\neq 0$$

and the q-integral is

$$\int_a^b f(t)d_qt:=\int_0^b f(t)d_qt-\int_0^a f(t)d_qt,$$

where

$$\int_0^b f(t)d_qt:=(1-q)\sum_{n=0}^{\infty} bq^n f(bq^n).$$

The fractional q-integral is

$$I_q^\alpha f(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)d_qt \quad \text{for } \alpha\in R^+,$$

and the fractional q-derivative is

$$D_q^\alpha f(x)=D_q^{[\alpha]}I_q^{\alpha-[\alpha]}f(x) \quad ([\alpha]:=\max\{n\in N_0:\alpha\geq n\}).$$

For $\alpha>0$, the Caputo fractional q-derivative of order α is defined by

$${}^C D_q^\alpha f(x):=I_q^{n-\alpha}D_q^n f(x) \quad (n:=[\alpha]).$$

For $\gamma\in R$ we define the space

$$C_\gamma[a,b]=\{g(x):x^\gamma g(x)\in C[a,b],\|g\|_{C_\gamma}:=\max_{a\leq x\leq b}|x^\gamma g(x)|\}.$$

Let $C_q^n[a,b]$ be the space of all continuous functions with continuous q-derivatives up to order $n-1$ on the interval $[a,b]$ and has the norm of function

$$\|f\|:=\sum_{k=0}^{n-1}\max_{a\leq x\leq b}|D_q^k f(x)|.$$

In our work, we obtain that for $0<\gamma<1$ and $g \in C_\gamma[0,a]$ holds:

(i) $I_q^\alpha g \in C_\gamma[0,a]$ and $\left\| I_q^\alpha g \right\|_{C_\gamma} \leq \dfrac{a^\alpha \Gamma_q(1-\gamma)}{\Gamma_q(\alpha+1-\gamma)} \left\| g \right\|_{C_\gamma}$.

(ii) If $\gamma \leq \alpha$, then $I_q^\alpha g \in C[0,a]$.

Also we showed that for $\alpha>0, n=[\alpha]$, if there exists $\gamma \leq \alpha-n+1$ such that $f \in C_\gamma[0,a]$ then $I_q^\alpha f \in C_q^n[0,a]$. And we used result of combining the Riemann-Liouville fractional q-derivative and Caputo fractional q-derivative of not necessarily equal orders and identity relation between them, which proved in [5]:

1. Let $\alpha, \beta>0$ and $n=[\alpha],\ m=[\beta]$. If $f \in C_\gamma[0,a]$ then for all $x \in (0,a]$:

$$I_q^\alpha {}^C D_q^\beta f(x) = \begin{cases} I_q^{\beta-\alpha} f(x) - \sum_{j=0}^{m-1} \dfrac{D_q^j f(0^+)}{\Gamma_q(\beta-\alpha+j+1)} x^{\beta-\alpha+j}, & \beta \geq \alpha, \\ D_q^{\beta-\alpha} f(x) - \sum_{j=0}^{m-1} \dfrac{D_q^j f(0^+)}{\Gamma_q(\beta-\alpha+j+1)} x^{\beta-\alpha+j}, & \beta < \alpha. \end{cases}$$

2. Let $\alpha>0$ and $n=[\alpha]$. If $f \in C_\gamma[0,a]$ and $D_q^n f \in C[0,a]$, then:

$${}^C D_q^\alpha f(x) = D_q^\alpha \left(f(x) - \sum_{k=0}^{n-1} \frac{D_q^k f(0^+)}{\Gamma_q(k+1)} x^k \right) \quad (x \neq 0)$$

We consider the Cauchy type problem

$${}^C D_q^\alpha y(x) = f(x, y(x)) \quad (\alpha>0), \tag{1}$$

$$D_q^m y(0^+) = b_m,\ b_m \in R \quad (m=0,1,\ldots,[\alpha]-1). \tag{2}$$

In the following, we prove the existence and uniqueness of the solutions of the Cauchy type problem (1)-(2) in the space $C_q^n[0,a]$.

Theorem. Let $\alpha>0, n=[\alpha]$. Let $G \subset C$ and let $f:(0,a]\times G \to R$ be a function such that $f(x,y) \in C_\gamma[0,a]$ for any $y \in G,\ \ \gamma \leq \alpha-n+1$. If $y \in C_q^n[0,a]$, then $y(x)$ satisfies (1)-(2) for all $x \in (0,a]$ if and only if $y(x)$ satisfies the q-integral equation

$$y(x) = \sum_{k=0}^{n-1} \frac{b_k}{\Gamma_q(k+1)} x^k + \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} f(t, y(t)) d_q t$$

for all $x \in (0,a]$.

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