UDC 517.52

THE GENERALIZED FRACTIONAL-MAXIMAL FUNCTION AND ESTIMATE OF ITS NON-INCREASING REARRANGEMENT

Abek Azhar Nartaikyzy
azhar_18@inbox.ru, azhar.abekova@gmail.com
2nd year doctoral student, L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan
Research supervisors – N.A. Bokayev, A.Gogatishvili

Let $L_0 = L_0(R^n)$ be the set of all Lebesgue measurable functions $f : R^n \to C$; $\tilde{L}_0 = \tilde{L}_0(R^n)$ is the set of functions $f \in L_0$, for which the non-increasing rearrangement of the f^* is not identical to infinity. Non-increasing rearrangement f^* defined by the equality:

$$f^*(t) = \inf\left\{y \in [0,\infty) : \lambda_f(y) \le t\right\}, \quad t \in R_+ = (0,\infty),$$

where

$$\lambda_f(y) = \mu_n\left\{x \in R^n : |f(x)| > y\right\}, \quad y \in [0,\infty)$$

is the Lebesgue distribution function [1].

The function $f^{**} : (0,\infty) \to [0,\infty]$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau; \quad t \in R_+$$

We define the following classes of function.

Definition 1. Let $R\in(0,\infty]$. We say that a function $\Phi:(0,R)\to R_+$ belongs to the class $A_n(R)$ if:

(1) Φ decreases and is continuous on $(0,R)$;

(2) $\Phi(r)r^n\uparrow,\ \ r\in(0,R)$.

For example, $\Phi(t)=t^{-\alpha}\in A_n(\infty)$, $0<\alpha<n$.

Definition 2 [2]. Let $R\in(0,\infty]$. A function $\Phi:(0,R)\to R_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ decreases and is continuous on $(0,R)$;

(2) There exists a constant $C\in R_+$ such that

$$\int_0^r\Phi(\rho)\rho^{n-1}d\rho\le C\Phi(r)r^n,\ r\in(0,R).$$

For example,

$$\Phi(\rho)=\rho^{\alpha-n}\in\Im_n(\infty)\ \ (0<\alpha<n);\quad \Phi(\rho)=\ln\frac{eR}{\rho}\in\Im_n(R),\ R\in R_+.$$

For $\Phi\in B_n(R)$ the following estimate also holds:

$$\int_0^r\Phi(\rho)\rho^{n-1}d\rho\ge n^{-1}\Phi(r)r^n,\ r\in(0,R).$$

Therefore

$$\int_0^r\Phi(\rho)\rho^{n-1}d\rho\cong\Phi(r)r^n,\ r\in(0,R).$$

$$\Phi(\rho)\in\Im_n(R)\Rightarrow\left\{0\le\Phi\downarrow;\ \Phi(r)r^n\uparrow,\ \ r\in(0,R)\right\}.$$

Lemma 1. $B_n(R)\subset A_n(R)$.

Definition 3. Let $R\in(0,\infty]$. We will say that $\Phi:(0,R)\to R_+$ belongs to the class $E_n(R)$ if

$$\int_0^{r^n}\frac{ds}{\Phi(s^{1/n})s}\le\frac{C}{\Phi(r)},\ \forall t\in(0,R). \qquad (1)$$

Note that relation (1) is equivalent to the inequality (can be obtained by a change of variables):

$$\int_0^r\frac{dt}{\Phi(t)t}\le\frac{C}{\Phi(t)}. \qquad (1')$$

For example the function $\Phi(t)=t^{\alpha-n}\in A_n(\infty)\ \ (0<\alpha<n)$. Indeed in this case

$$\int_0^{t^n} \frac{ds}{\Phi(s^{1/n})s} = \int_0^{t^n} \frac{ds}{s^{\alpha/n-1}} = C_{\alpha,n} \frac{1}{t^{\alpha-n}} = \frac{C_{\alpha,n}}{\Phi(t)}, \ \forall t \in R_+ .$$

Lemma 2. $E_n(R) \subset B_n(R)$.

Definition 4. Let $\Phi : R_+ \to R_+$. The *generalized fractional-maximal function* $M_\Phi f$ is defined for the function $f \in E(R^n) \cap L_1^{loc}(R^n)$ by the equality

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int_{B(x,r)} f(y)dy,$$

where $B(x,r)$ is a ball with the center at the point x and radius r. That is, consider the operator $M_\Phi : L_1^{loc}(R^n) \to \tilde{L}_0(R^n)$.

In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0,n)$ we obtain the classical fractional-maximal function $M_\alpha f$ [3]:

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(x,r)} |f(y)| dy$$

We denote by $M_E^\Phi = M_E^\Phi(R^n)$ the set of the functions u, for which there is a function $f \in E(R^n)$ such that

$$u(x) = (M_\Phi f)(x),$$

$$\|u\|_{M_E^\Phi} = \inf \left\{ \|f\|_E : f \in E(R^n),\ M_\Phi f = u \right\}.$$

Theorem 1. Let $\Phi \in A_n(\infty)$. Then there exist a positive constant C depending from n such that

$$(M_\Phi f)^*(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \ \ t \in (0,\infty)$$

for every $f \in L_1^{loc}(R^n)$.

Theorem 2. Let $\Phi \in A_n(\infty)$. Inequality (10) is sharp in the sense that for every $\varphi \in L_0^+(0,\infty;\downarrow)$ there exists a function $f \in L_1(R^n)$ such that $f^* = \varphi$ a.e. on $(0,\infty)$ and

$$(M_\Phi f)^*(t) \ge C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \ \ t \in (0,\infty),$$

where, C is a positive constant which depends only on n.

Theorem 3. Let $\Phi \in B_n(\infty)$. Then there exist a positive constant C depending from n such that

$$(M_\Phi f)^{**}(t) \le C \sup_{t<s<\infty} s\Phi(s^{1/n}) f^{**}(s), \ \ t \in (0,\infty)$$

for every $f \in L_1^{loc}(R^n)$.

References

1. C.Bennett, R.Sharpley, Interpolation of operators. Pure and applied mathematics, Volume 129. Boston, MA: Acad. Press Inc., 1988.
2. M.L. Goldman, E.G. Bakhtigareeva Some classes of operators in general Morrey-type spaces // Eurasian Mathematical Journal, Volume 11, №4, 2020, p.35-44.
3. R.Ch. Mustafayev, N.Bilgicli Generalized fractional maximal functions in Lorentz spaces A // Journal of Mathematical Inequalities, Volume 12, №3, 2018, p.827-851.