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Soliton surfaces induced by the coupled integrable dispersionless equation with self-consistent sources

Z. Shanina and R. Myrzakulov

Department of General and Theoretical Physics, L.N.Gumilyov Eurasian National University, Nur-Sultan, 010008, Kazakhstan

E-mail: `shaninazk@gmail.com, rmyrzakulov@gmail.com`

Abstract. The coupled integrable dispersionless equations have a significant interest because of structure, integrability, and the possibility of obtaining a soliton solution. In this paper, we construct soliton surfaces for integrable dispersionless equation with self-consistent sources in Riemann space. The surfaces, arising from M-XXXII equation and their reduction in R^3, are studied. We obtain Gaussian and mean curvatures and also evaluate the area of surface parametrically defined with the Riemannian metric. Using the scale transformation and transformation of dependent and independent variables of the coupled dispersionless equations we obtain the equation that describes a current-fed string interacting with an external magnetic field in three-dimensional Euclidean space.

1. Introduction

Surface theory in three dimensional Euclidean space is widely used in different branches of science, particularly mathematics (differential geometry, topology, Partial Differential Equations (PDEs)), theoretical physics (string theory, general theory of relativity), and biology [1]-[3]. There are some special subclasses of 2-surfaces which arise in the branches of science aforementioned. For the classification of surfaces in three dimensional Euclidean space, particular conditions are imposed on the Gaussian and mean curvatures. These conditions are sometimes given as algebraic relations between curvatures and sometimes given as differential equations for these two curvatures.

The soliton surfaces approach is very useful in construction of the so called integrable geometries. Indeed, any class of soliton surfaces is integrable. Geometrical objects associated with soliton surfaces (tangent vectors, normal vectors, foliations by curves etc.) usually can be identified with solutions to some nonlinear models (spins, chiral models, strings, vortices etc.) The classical action of the boson string (Nambu-Goto action) depends from the geometric point of view only on the internal geometry of the worldsheet (through the metric) and does not depend on the method of embedding the worldsheet in the enclosing D-dimensional space-time. The method of embedding a two-dimensional surface in D-dimensional space ($D \geq 3$) is characterized in differential geometry by a second fundamental form.

2. M-XXXII equation and its reductions

The M-XXXII equation reads as [4]

$$q_{xt} - 4aq + 2p_x = 0, \tag{1}$$

$$r_{xt} - 4ar - 2n_x = 0, \tag{2}$$
$$a_x - 0.5(rq)_t + qn - rp = 0, \tag{3}$$
$$\eta_x + rp - qn = 0, \tag{4}$$
$$p_x + 2i\omega p + 2\eta q = 0, \tag{5}$$
$$n_x - 2i\omega n - 2\eta r = 0. \tag{6}$$

where q, p are complex functions, r, n are conjugate complex functions of q and p, respectively; a, η are real functions, ω is a constant.

The M-XXXII equation (1)-(6) is integrable and its Lax representation (LR) is

$$\begin{aligned} \Phi_x &= U_1\Phi, \\ \Phi_t &= V_1\Phi. \end{aligned}$$

Here

$$U_1 = -i\lambda\sigma_3 + Q, \quad V_1 = \frac{i}{\lambda}F + \frac{i}{\lambda - \omega}G,$$

where

$$Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad F = \begin{pmatrix} a & -0.5q_t - p \\ 0.5r_t - n & -a \end{pmatrix}, \quad G = \begin{pmatrix} \eta & p \\ n & -\eta \end{pmatrix}.$$

Note that from Eqs. (4)-(6) we get the following important formula

$$\eta^2 + np = const,$$

or, for simplicity,

$$\eta^2 + np = 1.$$

2.1. Case $r = \sigma\bar{q}, \quad n = -\sigma\bar{p}$

We consider the case when

$$r = \sigma\bar{q}, \quad n = -\sigma\bar{p}. \tag{7}$$

Here $\sigma = \pm 1$. We take the case $\sigma = -1$. Then we have

$$\begin{aligned} q_{xt} - 4aq + 2p_x &= 0, \\ -\bar{q}_{xt} + 4a\bar{q} + 2\bar{p}_x &= 0, \\ a_x + 0.5(|q|^2)_t + q\bar{p} + \bar{q}p &= 0, \\ \eta_x - \bar{q}p - q\bar{p} &= 0, \\ p_x + 2i\omega p + 2\eta q &= 0, \\ \bar{p}_x - 2i\omega\bar{p} + 2\eta\bar{q} &= 0. \end{aligned}$$

This reduction of the M-XXXII equation is integrable with the following LR

$$\begin{aligned} \Phi_x &= U_2\Phi, \\ \Phi_t &= V_2\Phi. \end{aligned}$$

Here

$$U_2 = -i\lambda\sigma_3 + Q, \quad V_2 = \frac{i}{\lambda}F + \frac{i}{\lambda - \omega}G, \tag{8}$$

where

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} a & -0.5q_t - p \\ -0.5\bar{q}_t - \bar{p} & -a \end{pmatrix}, \quad G = \begin{pmatrix} \eta & p \\ \bar{p} & -\eta \end{pmatrix}.$$

2.2. Case $p = n = \eta = 0$

In the case when

$$p = n = \eta = 0,$$

the M-XXXII equation reduces to the generalized Konno-Oono equation [5]

$$
\begin{aligned}
q_{xt} - 4aq &= 0, \\
a_x + 0.5(\bar{q}q)_t &= 0.
\end{aligned}
$$

Its LR is

$$
\begin{aligned}
\Phi_x &= U_3 \Phi, \\
\Phi_t &= V_3 \Phi,
\end{aligned}
$$

where

$$U_3 = -i\lambda\sigma_3 + Q, \quad V_3 = \frac{i}{\lambda}F, \tag{9}$$

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} a & -0.5q_t \\ -0.5\bar{q}_t & -a \end{pmatrix}.$$

3. Soliton surfaces induced by M-XXXII equation

In this section we study the surfaces arising from M-XXXII equations in R^3.

The square of the interval between two infinitely close events in the case of a curved surface is given by

$$ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2 = g_{ij}dx^i dx^j,$$

where g_{ij} is the metric tensor.

In our case, the interval between two infinite events is an integral

$$S = \int \sqrt{g_{ij}dx^i dx^j} = \int ds.$$

To describe the surface in Euclidean space, we consider 2 fundamental forms. The first and the second fundamental forms (FF) are [6]:

$$
\begin{aligned}
I &= g_{ij}dx^i dx^j = d\mathbf{r} \cdot d\mathbf{r}, \tag{10} \\
II &= b_{ij}dx^i dx^j = -d\mathbf{n} \cdot d\mathbf{r}, \tag{11}
\end{aligned}
$$

where $x^1 = x, x^2 = t$, $i, j = 1, 2$ and

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j, \quad b_{ij} = \mathbf{r}_{ij} \cdot \mathbf{n}.$$

The first quadratic form defines the internal geometry, and the second quadratic form defines the external geometry of the surface. The main purpose of this section is to find the first and the second fundamental forms of surfaces, also the Gaussian and mean curvatures, area of surfaces using Lax representations for reductions of the M-XXXII equation. The surface is

given by position vector $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$. We consider the surface $M \in R^3$. The position vector $\mathbf{r}(x,t) \in so(3) \leftrightarrow r(x,t) \in su(2)$, where r is matrix.

Using the Sym-Tafel formula [7] we write r matrix

$$r = \Phi^{-1}\Phi_\lambda.$$

$$
\begin{aligned}
r_x &= \Phi^{-1}U_\lambda\Phi, \\
r_t &= \Phi^{-1}V_\lambda\Phi.
\end{aligned}
$$

The r matrix refers to algebra $su(2)$ and is given by

$$r = \begin{pmatrix} r_3 & r^- \\ r^+ & -r_3 \end{pmatrix},$$

and respectively

$$
\begin{aligned}
\mathbf{r}_x^2 &= \frac{1}{2}\,\mathrm{tr}\left(r_x^2\right) = \frac{1}{2}\,\mathrm{tr}\left(U_\lambda U_\lambda\right), \\
\mathbf{r}_t^2 &= \frac{1}{2}\,\mathrm{tr}\left(r_t^2\right) = \frac{1}{2}\,\mathrm{tr}\left(V_\lambda V_\lambda\right), \\
\mathbf{r}_x\mathbf{r}_t &= \frac{1}{2}\,\mathrm{tr}\left(r_x r_t\right) = \frac{1}{2}\,\mathrm{tr}\left(U_\lambda V_\lambda\right).
\end{aligned}
$$

The normal vector in R^3 is defined as

$$\mathbf{n} = \frac{\mathbf{r}_x \wedge \mathbf{r}_y}{|\mathbf{r}_x \wedge \mathbf{r}_y|}.$$

Using the isomorphism $so(3) \simeq su(2)$ we rewrite

$$\mathbf{r}_x \wedge \mathbf{r}_y \longleftrightarrow [r_x, r_y] = \Phi^{-1}\left[U_\lambda, V_\lambda\right]\Phi,$$

$$\mathrm{tr}(\Phi^{-1}\left[U_\lambda, V_\lambda\right]\Phi) = \mathrm{tr}([U_\lambda, V_\lambda]).$$

So the normal n is defined in following form

$$n = \pm\frac{\Phi^{-1}\left[U_\lambda, V_\lambda\right]\Phi}{\sqrt{\frac{1}{2}\,\mathrm{tr}([U_\lambda, V_\lambda]^2)}}$$

and

$$
\begin{aligned}
\mathbf{r}_{xx}\mathbf{n} &= \frac{1}{2}\,\mathrm{tr}\left(r_{xx}n\right) = \pm\frac{\mathrm{tr}\left([U_\lambda U]\,[U_\lambda V_\lambda]\right)}{2\sqrt{\frac{1}{2}\,\mathrm{tr}\left([U_\lambda V_\lambda]^2\right)}}, \\
\mathbf{r}_{xt}\mathbf{n} &= \frac{1}{2}\,\mathrm{tr}\left(r_{xt}n\right) = \pm\frac{\mathrm{tr}\left([U_\lambda V]\,[U_\lambda V_\lambda]\right)}{2\sqrt{\frac{1}{2}\,\mathrm{tr}\left([U_\lambda V_\lambda]^2\right)}}, \\
\mathbf{r}_{tt}\mathbf{n} &= \frac{1}{2}\,\mathrm{tr}\left(r_{tt}n\right) = \pm\frac{\mathrm{tr}\left([V_\lambda V]\,[U_\lambda V_\lambda]\right)}{2\sqrt{\frac{1}{2}\,\mathrm{tr}\left([U_\lambda V_\lambda]^2\right)}}.
\end{aligned}
$$

So, rewrite the formulas (10) and (11) in the form

$$
\begin{aligned}
I &= \mathbf{r}_x^2 dx^2 + 2\mathbf{r}_x\mathbf{r}_t dxdt + \mathbf{r}_t^2 dt^2 = \\
&= \frac{1}{2}\left[\mathrm{tr}(r_x r_x)dx^2 + \mathrm{tr}\left(r_x r_t + r_t r_x\right)dxdt + \mathrm{tr}\left(r_t r_t\right)dt^2\right], \\
II &= (\mathbf{r}_{xx} \cdot \mathbf{n})dx^2 + 2(\mathbf{r}_{xt} \cdot \mathbf{n})dxdt + (\mathbf{r}_{tt} \cdot \mathbf{n})dt^2 = \\
&= \frac{1}{2}\left[\mathrm{tr}(r_{xx}n)dx^2 + 2\,\mathrm{tr}\left(r_{xt}n\right)dxdt + \mathrm{tr}\left(r_{tt}n\right)dt^2\right].
\end{aligned}
$$

3.1. Reduction $r = \sigma\bar{q}$, $n = -\sigma\bar{p}$

The first fundamental form, generated by U_2, V_2 (8), is

$$
\begin{aligned}
I &= -[dx^2 + 2\left(\frac{a}{\lambda^2} + \frac{\eta}{(\lambda - \omega)^2}\right)dxdt + (\frac{1}{\lambda^4}(a^2 + 0.25|q_t|^2 + 0.5(q_t\bar{p} + \bar{q}_t p) + |p|^2) + \\
&+ \frac{1}{\lambda^2(\lambda - \omega)^2}(2a\eta - 2|p|^2 - 0.5(q_t\bar{p} + \bar{q}_t p)) + \frac{1}{(\lambda - \omega)^4})dt^2].
\end{aligned}
$$

The second fundamental form for U_2, V_2 is defined as

$$
II = \frac{1}{2}\left[\text{tr}(r_{xx}n)dx^2 + 2\,\text{tr}\,(r_{xt}n)dxdt + \text{tr}\,(r_{tt}n)dt^2\right].
$$

Here

$$
\begin{aligned}
\text{tr}(r_{xx}n) &= \frac{i}{\sqrt{D}}\left[(\lambda - \omega)^2\left(q\bar{q}_t - \bar{q}q_t + 2(\bar{p}q - \bar{q}p)\right) + 2\lambda^2\left(\bar{q}p - \bar{p}q\right)\right], \\
\text{tr}(r_{xt}n) &= \frac{1}{\lambda(\lambda - \omega)\sqrt{D}}\left[\lambda\,(\lambda - \omega)\,(2\lambda - \omega)\,(p\bar{q}_t + \bar{p}q_t + 2|p|^2) - \lambda^3|p|^2\right. \\
&- \left.(\lambda - \omega)^3(0.25|q_t|^2 + 0.5(\bar{p}q_t + p\bar{q}_t) + |p|^2)\right], \\
\text{tr}(r_{tt}n) &= \frac{\omega}{\sqrt{D}}\left[\frac{1}{\lambda^2}\left(a(p\bar{q}_t + \bar{p}q_t + 4|p|^2) + \eta(|q_t|^2 + 2(p\bar{q}_t + \bar{p}q_t) + 4|p|^2)\right) - \right. \\
&- \left.\frac{2}{(\lambda - \omega)^2}\left((2a|p|^2 + \eta(0.5(q_t\bar{p} + \bar{q}_t p)) + 2|p|^2)\right)\right],
\end{aligned}
$$

where

$$
D = (\lambda - \omega)^2\,\lambda^2(0.5(p\bar{q}_t + \bar{p}q_t) + 2|p|^2) - \lambda^4|p|^2 - (\lambda - \omega)^4\,(0.25|q_t|^2 + 0.5(p\bar{q}_t + \bar{p}q_t) + |p|^2).
$$

The Gaussian and mean curvatures can be found directly by the first and second fundamental forms. The Gaussian and mean curvatures of the surface are, respectively, shown by [6]

$$
K = \det{(g^{-1})}b,
$$
$$
H = \frac{1}{2}\,\text{tr}\,(g^{-1}b),
$$

where g and b denote the matrices (g_{ij}) and (b_{ij}), respectively, and g^{-1} stands for the inverse matrix g.

The area of surface

$$
S = \int\int \sqrt{g}dxdt = \int\int |\mathbf{r}_x \wedge \mathbf{r}_t|\,dxdt,
$$

where

$$
g = \det{(g_{ij})} = \det\begin{pmatrix} \mathbf{r}_x^2 & \mathbf{r}_x \cdot \mathbf{r}_t \\ \mathbf{r}_x \cdot \mathbf{r}_t & \mathbf{r}_t^2 \end{pmatrix}.
$$

Thus, the surface area for this case is defined as

$$
S = \int\int \sqrt{\frac{1}{\lambda^4}(0,25|q_t|^2 + 0,5(q_t\bar{p} + \bar{q}_t p) + |p|^2) - \frac{1}{\lambda^2(\lambda - \omega)^2}(2|p|^2 + 0,5(q_t\bar{p} + \bar{q}_t p)) + \frac{1 - \eta^2}{(\lambda - \omega)^4}}\,dxdt.
$$

3.2. Reduction $p = n = \eta = 0$

The surface, generated by U_3, V_3 (9), has the following first and second fundamental forms

$$
\begin{aligned}
I &= -\left[dx^2 + \frac{2a}{\lambda^2}dxdt + \frac{1}{\lambda^4}(a^2 + 0.25\,|q_t|^2)dt^2\right], \\
II &= \frac{i}{\sqrt{|q_t|^2}}(q\bar{q}_t - q_t\bar{q})dx^2 + \frac{\sqrt{|q_t|^2}}{\lambda}dxdt.
\end{aligned}
$$

Since we know the elements of the matrix, we can determine the Gaussian and mean curvature

$$
\begin{aligned}
K &= -4\lambda^2, \\
H &= \frac{2}{\sqrt{|q_t|^2}}\left[a\lambda - \frac{i(a^2 + 0,25|q_t|^2)(q\bar{q}_t - q_t\bar{q})}{|q_t|^2}\right].
\end{aligned}
$$

The area of surface for this case is

$$
S = \int\int \sqrt{g}dxdt = \int\int \sqrt{\frac{1}{4\lambda^4}|q_t|^2}dxdt = \frac{1}{2\lambda^2}\int\int |q_t|dxdt.
$$

In string theory, this integral describes the area of the string worldsheet.

4. The coupled dispersionless equations with self-consistent sources as equations of the current-fed string

At scale transformation

$$
a \to 0.25\rho, \qquad (q, r) \to 0.5(q, r)
$$

the set of equations (1)-(6) takes the form

$$
\begin{aligned}
q_{xt} - \rho q + 4p_x &= 0, & (12)\\
r_{xt} - \rho r - 4n_x &= 0, & (13)\\
\rho_x - 0.5(rq)_t + 2(qn - rp) &= 0, & (14)\\
\eta_x + 0.5(rp - qn) &= 0, & (15)\\
p_x + 2i\omega p + \eta q &= 0, & (16)\\
n_x - 2i\omega n - \eta r &= 0. & (17)
\end{aligned}
$$

In case (7) we rewrite the system of equations (12)-(17) in following form

$$
\begin{aligned}
q_{xt} - \rho q + 4p_x &= 0, & (18)\\
-\bar{q}_{xt} + \rho\bar{q} - 4\bar{p}_x &= 0, & (19)\\
\rho_x + 0.5(q\bar{q})_t + 2\left(q\bar{p} + \bar{q}p\right) &= 0, & (20)\\
\eta_x - 0.5(\bar{q}p + q\bar{p}) &= 0, & (21)\\
p_x + 2i\omega p + \eta q &= 0. & (22)\\
\bar{p}_x - 2i\omega\bar{p} + \eta\bar{q} &= 0. & (23)
\end{aligned}
$$

Equations (18)-(20) become the nonlinear Klein Gordon-type equations

$$
\begin{aligned}
X_{1\tau\tau} - X_{1\sigma\sigma} &= -(Z_{1\tau} - Z_{1\sigma})X_1 - 4(X_{2\tau} + X_{2\sigma}), & (24)\\
Y_{1\tau\tau} - Y_{1\sigma\sigma} &= -(Z_{1\tau} - Z_{1\sigma})Y_1 - 4(Y_{2\tau} + Y_{2\sigma}), & (25)\\
Z_{1\tau\tau} - Z_{1\sigma\sigma} &= (X_1(X_{1\tau} - X_{1\sigma}) + Y_1(Y_{1\tau} - Y_{1\sigma})) - 4(X_1 X_2 + Y_1 Y_2). & (26)
\end{aligned}
$$

under the transformation of dependent and independent variables

$$q = X_1 + iY_1, \quad \bar{q} = X_1 - iY_1 \tag{27}$$

$$p = X_2 + iY_2, \quad \bar{p} = X_2 - iY_2 \tag{28}$$

$$\rho = \xi_t, \quad \xi = -Z_1, \quad \eta = Z_2 \tag{29}$$

$$\tau = x + t, \quad \sigma = x - t. \tag{30}$$

The system of equations (24)-(26) can be rewritten in the following vector form

$$\mathbf{r}_{1\tau\tau} - \mathbf{r}_{1\sigma\sigma} = (\mathbf{r}_{1\tau} - \mathbf{r}_{1\sigma}) \wedge [\mathbf{J} \wedge \mathbf{r}_1] - 4(\mathbf{r}_{2\tau} + \mathbf{r}_{2\sigma}),$$

where $\mathbf{r}_1, \mathbf{r}_2$ are position vectors, $\mathbf{J}$ is the external electric current defined by

$$\mathbf{r}_1 = (X_1, Y_1, Z_1), \quad \mathbf{r}_2 = (X_2, Y_2, Z_2), \quad \mathbf{J} = (0, 0, 1).$$

From (21)-(23) we get following set of equations under the transformation (27)-(30)

$$\begin{aligned}
X_{2\tau} + X_{2\sigma} &= 2\omega Y_2 - Z_2 X_1, \\
Y_{2\tau} + Y_{2\sigma} &= 2\omega X_2 - Z_2 Y_1, \\
Z_{2\tau} + Z_{2\sigma} &= (X_1 X_2 + Y_1 Y_2).
\end{aligned}$$

5. Conclusion

In this paper we considered the M-XXXII equation and its reductions having the Lax representation, constructed the 1 and 2 fundamental forms of soliton surfaces for the reductions of the M-XXXII equation, evaluated the Gaussian and mean curvature, and also found the surface area that describes the string's worldsheet. With the scale transformation and transformation of dependent and independent variables of M-XXXII equation we obtained a nonlinear Klein-Gordon equation and its vector form, which describes the dynamics of the current-fed string.

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