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Reconstructed $f(R)$ Gravity and Its Cosmological Consequences in theChameleon Scalar Field with a Scale Factor Describing the Pre-Bounce Ekpyrotic Contraction

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Abstract: The present study reports a reconstruction scheme for $f(R)$ gravity with the scale factor $a(t) \propto (t_* - t)^{\frac{2}{c^2}}$ describing the pre-bounce ekpyrotic contraction, where t_* is the big crunch time. The reconstructed $f(R)$ is used to derive expressions for density and pressure contributions, and the equation of state parameter resulting from this reconstruction is found to behave like "quintom". It has also been observed that the reconstructed $f(R)$ has satisfied a sufficient condition for a realistic model. In the subsequent phase, the reconstructed $f(R)$ is applied to the model of the chameleon scalar field, and the scalar field ϕ and the potential $V(\phi)$ are tested for quasi-exponential expansion. It has been observed that although the reconstructed $f(R)$ satisfies one of the sufficient conditions for realistic model, the quasi-exponential expansion is not available due to this reconstruction. Finally, the consequences of pre-bounce ekpyrotic inflation in $f(R)$ gravity are compared to the background solution for $f(R)$ matter bounce.

Keywords: pre-bounce ekpyrotic contraction; $f(R)$ gravity; reconstruction

1. Introduction

Observational evidence in support of the late time acceleration of the universe is documented in a plethora of literature [1–4]. An exotic matter, characterized by negative pressure, is considered to be responsible for this accelerated expansion and is dubbed "dark energy" (DE) [5–7]. Reviews on various candidates of DE have been made in a considerable number of literature works. Some significant ones include [5,7–9]. An approach, an alternative to dark energy, also known as "modified gravity", has some relative advantages. Although DE and modified gravity theories have some similarities in their basic approach, modified gravity has some features that have made it attractive in the study of the late time acceleration of the universe. One very promising modified theory of gravity is $f(R)$ gravity [10–19].

A scheme of the reconstruction of modified gravity that is capable of realizing a unification of the acceleration of late time and early inflation was demonstrated in [20]. In this reconstruction scheme, Reference [20] reported models of modified gravity capable of presenting a successful transition

from the phase of matter dominance to the late time accelerated phase of the universe. Another noteworthy reference in this context is [21], which reported $f(R)$ gravities that are viable and are reproduced through e-folding numbers. The unification of late and early time acceleration was also reported in this study. Another $f(R)$ reconstruction by the imposition of the restrictions of a dynamical nature was presented in [22]. Realization of ΛCDM through $f(R)$ gravity was elaborated in [23]. Reference [24] demonstrated how reconstruction schemes can be extended to modified gravity models so as to realize the transition from matter dominance to dark energy dominance. This is due to the non-linear character of $f(R)$ gravity relevant to early inflation for large R [25]. The unification approach, as stated above, was also reported in [26–29].

To avoid the measurable corrections to the occurrence of the local gravity, the models have to apply the chameleon mechanism [30,31] to overcome the Solar System tests. References [32–35] suggested the models that satisfy both the cosmological and the local gravity constraints. Studies including [36–39] demonstrated exact solutions that are capable of explaining the current acceleration. For more details, see [40,41]. The exact behaviour of the exotic matter that is thought to be responsible for the acceleration of the current universe is yet to be fully understood. As a consequence, various candidates have been proposed for DE to date, and modified theories also have a considerable variability in the approach, and accordingly, the cosmological parameters have been studied. Among the several candidates proposed so far for DE, holographic dark energy (HDE) is considered to be of immense potential. HDE was proposed on the basis of the holographic principle [42–45]. The present work, as one of its primary objectives, aims to reconstruct modified gravity based on holographic dark energy. Holographic reconstruction of modified gravity has already been reported in the cosmological literature. Various authors [40,41,46–48] have reconstructed different candidates of modified gravity from holographic dark energy (HDE) with different IR cut-offs.

One natural query of the present day cosmology is whether the universe's evolution actually follows the standard inflationary paradigm. In order for the standard inflation to follow, an initial singularity has to exist. Otherwise, the cosmological bounce is considered, in which case there does not exist any initial singularity [49]. Big bounce scenarios provide us with an alternative to the Big Bang cosmologies. A bounce cosmology in $f(R)$ cosmological settings was reported in [49], where it was observed that the bouncing point is characterized by a type-IV singularity. In another interesting study, Odintsov and Oikonomou [50] provided an $f(R)$ gravity description of a ΛCDM bouncing model without depending on any matter fluid or cosmological constant. In another study, the authors [51] demonstrated by conformally transforming the Jordan frame singular bounce that the Einstein frame metric leads to a Big Rip singularity. The present work endeavours to demonstrate a reconstruction scheme for $f(R)$ gravity with a scale factor describing the pre-bounce ekpyrotic contraction and to study the cosmological consequences for the chameleon scalar field model. The rest of the paper is organised as follows: In Section 1, a brief overview of $f(R)$ gravity is presented. A reconstruction scheme for $f(R)$ gravity is presented in Sections 2 and 3, demonstrating the cosmology of the chameleon scalar field under reconstructed $f(R)$ gravity. In Section 4, we discuss the background evolution of matter bounce in $f(R)$ gravity for a different choice of the bouncing scale factor. The results and concluding remarks are presented in Section 5. Throughout the study, t represents cosmic time, and it is known that, as per the Big Bang theory, the age of the universe is 13.799 ± 0.021 billion years.

2. Brief Overview of $f(R)$ Gravity

The action of $f(R)$ gravity is given by [15,17]:

$$S = d^4x\sqrt{-g}\left[\frac{f(R)}{2\kappa^2} + \mathcal{L}_{matter}\right] \tag{1}$$

where $\kappa^2 = \frac{8\pi G}{c^4}$, where c stands for the velocity of light, $g = \det g_{\mu\nu}$ is the determinant of the metric tensor and $\mathcal{L}_{matter}$ is the matter Lagrangian. The $f(R)$ is some function of the Ricci scalar. In the remaining part of the paper, we shall take $\kappa^2 = 1$. $f(R)$ is a non-linear function of R that contains corrections to the EHaction. The gravitational field equations are given by:

$$H^2 = \frac{1}{3f'(R)}\left(\rho_m + \rho_R\right) \tag{2}$$

$$\dot{H} = -\frac{1}{2f'(R)}\left(\rho_m + p_m + \rho_R + p_R\right) \tag{3}$$

where ρ_R and p_R are the density and pressure generated due to $f(R)$ gravity, and they have the forms:

$$\rho_R = \frac{1}{2}\left[-f(R) + Rf'(R)\right] - 3H\dot{R}f''(R) \tag{4}$$

and:

$$p_R = \frac{1}{2}\left[f(R) - Rf'(R)\right] + \left[2H\dot{R} + \ddot{R}\right]f''(R) + \dot{R}^2 f'''(R) \tag{5}$$

respectively. The density due to dark matter in $f(R)$ gravity is:

$$\rho_m = 3H^2 f'(R) - \rho_R \tag{6}$$

and we consider pressureless dark matter, $p_m = 0$. In the subsequent section, we will consider $f(R)$ gravity with the scale factor describing the pre-bounce ekpyrotic contraction.

3. A Reconstruction Scheme for $f(R)$ Gravity

In this section, we describe the reconstruction scheme for $f(R)$ gravity in a bounce model developed by Cai et al. [52] and further demonstrated by Koehn et al. [53], Odintsov and Oikonomou [54] and Odintsov et al. [55]. In the model demonstrated by [53], a scalar field ϕ with non-canonical kinetic terms and a potential $V(\phi)$ were used to develop the cosmological model in a non-supersymmetric framework. In the present work, instead of taking the potential as the ekpyrotic potential, i.e., $V(\phi) \approx -V_0 e^{-c(\phi)\phi}$ we consider the chameleon scalar field in the framework of $f(R)$ gravity reconstructed for a scale factor describing a pre-bounce ekpyrotic contraction, as was mentioned in [53]. It may also be noted that the reconstruction procedure is similar to that of [55]. However, instead of introducing the e-folding number N, contrary to what was presented in [55], we demonstrate a reconstruction scheme for $f(R)$ through the cosmic time t and subsequently in terms of the Ricci scale factor R. The presence of dark matter is also considered. In Equation (7), t_* is the big crunch time. However, instead of constraining c by the lower bound of $\sqrt{6}$, we constrained it by non-negativity. The scale factor describing a pre-bounce ekpyrotic contraction is [53]:

$$a(t) \propto (t_* - t)^{\frac{2}{c^2}} \tag{7}$$

giving rise to:

$$\dot{a} = -\frac{2(t_* - t)^{\frac{2}{c^2}-1}}{c^2} \tag{8}$$

It may be noted that t_* is the big crunch time if the ekpyrotic phase were to continue until that time and c is a parameter constrained to $c > \sqrt{6}$ [55]. In this connection, we would like to mention that in the present work, we do not impose any prior constraint on c; rather, we generate a constraint on c based on the solution of the reconstructed $f(R)$. It may further be mentioned that the constraint on c was initially

presented in [53], where the constraint of $c > \sqrt{6}$ is needed for the assumed form of ekpyrotic potential. In this work, instead of considering a specific form of potential, firstly we reconstruct $f(R)$ and then consider the scalar field a chameleon one that has a non-minimal coupling with the matter Lagrangian through an analytic function. For the above choice of scale factor, the Hubble parameter and its derivatives turn out to be as follows:

$$H = \frac{\dot{a}}{a} = -\frac{2}{c^2(t_* - t)}, \quad \dot{H} = -\frac{2}{c^2(t_* - t)^2}, \quad \ddot{H} = \frac{4}{c^2(t_* - t)^3}. \tag{9}$$

Using the form of the Hubble parameter derived above the Ricci scalar, its derivatives are derived below:

$$R = 6\left(2H^2 + \dot{H}\right) = -\frac{12(c^2 - 4)}{c^4(t_* - t)^2}, \quad \dot{R} = \frac{24(4 - c^2)}{c^4(t_* - t)^3}, \quad \ddot{R} = \frac{72(4 - c^2)}{c^4(t_* - t)^4}. \tag{10}$$

It was already stated that the purpose of the present work is to reconstruct $f(R)$ gravity and to demonstrate the cosmology of the chameleon scalar field under this reconstruction. In view of the above, $f(R)$ gravity is reconstructed using Equations (9) and (10). Hence, the modified field equations give rise to the Friedmann equation in FRW geometry as follows:

$$-18\left(4H^2\dot{H} + H\ddot{H}\right) f''(R) + 3\left(H^2 + \dot{H}\right) f'(R) - \frac{f(R)}{2} + \rho_m = 0, \tag{11}$$

where $\rho_m = \rho_{m_0} a(t)^{-3}$ indicates the dark matter density and R is the Ricci scalar as stated in Equation (10). In Equation (10), we show how R can be expressed in terms of t based on the choice of the scale factor. Hence, the modified Friedmann Equation (11) written above gives rise to a differential equation with t as the independent variable, which upon solving gives us the solution for $f(R)$ in terms of cosmic time t as follows:

$$F(t) = 1728^{\frac{1}{c^2}} A_1 \left[-\frac{-4 + c^2}{c^4(t_* - t)^2}\right]^{\frac{3}{c^2}} + 12^{A_2}\left[-\frac{-4 + c^2}{c^4(t_* - t)^2}\right]^{A_2} C_1 + 12^{A_3}\left[-\frac{-4 + c^2}{c^4(t_* - t)^2}\right]^{A_3} C_2 \tag{12}$$

where $F(t) = f(R(t))$. The above equation (12) is converted to a function of R, and finally, the reconstructed $f(R)$ takes the following form:

$$f(R) = A_1 R^{\frac{3}{c^2}} + R^{A_2} C_1 + R^{A_3} C_2, \tag{13}$$

where:

$$\begin{aligned}
A_1 &= \frac{1}{24 - 13c^2 + c^4}\left[2^{1-\frac{6}{c^2}} 3^{-\frac{3}{c^2}} \rho_{m_0} c^2(-4 + c^2)\left(\frac{4 - c^2}{c^4}\right)^{-\frac{3}{c^2}}\right], \\
A_2 &= -\frac{1}{4c^2}\left[2 + c\left\{-3c + \frac{(4 + c)\sqrt{c^4 + 20c^2 + 4}}{c^2 - 4}\right\}\right] + \frac{2\sqrt{c^4 + 20c^2 + 4}}{c^2(c^2 - 4)}, \\
A_3 &= -\frac{1}{4c^2}\left[2 + c\left\{-3c + \frac{(4 + c)\sqrt{c^4 + 20c^2 + 4}}{c^2 - 4}\right\}\right] + \frac{\sqrt{c^4 + 20c^2 + 4}}{2(c^2 - 4)}.
\end{aligned} \tag{14}$$

Equation (14) shows that for A_1 to be real, one needs $0 < c^2 < 4$. This choice does not affect A_2 and A_3.

The derivatives of $f(R)$ up to various orders are computed below:

$$f'(R) = \frac{1}{R}\left[\frac{3A_1R^{\frac{3}{c^2}}}{c^2} + A_2R^{A_2}C_1 + A_3R^{A_3}C_2\right], \tag{15}$$

$$f''(R) = \frac{1}{R^2}\left[\frac{3A_1(3-c^2)R^{\frac{3}{c^2}}}{c^4} + (-1+A_2)A_2R^{A_2}C_1 + (-1+A_3)A_3R^{A_3}C_2\right], \tag{16}$$

$$f'''(R) = \frac{1}{R^3}[\frac{3A_1(9-9c^2+2c^4)R^{\frac{3}{c^2}}}{c^6} + (-2+A_2)(-1+A_2)A_2R^{A_2}C_1 + (-2+A_3)(-1+A_3)A_3R^{A_3}C_2]. \tag{17}$$

It is clear from the above expressions that the reconstructed $f(R)$ is a real solution to Equation (11), and derivatives up to the orders shown above exist. In Figure 1, we observe that $f(R)$ shows an increasing pattern with cosmic time for $0 < c^2 < 4$. This indicates that $\frac{d}{dt}f(R) > 0$ for $0 < c^2 < 4$. Furthermore, Equation (10) shows that $\dot{R} > 0$ for $0 < c^2 < 4$. Thus, $f'(R) = \frac{\dot{f(R)}}{\dot{R}} > 0$ for $0 < c^2 < 4$. Hence, it is observed that $f(R)$ gravity so obtained is free from ghost instability. Furthermore, from Figure 2, we observe that for $0 < c^2 < 4$, we have an increasing pattern of $f'(R)$ with cosmic time t. Hence, the time derivative of $f'(R)$ would be positive. Therefore, we will have $\frac{\dot{f'(R)}}{\dot{R}} = f''(R) > 0$. In this connection, we would like to mention that $f(R)$ gravity belongs to the class of modified gravity models for which the stability conditions have been investigated in the literature in an extensive manner (e.g., [56–58]). These literature works have demonstrated the high curvature regime stable against small perturbations to be a condition for the stability of $f(R)$ gravity. This leads to the requirement of a positive mass squared for the scalaron, i.e., $m^2_{fR} \approx \frac{1+f'(R)}{f''(R)} > 0$. In the present case, both $f'(R)$ and $f''(R)$ are positive, $m^2_{fR} > 0$. The tachyonic instability manifests in the negative mass terms leading to an unstable low-k regime [58]. In the present case, squared mass comes out to be positive for the reconstructed $f(R)$. Thus, the model of $f(R)$ obtained through the reconstruction scheme presented above is free from tachyonic instability. This proves the physical viability of the reconstructed $f(R)$ model.

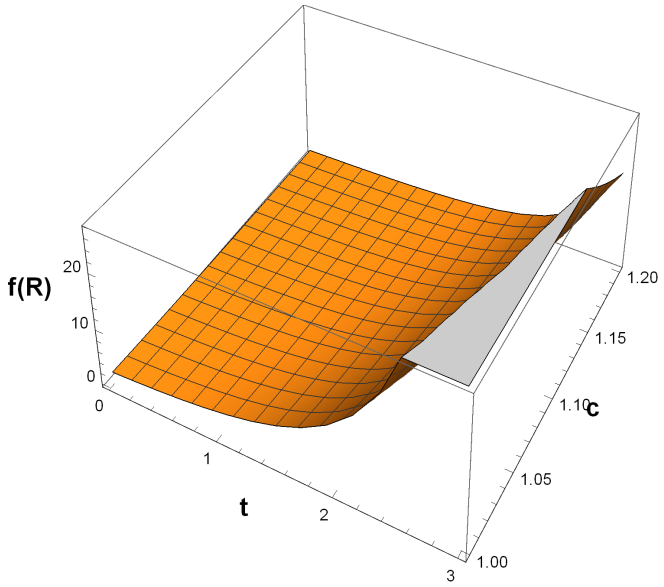

Figure 1. Evolution of reconstructed $f(R)$ (Equation (12)).

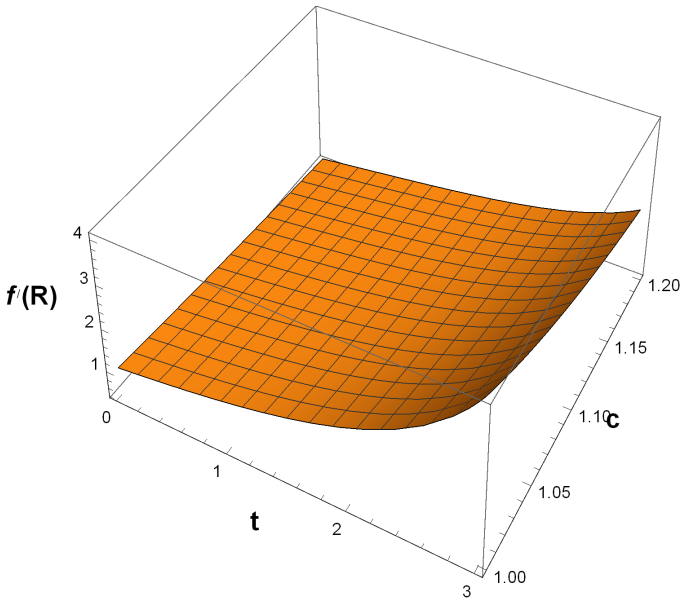

Figure 2. Evolution of reconstructed $f'(R)$.

4. Chameleon Scalar Field under Modified $f(R)$ Gravity

In this section, the reconstructed $f(R)$ gravity will be considered to reconstruct the chameleon scalar field and chameleon potential to investigate whether they can lead to quasi-exponential expansion in the framework of the reconstructed $f(R)$ gravity. In this connection, it may be noted that an equivalence between $f(R)$ and scalar–tensor theories was discussed in [59,60]. In these references, it was discussed how one may consider $\phi = R$ to reproduce the original action of the chameleon mechanism [61]. In a flat homogeneous universe, the action for the relevant scalar field and potential is given by [62]:

$$S = \sqrt{-g}d^4x \left[f(\phi)\mathcal{L} + \tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} + \tfrac{R}{16\pi G} - V(\phi) \right] \tag{18}$$

where ϕ is the chameleon scalar field and $V(\phi)$ is the chameleon potential. R and G represent the Ricci scalar and Newtonian constant of gravity, respectively. $f(\phi)$ is an analytic function of ϕ, and $f(\phi)\mathcal{L}$ is the modified matter Lagrangian. Variation of the action with respect to metric tensor components in an FRW cosmology leads to the following modified field equation (assuming $8\pi G = 1$).

$$H^2 = \frac{1}{3}\left[(\rho_m + \rho_R) f(R) + \frac{1}{2}\dot{\phi}^2 + V(\phi) \right], \tag{19}$$

$$2\dot{H} + 3H^2 = \frac{1}{2}\left[-p_R f(R) - \frac{1}{2}\dot{\phi}^2 + V(\phi) \right]. \tag{20}$$

It may be noted that as we consider the chameleon scalar field in the $f(R)$ gravity framework, we chose the density to be $\rho_m + \rho_R$, the pressure to be p_R and the analytic function to be replaced by a function of the Ricci scalar R.

Solving Equations (4), we get the reconstructed density contribution due to $f(R)$ gravity as follows:

$$\rho_R = -\frac{1}{2c^2(-4+c^2)}\left[1728^{\frac{1}{c^2}} A_1 \left(30-13c^2+c^4\right)\left\{\frac{4-c^2}{c^4(t_*-t)^2}\right\}^{\frac{3}{c^2}} + c^2\left\{12^{A_2}(-1+A_2)\left\{4+(-1+2A_2)c^2\right\}\left\{-\frac{-4+c^2}{c^4(t_*-t)^2}\right\}^{A_2} C_1 + 12^{A_3}(-1+A_3)\left\{4+(-1+2A_3)c^2\right\}\left\{-\frac{-4+c^2}{c^4(t_*-t)^2}\right\}^{A_3} C_2\right\}\right]. \tag{21}$$

Similarly, from Equation (5), we derive the expression for pressure contribution of this reconstructed $f(R)$ gravity as follows:

$$p_R = \frac{1}{6c^2(-4+c^2)}\left[-2^{1+\frac{6}{c^2}} 3^{1+\frac{3}{c^2}} A_1 \left(-3+c^2\right)\left\{\frac{4-c^2}{c^4(t_*-t)^2}\right\}^{\frac{3}{c^2}} + c^2\left\{-12^{A_2}(-1+A_2)\left\{-12+(3-4A_2)c^2+A_2(-1+2A_2)c^4\right\}\left\{-\frac{-4+c^2}{c^4(t_*-t)^2}\right\}^{A_2} C_1 - 12^{A_3}(-1+A_3)\left\{-12+(3-4A_3)c^2+A_3(-1+2A_3)c^4\right\}\left\{-\frac{-4+c^2}{c^4(t_*-t)^2}\right\}^{A_3} C_2\right\}\right] \tag{22}$$

Now, using the reconstructed $f(R)$ and ρ_R in Equation (6), the form of dark matter density in the $f(R)$ gravity framework turns out to be the following:

$$\rho_m = \frac{1}{2(-4+c^2)}\left[\frac{1728^{\frac{1}{c^2}} A_1\left(24-13c^2+c^4\right)\left\{\frac{4-c^2}{c^4(t_*-t)^2}\right\}^{\frac{3}{c^2}}}{c^2} + 12^{A_2}\left\{-4+c^2+A_2\left\{2+(-3+2A_2)c^2\right\}\right\}\left\{-\frac{-4+c^2}{c^4(t_*-t)^2}\right\}^{A_2} C_1 + 12^{A_3}\left\{-4+c^2+A_3\left\{2+(-3+2A_3)c^2\right\}\right\}\left\{-\frac{-4+c^2}{c^4(t_*-t)^2}\right\}^{A_3} C_2\right]. \tag{23}$$

The equation of the state parameters, as defined below, can be modified using the reconstructed pressure and densities elaborated above.

$$w_R = \frac{p_R}{\rho_R} \tag{24}$$

$$w_{eff} = \frac{p_R}{\rho_m + \rho_R} \tag{25}$$

At this juncture, we consider the modified field equations for the reconstruction of the chameleon scalar field. Using modified field Equations (19) and (20), we can have:

$$2V - \dot{\phi}^2 = 8\dot{H} + 12H^2 + 2p_R f(R) \tag{26}$$

where H, p_R and $f(R)$ were already derived. The cosmological parameters derived above will now be explored through plots, and the outcomes will be discussed in the subsequent section.

5. Background Solution for $f(R)$ Matter Bounce

In the previous sections, we discussed the various outcomes of a scale factor describing the pre-bounce ekpyrotic contraction. In this section, we demonstrate a comparison of the results with the bounce scale factor proposed in [63]:

$$a(t) = a_0\left(1+\frac{3\sigma t^2}{2}\right)^{\frac{1}{3}} \tag{27}$$

where a_0 is a scale factor at the bouncing point and σ is a positive parameter. It was demonstrated in [63] that apart from presenting bouncing behaviour corresponding to matter-dominated contraction and expansion, such an ansatz presented in Equation (27) additionally exhibits the advantage of allowing for semi-analytic solutions.

With the help of this scale factor, we derive the Hubble parameter (H), the Ricci scalar (R), and their derivatives in terms of cosmic time t as follows:

$$H = \frac{2t\sigma}{2+3t^2\sigma}, \quad \dot{H} = \frac{2\sigma(2-3t^2\sigma)}{(2+3t^2\sigma)^2}, \quad \ddot{H} = \frac{36t\sigma^2(t^2\sigma-2)}{(2+3t^2\sigma)^3} \tag{28}$$

$$R = \frac{12\sigma(2+t^2\sigma)}{(2+3t^2\sigma)^2}, \quad \dot{R} = -\frac{24t\sigma^2(10+3t^2\sigma)}{(2+3t^2\sigma)^3}, \quad \ddot{R} = \frac{24\sigma^2\left(-20+3t^2\sigma(44+9t^2\sigma)\right)}{(2+3t^2\sigma)^4} \tag{29}$$

This choice of $a(t)$ presents the bouncing behaviour corresponding to matter-dominated contraction and expansion. Furthermore, t ranges from $-\infty$ to $+\infty$. Bounce occurs at $t = 0$. Using the expression for R, we can straight away express t as:

$$t(R) = \pm\left(\sqrt{\frac{2}{3}}\sqrt{\frac{-R\sigma+\left(\sigma^2 \pm \sqrt{\sigma^3(4R+\sigma)}\right)}{R\sigma^2}}\right) \tag{30}$$

The above inversion is valid if $-\sqrt{\frac{2}{3\sigma}} < t < \sqrt{\frac{2}{3\sigma}}$. At the bouncing point, we take $a_0 = 1$ and $\rho_{m0} = 1.41 \times 10^{-5}$. Roughly based on the CMB spectrum, $\sigma = 7 \times 10^{-6}$.

With H, R and $a(t)$ discussed above, clearly, with this choice,

$$f'(R) = -\frac{\left(2+3t^2\sigma\right)^3 f'(t)}{24t\sigma^2\left(10+3t^2\sigma\right)}$$

and:

$$f''(R) = \frac{\left(2+3t^2\sigma\right)^5\left(\left(-20+132t^2\sigma+27t^4\sigma^2\right)f'(t)+t\left(20+36t^2\sigma+9t^4\sigma^2\right)f''(t)\right)}{576t^3\sigma^4\left(10+3t^2\sigma\right)^3}.$$

Because of the complicated form of the coefficients, the differential equation (Equation (11)) with $f'(R)$, $f''(R)$ in the above forms cannot be solved analytically. The reconstructed $f(R)$ is obtained numerically, and the solution in terms of cosmic time t is graphically presented in Figure 3. Here, in the numerical solutions, the positive parameter σ plays a significant role. Therefore, we take three values of $\sigma = 0.75,\ 1.25,\ 1.75$ from the interval $(0.7, 2)$ for the entire calculation. In the following figures, the red, green and blue lines indicate the values $\sigma = 0.75, 1.25$ and 1.75, respectively. It is observed in Figure 3 that $f(R) \to 0$ as $t \to 0$ before, as well as after bounce. Before bounce, the reconstructed $f(R)$ tends to zero from the negative side. However, after attaining zero at the bouncing point, it starts increasing towards the positive direction. Hence, $f(R) \to 0$ irrespective of the pre- or post-bounce scenario. Hence, it is apparent that a realistic solution is available with this choice of the bounce scale factor. Secondly, Figure 4 shows that $\frac{f'(R)}{R} > 0$ before bounce and < 0 after bounce. Hence, in the pre-bounce phase, the model is not affected by tachyon instability, and this is consistent with the pre-bounce ekpyrotic contraction presented in the previous section. However, after bounce, $\frac{f'(R)}{R} < 0$. Hence, the post-bounce scenario is characterized by tachyon instability. As we study $2V - \dot{\phi}^2$, we observe that it is positive in the pre-, as well as post-bounce scenario (see Figure 5). However, for $t^- < t < t^+$, i.e., in the vicinity of the bouncing point, $2V - \dot{\phi}^2$ is

nearly flat, and after bounce, it starts increasing sharply; hence, we may consider it to be consistent with the inflationary expansion.

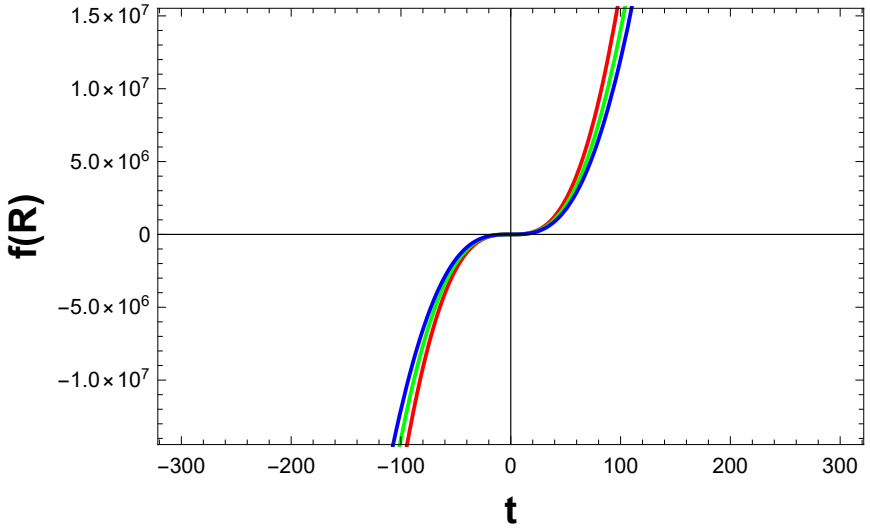

Figure 3. Evolution of the reconstructed $f(R)$ gravity over cosmic time t for scale factor $a(t) = a_0\left(1 + \frac{3\sigma t^2}{2}\right)^{\frac{1}{3}}$.

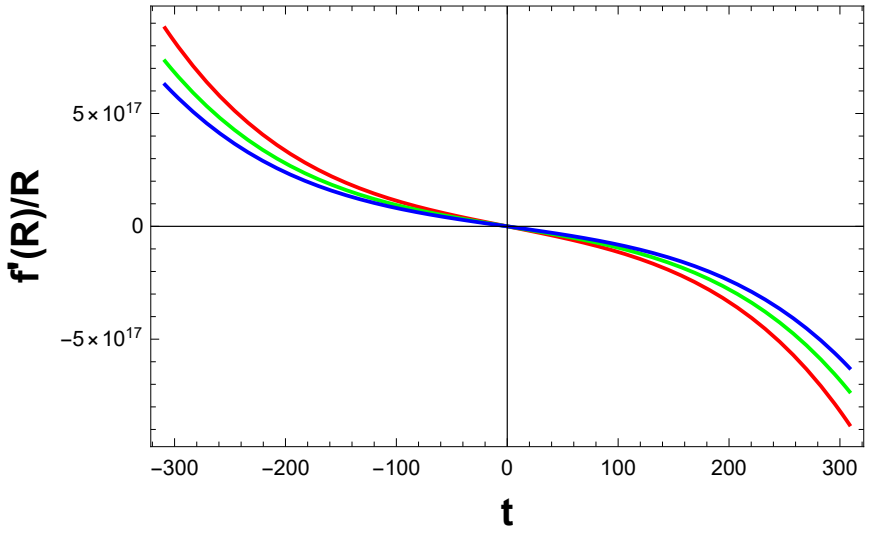

Figure 4. Evolution of $\frac{f'(R)}{R}$.

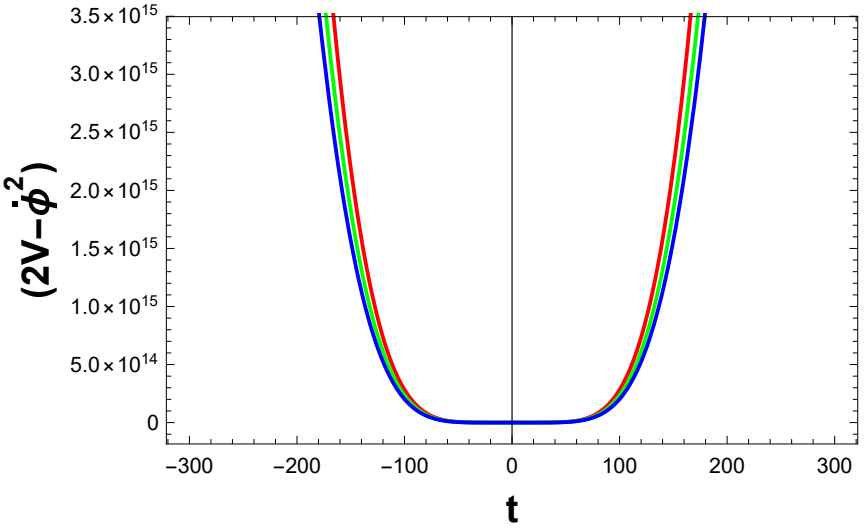

Figure 5. Evolution of the reconstructed $2V - \dot{\phi}^2$ over cosmic time t in the reconstructed $f(R)$ gravity (Figure 3) framework.

6. Results and Conclusions

In the present work, we carried out a reconstruction scheme for $f(R)$ gravity with the scale factor in the form $a(t) = (t_* - t)^{\frac{2}{c^2}}$. Initially, a Hubble parameter H was computed, and its time derivatives of different orders also were derived for this scale factor to get the Ricci scalar R. Subsequently, the field equation for the $f(R)$ gravity Equation (11) were solved in the presence of dark matter. The reconstructed $f(R)$ was obtained this way as a function of t in Equation (12), which was reexpressed as a function of R in Equation (13). This reconstructed $f(R)$ is plotted against cosmic time t for a range of values of $0 < c^2 < 4$ in Figure 1. It is observed in Figure 1 that $f(R)$ has an increasing pattern with t. Furthermore, Equation (13) shows that $\lim_{R\to 0} f(R) = 0$. This satisfies a sufficient condition for a realistic model [64,65]. Furthermore, the non-existence of ghost and tachyonic instability was established. Hence, it can be said that the reconstructed $f(R)$ model is a realistic model. It may be noted that $C_1 = 0.5, C_2 = 0.3, t_* = 4.1, \rho_{m_0} = 0.32$ and $0 < c^2 < 4$ were used while creating the plots. The choice of C_1, C_2 was made through trial and error; t_* and c^2 were set keeping the real solution for $f(R)$ in mind. It may be noted that the formulation of the reconstruction approach was inspired by [24].

It was further observed that the increasing pattern of $f(R)$ is influenced by the value of c. For larger values of c, $f(R)$ maintains approximately a flat pattern for a considerable period of cosmic time and exhibits a sudden increase at a later stage of the universe. Different derivatives of this reconstructed $f(R)$ were also computed and presented in Equations (15)–(17). In the next part of this study, we demonstrated the behaviour of the chameleon scalar field under this modified form of $f(R)$. The modified field equations in the presence of the chameleon scalar field and potential were presented in Equations (19) and (20). While reconstructing the chameleon scalar field and potential under this reconstructed $f(R)$ model, we first reconstructed the density and pressure due to $f(R)$ using Equation (13). Based on the reconstructed p_R and ρ_R, we demonstrated the behaviour of the EoS parameter w_R due to $f(R)$ gravity in Figure 6 and the effective EoS parameter w_{eff} in Figure 7. It is observed in Figure 6 that w_R stays at the negative level for the evolution of the universe. This holds true for the entire range of c. Hence, it may be concluded that w_R due to reconstructed $f(R)$ gravity behaves like the phantom. The phantom behaviour is stronger for lower values of c than the higher values. However, if we consider w_{eff} with the same choice of parameters, it is found that for $0 < c \lessapprox 0.5$, $w_{eff} \approx -1$, i.e., behaving approximately like the cosmological constant. In Figure 8, it is observed that the fractional energy density $\Omega_m = \frac{\rho_m}{3H^2}$ based on the reconstructed ρ_m, and

the scale factor describing the pre-bounce ekpyrotic contraction stays at the positive level. This satisfies the weak energy condition (WEC). Furthermore, the model produces fractional matter density that decays with the evolution of the universe. This indicates a transition from matter domination in the early stage to dark energy domination in the late stage.

In Figure 9, the evolution of $2V - \dot{\phi}^2$ is studied against z for a range of values of c. This figure shows that $2V - \dot{\phi}^2 \leq 0$, which violates the condition for inflationary expansion. Hence, quasi-exponential expansion is not available with the chameleon scalar field considered in the framework of modified $f(R)$ gravity in the presence of the scale factor $a(t) = (t_* - t)^{\frac{2}{c^2}}$ describing the pre-bounce ekpyrotic contraction. This observation is in contradiction with the study of Chattopadhyay et al. [66], where quasi-exponential expansion was found to be possible for a linear $f(T)$ gravity based on holographic Ricci dark energy.

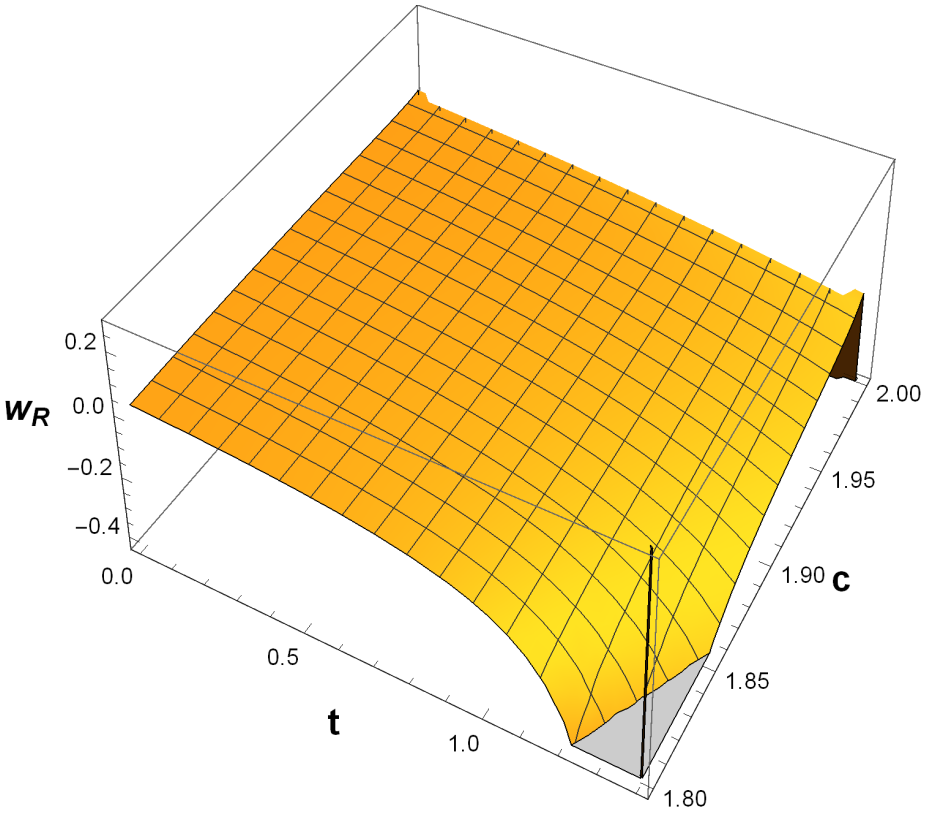

Figure 6. The EoS parameter $w_R = \frac{p_R}{\rho_R}$ (Equations (21) and (22)) due to the reconstructed $f(R)$ gravity.

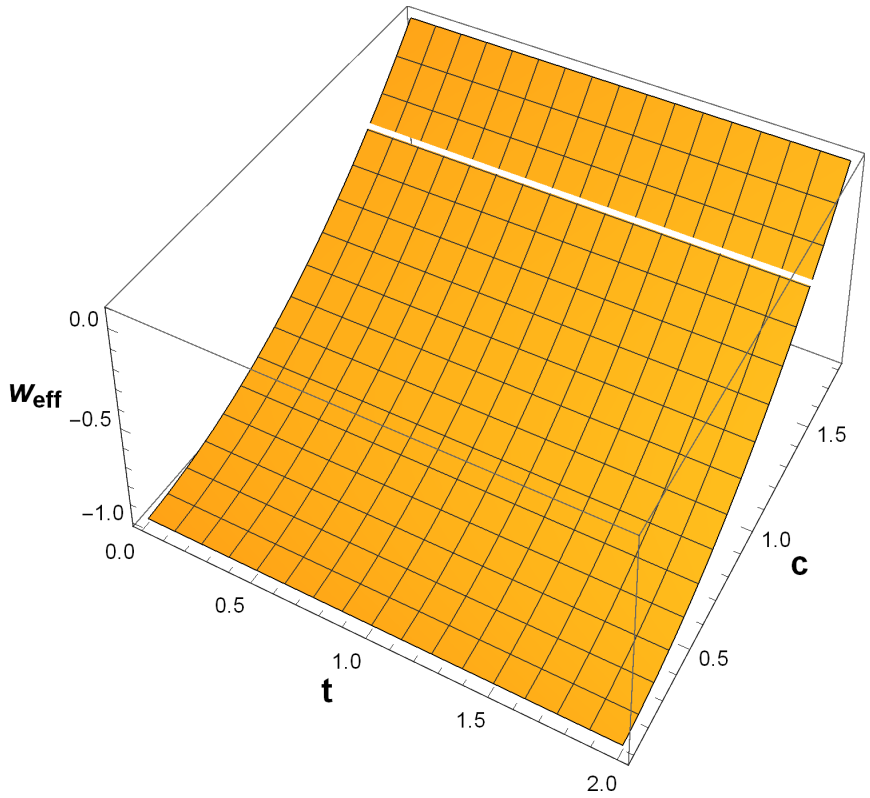

Figure 7. The behaviour of the effective EoS parameter $(w_{eff} = \frac{p_R}{\rho_R + \rho_m})$ (Equations (21)–(23)).

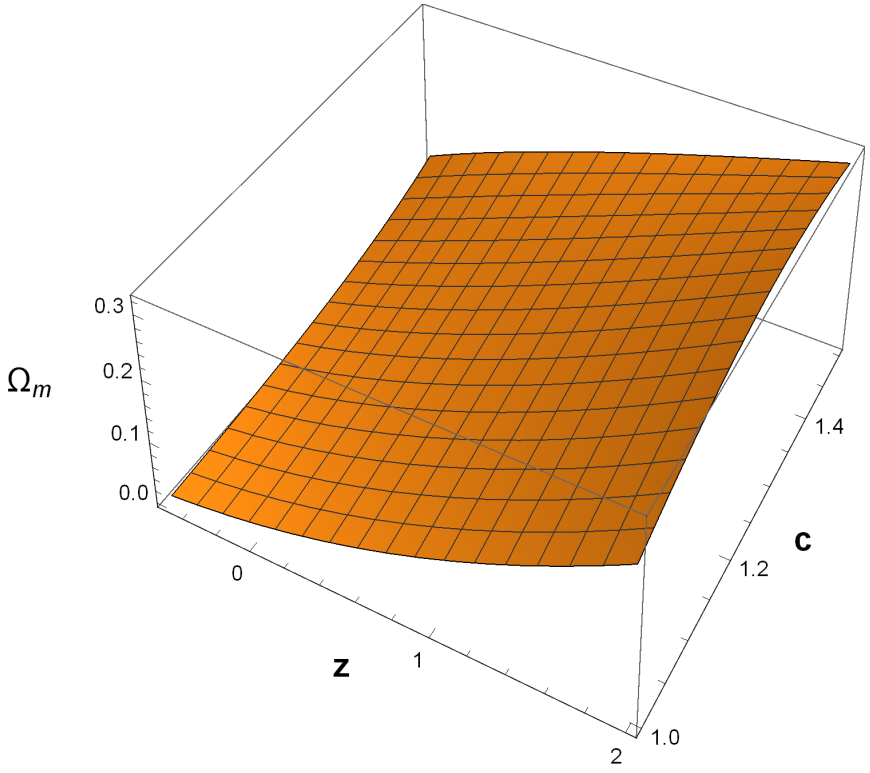

Figure 8. The evolution of the fractional matter density $\Omega_m = \frac{\rho_m}{3H^2}$ in the reconstructed $f(R)$ gravity.

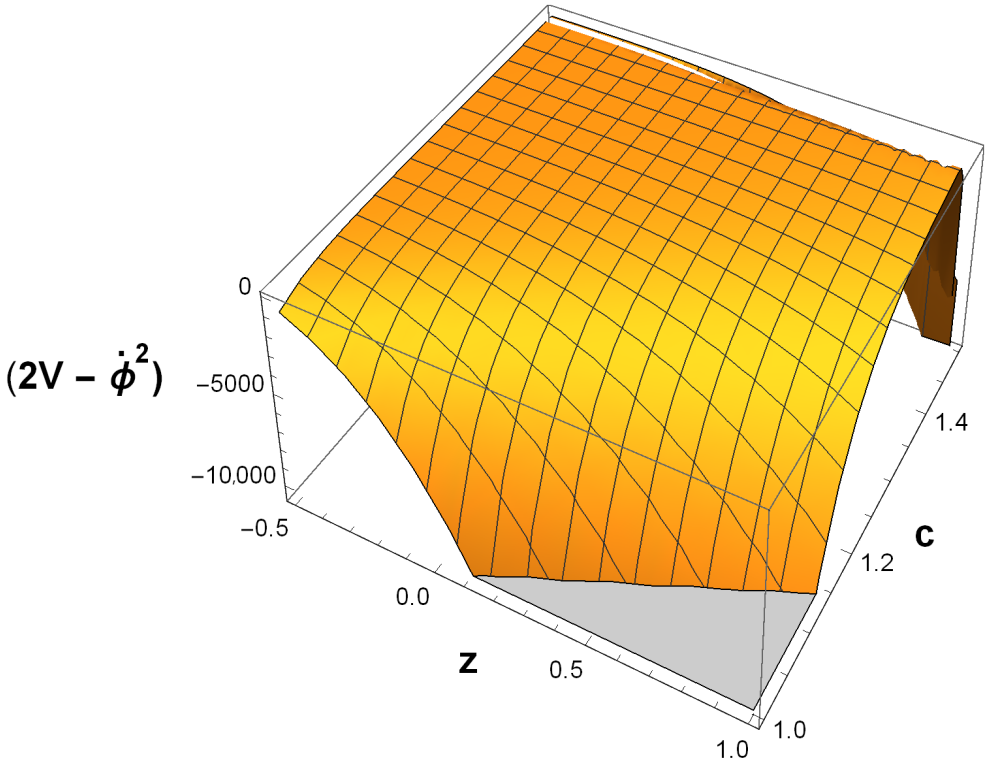

Figure 9. The evolution of $2V - \dot{\phi}^2$ in the reconstructed $f(R)$ gravity framework.

While concluding, we should mention that the reconstruction of $f(R)$ gravity was demonstrated in the model to unify the bouncing behaviour in the early universe, and the late time accelerated expansion of the universe at the dark energy-dominated stage can occur within a unified model. In view of the same, a scale factor describing the pre-bounce ekpyrotic contraction was chosen, and the reconstructed $f(R)$ was demonstrated to be capable of transiting the universe from the matter-dominated to the dark energy-dominated phase. Furthermore, the reconstructed $f(R)$ was found to be free from ghost and tachyonic instabilities, and hence, it was interpreted that the reconstructed $f(R)$ is realistic. We could explicitly derive the $f(R)$ gravity model capable of demonstrating the pre-bounce ekpyrotic contraction and the late time acceleration in a single model framework. This kind of reconstruction approach was earlier demonstrated in [67] in the framework of $f(G)$ gravity. As a future study, we propose to demonstrate whether this reconstruction scheme works for any arbitrary choice of the scale factor and to compare the results with the conventional reconstruction scheme using e-foldings.

We demonstrated the background matter bound for $f(R)$ with another bouncing scale factor introduced in [63] and found that although in the pre-bounce scenario, the model has tachyonic stability, the tachyonic stability is lost after the bouncing point. At the bouncing point, we found that $f(R) \to 0$ for $t \to 0^-$, as well as $t \to 0^+$. At this juncture, we must mention some important works in the direction of matter bounce solutions in the modified gravity framework. With the scale factor in the form $a(t) = (a_0 t^2 + 1)^n$, Reference [68] demonstrated that the Lagrange multiplier $f(R)$ is more adequate than the standard $f(R)$ in realizing the cosmological bounce. The matter bounce scenario, the singular bounce, the superbounce, and a symmetric bounce scenario in modular $f(R)$ were discussed by [69]. The present work primarily focused on the pre-bounce ekpyrotic contraction scenario. Nevertheless, the bouncing scenarios demonstrated in [69] are proposed to be incorporated into the study of exit from the pre-bounce ekpyrotic contraction in the modified gravity framework.

Finally, let us make a note of the behaviour of the EoS parameter obtained while reconstructing $f(R)$ gravity. In this connection, let us mention the very recent work of [70], where the authors presented unification models of gravity realizing the inflationary era along with a post-inflationary early dark energy era, with the late-time dark energy era. In line with [70], the current study can be extended to have a further insight into the unification approach by fine- tuning the EoS parameter towards a more successful unification of dark energy of the different eras.

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