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On the two-component generalization of the (2+1)-dimensional Davey-Stewartson I equation

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Abstract. The geometric-gauge equivalent of the famous Ishimori spin equation is the (2+1)-dimensional Davey-Stewartson equation, which in turn is one of the (2+1)-dimensional generalizations of the nonlinear Schrodinger equation. Multicomponent generalization of nonlinear integrable equations attract considerable interest from both physical and mathematical points of view. In this paper, the two-component integrable generalization of the (2+1)-dimensional Davey-Stewartson I equation is obtained based on its one-component representation, and the corresponding Lax representation is also obtained.

1. Introduction

It is known that integrable nonlinear Schrodinger type equations (NSE) are key models in the theory of integrable equations. Recently, their multicomponent generalizations have been actively studied. In the work [1] it is shown that the Manakov two-step system is integrable. The geometrical connection with the last system and the two-layer spin model was established in [2] - [4]. The geometric gauge equivalent of the Ishimori spin equation [5] is a (2+1)-dimensional Davy-Stewartson (DS) equation, which is one of the (2+1) -dimensional generalizations of the NSE [6].

Consider the (2+1)-dimensional DS equation

$$iq_t + \frac{1}{2}(\sigma^2 q_{xx} + q_{yy}) = (v - qr)q, \tag{1}$$

$$-ir_t + \frac{1}{2}(\sigma^2 r_{xx} + r_{yy}) = (v - qr)r, \tag{2}$$

$$v_{xx} - \sigma^2 v_{yy} = 2(qr)_{xx}, \tag{3}$$

where $r = \pm q^*$, q^* is the complex conjugate of q. For $\sigma^2 = 1$, the system of equations 1-2 is called the Davy-Stewartson Type I equation (DSI), and for $\sigma^2 = -1$ – the Davy-Stewartson Type II equation (DSII) [7]. We focus on the (2+1)-dimensional DSI equation, and in the next section we give some well-known data on the single-component (2+1)-dimensional DSI equation. Our goal is to derive a two-component generalization for the last equation.

The result of the study is formed in the form of approval and is proved in the second paragraph.

2. A single-component (2+1)-dimensional DSI equation

As it is known, the standard Lax representation of the DSI equation is

$$F_y = \sigma_3 F_x + QF, \tag{4}$$
$$F_t = A_2 F_{xx} + A_1 F_x + A_0 F, \tag{5}$$

where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix},$$

$$A_0 = i\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad A_1 = 2i\begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \quad A_2 = 2i\sigma_3.$$

The elements of the matrix A_0 satisfy the following conditions:

$$c_{12} = \frac{1}{2}(\partial_x + \partial_y)q,$$
$$c_{21} = -\frac{1}{2}(\partial_x - \partial_y)r,$$
$$(\partial_x - \partial_y)c_{11} = -\frac{1}{2}(\partial_x + \partial_y)(qr),$$
$$(\partial_x + \partial_y)c_{22} = \frac{1}{2}(\partial_x - \partial_y)(qr),$$

here $\partial_x \equiv \partial/\partial_x$, $\partial_y \equiv \partial/\partial_y$ and the field v in the system of DS equations (1)-(2) is defined as

$$v = -i(c_{22} - c_{11}) + qr.$$

The compatibility condition for the equation (4) and (5) $F_{yt} = F_{ty}$ implies the following series of equations:

$$[\sigma_3, A_2] = 0, \tag{6}$$
$$\sigma_3 A_{2x} - A_{2y} + [\sigma_3, A_1] + [Q, A_2] = 0, \tag{7}$$
$$\sigma_3 A_{1x} - A_{1y} + [\sigma_3, A_0] + [Q, A_1] - 2A_2 Q_x = 0, \tag{8}$$
$$\sigma_3 A_{0x} - A_{0y} + Q_t + [Q, A_0] - A_2 Q_{xx} - A_1 Q_x = 0. \tag{9}$$

From the system of equations (6)-(9) it is not difficult to obtain a one-component (2+1)-dimensional DSI equation [8].

3. A two-component (2+1)-dimensional DSI equation

Theorem. *If the matrices Σ, Q and A belong to the group $SU(3)$, then a two-component (2+1)-dimensional DSI equation has the following form:*

$$iq_{1t} + q_{1xx} + q_{1yy} - v_1 q_1 - w_1 q_2 = 0, \tag{10}$$
$$iq_{2t} + q_{2xx} + q_{2yy} - w_2 q_1 - v_2 q_2 = 0, \tag{11}$$
$$-ir_{1t} + r_{1xx} + r_{1yy} - v_1 r_1 - w_1 r_2 = 0, \tag{12}$$
$$-ir_{2t} + r_{2xx} + r_{2yy} - w_2 r_1 - v_2 r_2 = 0, \tag{13}$$
$$v_{1xx} - v_{1yy} = (2r_1 q_1 + r_2 q_2)_{xx} + 2(r_2 q_2)_{xy} + (2r_1 q_1 + r_2 q_2)_{yy}, \tag{14}$$
$$v_{2xx} - v_{2yy} = (r_1 q_1 + 2r_2 q_2)_{xx} + 2(r_1 q_1)_{xy} + (r_1 q_1 + 2r_2 q_2)_{yy}, \tag{15}$$
$$w_{1xx} - w_{1yy} = (q_1 r_2)_{xx} - 2(q_1 r_2)_{xy} + (q_1 r_2)_{yy}, \tag{16}$$
$$w_{2xx} - w_{2yy} = (q_2 r_1)_{xx} - 2(q_2 r_1)_{xy} + (q_2 r_1)_{yy}, \tag{17}$$

where q and r are complex-valued functions, and v_j and w_j are real functions.

Proof. For proof, we require that the column matrix F satisfies the following Lax representation

$$F_y = \Sigma F_x + PF, \tag{18}$$
$$F_t = B_2 F_{xx} + B_1 F_x + B_0 F, \tag{19}$$

where

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

the remaining matrices belong to the group $su(3)$:

$$P = \begin{pmatrix} 0 & q_1 & q_2 \\ -r_1 & 0 & 0 \\ -r_2 & 0 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \ B_1 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \ B_2 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Then, from the compatibility condition $F_{yt} = F_{ty}$ of the system (18) and (19), we obtain

$$[\Sigma, B_2] = 0, \tag{20}$$
$$\Sigma B_{2x} - B_{2y} + [\Sigma, B_1] + [P, B_2] = 0, \tag{21}$$
$$\Sigma B_{1x} - B_{1y} + [\Sigma, B_0] + [P, B_1] - 2B_2 P_x = 0, \tag{22}$$
$$\Sigma B_{0x} - B_{0y} + P_t + [P, B_0] - B_2 P_{xx} - B_1 P_x = 0. \tag{23}$$

Now we define elements of the matrices B_0, B_1 and B_2. From the equation 20 it is determined that

$$a_{12} = a_{21} = a_{13} = a_{31} = 0.$$

Therefore, we have

$$B_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Similarly, from the equations (21) and (22) we get a number of restrictions for the elements of the matrices B_0, B_1 and B_2.

Namely, for the elements a_{ij} $(i, j = 1, 2, 3)$ of the matrix B_2:

$$a_{11x} + a_{11y} = 0,$$
$$a_{22x} + a_{22y} = 0,$$
$$a_{33x} + a_{33y} = 0,$$
$$a_{23x} + a_{23y} = 0,$$
$$a_{32x} + a_{32y} = 0,$$

for the elements b_{ij} $(i, j = 1, 2, 3)$ of the matrix B_1:

$$b_{12} = -\frac{1}{2}(a_{2211}q_1 + a_{32}q_2),$$

$$
\begin{aligned}
b_{21} &= \frac{1}{2}(a_{2211}r_1 + a_{23}r_2),\\
b_{13} &= -\frac{1}{2}(a_{3311}q_2 + a_{23}q_1),\\
b_{31} &= \frac{1}{2}(a_{3311}r_2 + a_{32}r_1),\\
b_{11x} - b_{11y} &= 0,\\
b_{22x} + b_{22y} &= -\frac{1}{2}(a_{23}q_1r_2 - a_{32}q_2r_1),\\
b_{33x} + b_{33y} &= \frac{1}{2}(a_{23}q_1r_2 - a_{32}q_2r_1),\\
b_{23x} + b_{23y} &= \frac{1}{2}(a_{3322}q_2r_1 + a_{23}(q_1r_1 - q_2r_2)),\\
b_{32x} + b_{32y} &= -\frac{1}{2}(a_{3322}q_1r_2 + a_{32}(q_1r_1 - q_2r_2)),
\end{aligned}
$$

where $a_{iijj} = a_{ii} - a_{jj}$ $(i > j)$. And for the elements c_{ij} $(i, j = 1, 2, 3)$ of the matrix B_0, we have

$$
\begin{aligned}
c_{12} &= \frac{1}{4}[(3a_{11} + a_{22})q_{1x} - a_{2211}q_{1y} + a_{32}q_{2x} - a_{32}q_{2y} + (a_{22x} - a_{22y} - 2b_{2211})q_1 + (a_{32x} - a_{32y} - 2b_{32})q_2],\\
c_{21} &= \frac{1}{4}[(3a_{11} + a_{22})r_{1x} - a_{2211}r_{1y} + 3a_{32}r_{2x} - a_{23}r_{2y} + (a_{11x} + a_{11y} + 2b_{2211})r_1 + 2b_{23}r_2)],\\
c_{13} &= \frac{1}{4}[a_{23}q_{1x} - a_{23}q_{1y} + (3a_{11} + a_{33})q_{2x} - a_{3311}q_{2y} + (a_{23x} - a_{23y} - 2b_{32})q_1 + (a_{33x} - a_{33y} - 2b_{3311})q_2],\\
c_{31} &= \frac{1}{4}[3a_{32}r_{1x} - a_{32}r_{1y} + (a_{11} + 3a_{33})r_{2x} - a_{3311}r_{2y} + 2b_{32}r_1 + (a_{11x} + a_{11y} + 2b_{3311})r_2].
\end{aligned}
$$

From the equation (**23**) we get the following system of equations:

$$
\begin{aligned}
q_{1t} &= a_{11}q_{1xx} + b_{11}q_{1x} - c_{12x} + c_{12y} - c_{2211}q_1 - c_{32}q_2, && (24)\\
q_{2t} &= a_{11}q_{2xx} + b_{11}q_{2x} - c_{13x} + c_{13y} - c_{3311}q_2 - c_{23}q_1, && (25)\\
r_{1t} &= a_{22}r_{1xx} + a_{23}r_{2xx} + b_{22}r_{1x} + b_{23}r_{2x} - c_{21x} - c_{21y} + c_{2211}r_1 + c_{23}r_2, && (26)\\
r_{2t} &= a_{32}r_{1xx} + a_{33}r_{2xx} + b_{32}r_{1x} + b_{33}r_{2x} - c_{31x} - c_{21y} + c_{3311}r_2 + c_{32}r_1, && (27)\\
&c_{11x} + c_{21}q_1 + c_{31}q_2 + c_{12}r_1 + c_{13}r_2 + b_{12}r_{1x} + b_{13}r_{2x} - c_{11y} = 0, && (28)\\
&-c_{22x} - c_{12}r_1 - c_{21}q_1 - b_{21}q_{1x} - c_{22y} = 0, && (29)\\
&-c_{23x} - c_{13}r_1 - c_{21}q_2 - b_{21}q_{2x} - c_{23y} = 0, && (30)\\
&-c_{32x} - c_{12}r_2 - c_{31}q_1 - b_{31}q_{1x} - c_{32y} = 0, && (31)\\
&-c_{33x} - c_{13}r_2 - c_{31}q_2 - b_{31}q_{2x} - c_{33y} = 0. && (32)
\end{aligned}
$$

Then, taking into account the above results, the equation (**10**) can be rewritten as

$$
\begin{aligned}
&iq_{1t} + \frac{i}{4}a_{2211}q_{1xx} + \frac{i}{4}a_{2211}q_{1yy} + \frac{i}{4}a_{32}q_{2xx} + \frac{i}{4}a_{32}q_{2yy} + \frac{i}{2}(-a_{22} - a_{11})q_{1xy} -\\
&-\frac{i}{2}a_{32}q_{2xy} + \frac{i}{2}(a_{32x} - a_{32y} - b_{32})q_{2x} + \frac{i}{2}(-a_{32x} + a_{32y} + b_{32})q_{2y} +\\
&+\frac{i}{2}(a_{22x} - a_{22y} - b_{22} - b_{11})q_{1x} + \frac{i}{2}(a_{22x} + a_{22y} + b_{22} - b_{11})q_{1y} +\\
&+\frac{i}{4}[a_{22xx} - 2a_{22xy} + a_{22yy} - 2(b_{22x} - b_{22y}) + 4c_{2211}]q_1 +\\
&+\frac{i}{4}[a_{32xx} - 2a_{32xy} + a_{32y} - 2(b_{32x} - b_{32y}) + 4c_{32}]q_2 = 0. \qquad (33)
\end{aligned}
$$

Assuming that the coefficients of the second derivative of q_1 with respect to x and y in the equation (33) are equal to unity, i. e. $\frac{i}{4}a_{2211} = 1$, we obtain $a_{2211} = a_{22} - a_{11} = -4i$. Without loss of generality, take $a_{11} = 2i$ and $a_{22} = -2i$.

Similarly, we consider the equation (25). We get that

$$\begin{aligned} iq_{2t} + \frac{i}{4}a_{3311}q_{2xx} + \frac{i}{4}a_{3311}q_{2yy} + \frac{i}{4}a_{23}q_{1xx} + \frac{i}{4}a_{23}q_{1yy} + \frac{i}{2}(-a_{33} - a_{11})q_{2xy} - \\ -\frac{i}{2}a_{23}q_{1xy} + \frac{i}{2}(a_{23x} - a_{23y} - b_{32})q_{1x} + \frac{i}{2}(-a_{23x} + a_{23y} + b_{32})q_{1y} + \\ +\frac{i}{2}(a_{33x} - a_{33y} - b_{33} - b_{11})q_{2x} + \frac{i}{2}(a_{33x} + a_{33y} + b_{33} - b_{11})q_{2y} + \\ +\frac{i}{4}[a_{33xx} - 2a_{33xy} + a_{33yy} - 2(b_{33x} - b_{33y}) + 4c_{3311}]q_2 + \\ +\frac{i}{4}[a_{23xx} - 2a_{23xy} + a_{23y} - 2(b_{23x} - b_{23y}) + 4c_{23}]q_1 = 0. \end{aligned} \tag{34}$$

We assume that in the equation $34\frac{i}{4}a_{3311} = 1$. Then we determine that $a_{33} = -2i$. Given the above, the matrix B_2 takes the form

$$B_2 = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & a_{23} \\ 0 & a_{32} & -2i \end{pmatrix} = 2i\Sigma + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}.$$

In the case when $a_{23} = a_{32} = 0$, the matrix B_2 takes the form

$$B_2 = 2i\Sigma.$$

Now we list all the equations obtained from 20 - 23 for the elements b_{ij} and c_{ij}. We have

$$\begin{aligned} b_{12} &= 2iq_1, \\ b_{13} &= 2iq_2, \\ b_{21} &= -2ir_1, \\ b_{31} &= -2ir_2, \\ b_{11x} - b_{11y} &= 0, \\ b_{22x} + b_{22y} &= 0, \\ b_{33x} + b_{33y} &= 0, \\ b_{23x} + b_{23y} &= 0, \\ b_{32x} + b_{32y} &= 0. \end{aligned}$$

The last five equations have a solution

$$b_{11} = b_{22} = b_{23} = b_{32} = b_{33} = 0.$$

Thus, for the elements of the matrix B_1 we get

$$B_1 = \begin{pmatrix} 0 & 2iq_1 & 2iq_2 \\ -2ir_1 & 0 & 0 \\ -2ir_2 & 0 & 0 \end{pmatrix} = 2iP.$$

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Similarly, we define the expressions for the elements c_{ij} of the matrix B_0 in the form

$$\begin{aligned} c_{12} &= i(q_{1x} + q_{1y}), \\ c_{13} &= i(q_{2x} + q_{2y}), \\ c_{21} &= -i(r_{1x} - r_{1y}), \\ c_{31} &= -i(r_{2x} - r_{2y}). \end{aligned}$$

The remaining five elements c_{11}, c_{22}, c_{33}, c_{23}, c_{32} satisfy the following nontrivial equations:

$$c_{11x} - c_{11y} = -i(r_1q_1 + r_2q_2)_x - i(r_1q_1 + r_2q_2)_y, \quad (35)$$
$$c_{22x} + c_{22y} = i(r_1q_1)_x - i(r_1q_1)_y, \quad (36)$$
$$c_{33x} + c_{33y} = i(r_2q_2)_x - i(r_2q_2)_y, \quad (37)$$
$$c_{23x} + c_{23y} = i(r_1q_2)_x - i(r_1q_2)_y, \quad (38)$$
$$c_{32x} + c_{32y} = i(r_2q_1)_x - i(q_1r_2)_y. \quad (39)$$

Let us introduce the notations $v_1 = -ic_{2211}$, $v_2 = -ic_{3311}$, $w_1 = -ic_{32}$ and $w_2 = -ic_{23}$. Acting on v_1 with operators $D^+ = \partial_x + \partial_y$ $D^- = \partial_x - \partial_y$, we get

$$D^-D^+(-ic_{22}) = -i(c_{22xx} - c_{22yy}),$$

$$D^+D^-(-ic_{22}) = (r_1q_1)_{xx} - (r_1q_1)_{yx} - (r_1q_1)_{xy} + i(r_1q_1)_{yy} = (r_1q_1)_{xx} - 2(r_1q_1)_{xy} + i(r_1q_1)_{yy},$$

$$D^+D^-v_1 = (2r_1q_1 + r_2q_2)_{xx} + 2(r_2q_2)_{xy} + (2r_1q_1 + r_2q_2)_{yy}.$$

As a result, we have

$$v_{1xx} - v_{1yy} = (2r_1q_1 + r_2q_2)_{xx} + 2(r_2q_2)_{xy} + (2r_1q_1 + r_2q_2)_{yy},$$

which in turn gives the equation (14).

Similarly, acting by the operators D^+y D^- and v_2, w_1 and w_2, we obtain the following equations, respectively:

$$v_{2xx} - v_{2yy} = (r_1q_1 + 2r_2q_2)_{xx} + 2(r_1q_1)_{xy} + (r_1q_1 + 2r_2q_2)_{yy},$$

$$w_{1xx} - w_{1yy} = (q_1r_2)_{xx} - 2(q_1r_2)_{xy} + (q_1r_2)_{yy},$$
$$w_{2xx} - w_{2yy} = (q_2r_1)_{xx} - 2(q_2r_1)_{xy} + (q_2r_1)_{yy}.$$

As we can see, the last three equations obtained above turn out to be the equivalent equations (15)-(17). Furthermore, using the above notation (35)-(39), we can write for equations for q_{1t}, q_{2t}, r_{1t} and r_{2t} in the form

$$\begin{aligned} iq_{1t} + q_{1xx} + q_{1yy} - v_1q_1 - w_1q_2 &= 0, \\ iq_{2t} + q_{2xx} + q_{2yy} - w_2q_1 - v_2q_2 &= 0, \\ -ir_{1t} + r_{1xx} + r_{1yy} - v_1r_1 - w_1r_2 &= 0, \\ -ir_{2t} + r_{2xx} + r_{2yy} - w_2r_1 - v_2r_2 &= 0. \end{aligned}$$

Thus, we obtained the system of equations (10)-(12), which was required to prove.

4. Conclusion

In conclusion, we note that the obtained two-component generalization of the DSI (10)-(17) equation and its Lax representation (18)-(19) are new. A detailed study of the algebraic and geometric properties of the (10)-(17) system is the subject of our further research.

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