Journal of Physics: Conference Series

PAPER • OPEN ACCESS

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To cite this article: A T Rakhymova *et al* 2021 *J. Phys.: Conf. Ser.* **1988** 012027

Determination of trigonometric Fourier's series by the method of four-dimensional mathematics

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Abstract. This paper is devoted to the determination of the trigonometric Fourier series by the method of four-dimensional mathematics. In this paper, a new four-dimensional method for evaluation of the sums containing trigonometric function is proposed. Current work provides a first study and finding towards Fourier's series in four-dimensional space.

Key words: four-dimensional variable, Fourier series, trigonometrical functions, spectrum.

1. Introduction

In this paper, we study a trigonometric form of Fourier series in four-dimensional mathematics. Skills on trigonometry functions play an important role in a wide variety of careers, including architecture and engineering. By referring to this, it's important for students that are interested in the scientific or engineering fields to understand trigonometry. The purpose of this work is to formulate a new formulation of Fourier's series by using new approach theory of four-dimensional variables. To achieve this goal, it is necessary to study a theory of four-dimensional variables, determination of main formulas in this theory in the shade of the functional analysis, spectral theory and modern methods of mathematical analysis.

The theory of four-dimensional functions is the new method in mathematics and due to this is scantily explored. In 2015 it was possible to describe the initial chapters of this theory by Kazakh researcher M.M.Abenov [1-2]. The discovery of the spectral theory made it possible to systematize the available results and obtain new information about theory of four-dimensional variables and solved three dimensional problems analytically [2-3]. Full substantiated analysis of this theory was published by Abenov M. (Al-Farabi Kazakh National University). While developing this theory, Abenov M.M. and Gabbasov M.B. found all four-dimensional Abenov spaces with commutative multiplication, which were assigned the designations M2-M7, and it became necessary to study these spaces. In this paper we performed a research in Abenov space of four-dimensional numbers M3 [4-5].

The theory of four-dimensional functions is based on the principles of commonality of its key concepts with similar categories of one-dimensional and two-dimensional analysis [2]. This approach is justified by the fact that linear spaces of one-dimensional and two-dimensional numbers can initially be considered as eigenspaces of a more general space of four-dimensional numbers. This leads to the study

of these one-dimensional, two-dimensional and four-dimensional numbers as elements of a triune system, which allows to uniformly define key concepts in the theory of these numbers (for example, arithmetic operations on the set of these numbers) [2-5].
Note that a similar approach was previously used by the physicist W. Hamilton when he constructed the quaternion algebra. He defined a non-commutative operation of multiplication of elements, which allowed avoiding zero divisors [2].

2. Methodology

Let consider a linear space $\mathfrak{R}^4$, which contains vectors with four components. Following the standard convention in linear algebra we shall denote an element $X \in \mathfrak{R}^4$ as a column vector

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{R}^4.$$

We define addition of the vectors and multiplication of the vector by scalar by components, thus,

$$\text{for } X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{R}^4,\ Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \in \mathcal{R}^4 \text{ and } \lambda \in \mathbb{R}$$

$$X + Y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \\ x_4 + y_4 \end{pmatrix}$$

$$\lambda \cdot X = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_3 \\ \lambda x_4 \end{pmatrix}.$$

and the product of elements X and Y from $\mathcal{R}^4$ we define by the following formula

$$X \cdot Y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4 \\ x_2y_1 + x_1y_2 + x_4y_3 + x_3y_4 \\ x_3y_1 - x_4y_2 + x_1y_3 - x_2y_4 \\ x_4y_1 + x_3y_2 + x_2y_3 + x_1y_4 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = Z \quad (1)$$

Let X, Y and Z be elements of linear space $\mathcal{R}^4$, then multiplication operation (1) has following properties:

i. $X \cdot Y = Y \cdot X$ (commutative multiplication);
ii. $(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)$ (associativity of multiplication);
iii. $(X + Y) \cdot Z = X \cdot Z + Y \cdot Z$ (associativity of multiplication over addition).

Simplex module of the element $X \in \mathcal{R}^4$, is said to be a number defined by

$$|X| = \sqrt[4]{[(x_1 - x_3)^2 + (x_2 - x_4)^2] \cdot [(x_1 + x_3)^2 + (x_2 + x_4)^2]}.$$

We define a space $M(4, \mathbb{R})$ as a linear space of all matrices of dimension (4×4):

$$A_X = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix}, x_k \in \mathbb{R}, k = \overline{1,4}$$

The inverse of the A_X is defined as follows

$$A_X^{-1} = A_W$$

where elements of A_w are evaluated using the following formulas

$$w_1 = \frac{1}{2}\left[\frac{x_1 + x_3}{(x_1 + x_3)^2 + (x_2 + x_4)^2} + \frac{x_1 - x_3}{(x_1 - x_3)^2 + (x_2 - x_4)^2}\right]$$

$$w_2 = \frac{1}{2}\left[\frac{x_2 + x_4}{(x_1 + x_3)^2 + (x_2 + x_4)^2} + \frac{x_2 - x_4}{(x_1 - x_3)^2 + (x_2 - x_4)^2}\right]$$

$$w_3 = \frac{1}{2}\left[\frac{x_1 + x_3}{(x_1 + x_3)^2 + (x_2 + x_4)^2} - \frac{x_1 - x_3}{(x_1 - x_3)^2 + (x_2 - x_4)^2}\right]$$

$$w_4 = \frac{1}{2}\left[\frac{x_2 + x_4}{(x_1 + x_3)^2 + (x_2 + x_4)^2} - \frac{x_2 - x_4}{(x_1 - x_3)^2 + (x_2 - x_4)^2}\right]$$

Division of $X \in \mathcal{R}^4$ is defined by multiplying to inverse element. Note that division fulfill only for non-generate numbers [2].

Theorem 1. Linear spaces $\mathcal{R}^4$ and $M(4, \mathbb{R})$ are isomorphic:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{R}^4 \Leftrightarrow A_X = \begin{pmatrix} x_1 & -x_2 & x_3 & -x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & -x_4 & x_1 & -x_2 \\ x_4 & x_3 & x_2 & x_1 \end{pmatrix} \in M(4, \mathbb{R})$$

The linear combination will be mapped to linear combination of the corresponding matrices:

$$\lambda X + \mu Y \in \mathcal{R}^4 \Leftrightarrow A_{\lambda X + \mu Y} = \lambda A_X + \mu A_Y \in M(4, \mathbb{R}).$$

The multiplication of the basic numbers

$$J_1^2 = J_3^2 = J_1;$$

$$J_2^2 = J_4^2 = -J_1;$$

$$J_1 * J_k = J_k;\ k = \overline{1,4}$$

$$J_2 * J_3 = J_4;$$

$$J_2 * J_4 = -J_3;$$

It is not difficult to find the spectrum of the elements of the space $\mathcal{R}^4$ by solving the equation $\det(A_X - \lambda E) = 0$, which leads us to the following

$$\begin{cases} \lambda_1 = x_1 - x_3 + (x_2 - x_4)i \\ \lambda_2 = x_1 - x_3 - (x_2 - x_4)i \\ \lambda_3 = x_1 + x_3 + (x_2 + x_4)i \\ \lambda_4 = x_1 + x_3 - (x_2 + x_4)i \end{cases} \tag{2}$$

The vector with the components (2) will be denoted by Λ_X:

$$\Lambda_X = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}$$

Let denote by $\Lambda = \{\Lambda_X : X \in \mathcal{R}^4\}$. Let define a map $T: \mathcal{R}^4 \to \Lambda$, by assigning to each element $X \in \mathcal{R}^4$ its spectral number $\Lambda_X \in \Lambda$. It is not difficult to show that the map T is bijection.

$$X \in \mathcal{R}^4 \Leftrightarrow \Lambda_X[\lambda_1(X), \lambda_2(X), \lambda_3(X), \lambda_4(X)] \in \Lambda(\mathcal{R}^4)$$

Main properties of spectrum: for all $X, Y \in \mathcal{R}^4$ we have

$\Lambda_{X\pm Y} = \Lambda_X \pm \Lambda_Y = [{\lambda_1}^x \pm {\lambda_1}^y;\ {\lambda_2}^x \pm {\lambda_2}^y;\ {\lambda_3}^x \pm {\lambda_3}^y;\ {\lambda_4}^x \pm {\lambda_4}^y]$

$\Lambda_{X*Y} = [{\lambda_1}^x \cdot {\lambda_1}^y;\ {\lambda_2}^x \cdot {\lambda_2}^y;\ {\lambda_3}^x \cdot {\lambda_3}^y;\ {\lambda_4}^x \cdot {\lambda_4}^y]$

$\Lambda_{X^{-1}} = \left[\frac{1}{{\lambda_1}^x};\ \frac{1}{{\lambda_2}^x};\ \frac{1}{{\lambda_3}^x};\ \frac{1}{{\lambda_4}^x}\right]$, for all X −non-degenerate numbers

A spectral norm is defined as

$$\|X\|_\Lambda = \frac{1}{4}\sum_{k=1}^{4} |\lambda_k^x|$$

Let $\mathcal{M}$ and $\mathcal{N}$ non-empty sets of four-dimensional numbers [2]. If for any $X \in \mathcal{M}$, it assigned uniquely defined $U = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \in \mathcal{N}$, then we say that it is given four-dimensional function $U = U(X)$, where $U(X) = \begin{pmatrix} u_1(x_1, x_2, x_3, x_4) \\ u_2(x_1, x_2, x_3, x_4) \\ u_3(x_1, x_2, x_3, x_4) \\ u_4(x_1, x_2, x_3, x_4) \end{pmatrix}$.

Definition 1. Four-dimensional function $U(x_1, x_2, x_3, x_4) = (u_1, u_2, u_3, u_4)$ is called differentiable at point $X = (x_1, x_2, x_3, x_4)$, if there is a limit of the form

$$u'(X) = \lim_{\triangle X \to 0} \frac{u(X + \triangle X) - u(X)}{\triangle X}.$$

To determine an exponential function in four-dimensional space we use the results from [2].

Theorem 2. There exists a single regular function satisfying the differential equation:

$$\frac{du}{dX} = u$$

and initial condition

$$u\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The function which satisfies conditions of theorem is called exponential function. If $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{R}^4$, then the exponential function can be written as follows [2]:

$$e^X = \begin{pmatrix} \frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) + e^{x_1-x_3}\cos(x_2-x_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3}\sin(x_2+x_4) + e^{x_1-x_3}\sin(x_2-x_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) - e^{x_1-x_3}\cos(x_2-x_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3}sin(x_2+x_4) - e^{x_1-x_3}sin(x_2-x_4)\right) \end{pmatrix}$$

To justify the name of exponential function we need to verify that newly defined function will inherit main properties of the normal exponential function in one dimensional case. First we prove the following formula:

$$e^{X+Y} = e^X \cdot e^Y \tag{3}$$

Proof. The left hand side is

$$e^{X+Y} = \begin{pmatrix} \frac{1}{2}\left(e^{x_1+x_3+y_1+y_3}\cos(x_2+x_4+y_2+y_4) + e^{x_1-x_3+y_1-y_3}\cos(x_2-x_4+y_2-y_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3+y_1+y_3}\sin(x_2+x_4+y_2+y_4) + e^{x_1-x_3+y_1-y_3}\sin(x_2-x_4+y_2-y_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3+y_1+y_3}\cos(x_2+x_4+y_2+y_4) - e^{x_1-x_3+y_1-y_3}\cos(x_2-x_4+y_2-y_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3+y_1+y_3}\sin(x_2+x_4+y_2+y_4) - e^{x_1-x_3+y_1-y_3}\sin(x_2-x_4+y_2-y_4)\right) \end{pmatrix}$$

We find the right-hand side of (3)

$$e^X \cdot e^Y = \begin{pmatrix} \frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) + e^{x_1-x_3}\cos(x_2-x_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3}\sin(x_2+x_4) + e^{x_1-x_3}\sin(x_2-x_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) - e^{x_1-x_3}\cos(x_2-x_4)\right) \\ \frac{1}{2}\left(e^{x_1+x_3}sin(x_2+x_4) - e^{x_1-x_3}sin(x_2-x_4)\right) \end{pmatrix} \times$$

$$\times \begin{pmatrix} \frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) + e^{y_1-y_3}\cos(y_2-y_4)\right) \\ \frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) + e^{y_1-y_3}\sin(y_2-y_4)\right) \\ \frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) - e^{y_1-y_3}\cos(y_2-y_4)\right) \\ \frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) - e^{y_1-y_3}\sin(y_2-y_4)\right) \end{pmatrix} =$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}.$$

After the multiplication is carried out, we obtain

$$u_1 \times w_1 = u_1w_1 - u_2w_2 + u_3w_3 - u_4w_4 =$$

$$= \left[\frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) + e^{x_1-x_3}\cos(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) + e^{y_1-y_3}\cos(y_2-y_4)\right)\right] -$$
$$-\left[\frac{1}{2}\left(e^{x_1+x_3}\sin(x_2+x_4) + e^{x_1-x_3}\sin(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) + e^{y_1-y_3}\sin(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) - e^{x_1-x_3}\cos(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) - e^{y_1-y_3}\cos(y_2-y_4)\right)\right] -$$
$$-\left[\frac{1}{2}\left(e^{x_1+x_3}sin(x_2+x_4) - e^{x_1-x_3}sin(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) - e^{y_1-y_3}\sin(y_2-y_4)\right)\right] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}(\cos(x_2+x_4)\cos(y_2+y_4) - sin(x_2+x_4)\,sin(y_2+y_4)) +$$
$$+e^{x_1-x_3+y_1-y_3}(\cos(x_2-x_4)\cos(y_2-y_4) - sin(x_2-x_4)\,sin(y_2-y_4)] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}\cos(x_2+x_4+y_2+y_4) + e^{x_1-x_3+y_1-y_3}\cos(x_2-x_4+y_2-y_4)$$

Similarly, we derive second component

$$u_2 \times w_2 = u_2w_1 + u_1w_2 + u_4w_3 + u_3w_4 =$$
$$= \left[\frac{1}{2}\left(e^{x_1+x_3}\sin(x_2+x_4) + e^{x_1-x_3}\sin(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) + e^{y_1-y_3}\cos(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) + e^{x_1-x_3}\cos(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) + e^{y_1-y_3}\sin(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}\left(e^{x_1+x_3}sin(x_2+x_4) - e^{x_1-x_3}sin(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) - e^{y_1-y_3}\cos(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) - e^{x_1-x_3}\cos(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) - e^{y_1-y_3}\sin(y_2-y_4)\right)\right] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}(\cos(x_2+x_4)sin(y_2+y_4) + sin(x_2+x_4)\cos(y_2+y_4)) +$$
$$+e^{x_1-x_3+y_1-y_3}(\cos(x_2-x_4)sin(y_2-y_4) + sin(x_2-x_4)\cos(y_2-y_4)] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}sin(x_2+x_4+y_2+y_4) + e^{x_1-x_3+y_1-y_3}sin(x_2-x_4+y_2-y_4)$$

And third component

$$u_3 \times w_3 = u_3w_1 - u_4w_2 + u_1w_3 - u_2w_4 =$$
$$= \left[\frac{1}{2}(e^{x_1+x_3}\cos(x_2+x_4) - e^{x_1-x_3}\cos(x_2-x_4))\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) + e^{y_1-y_3}\cos(y_2-y_4)\right)\right] -$$
$$-\left[\frac{1}{2}\left(e^{x_1+x_3}sin(x_2+x_4) - e^{x_1-x_3}sin(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) + e^{y_1-y_3}\sin(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) + e^{x_1-x_3}\cos(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) - e^{y_1-y_3}\cos(y_2-y_4)\right)\right] -$$
$$-\left[\frac{1}{2}(e^{x_1+x_3}\sin(x_2+x_4) + e^{x_1-x_3}\sin(x_2-x_4))\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) - e^{y_1-y_3}\sin(y_2-y_4)\right)\right] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}(\cos(x_2+x_4)\cos(y_2+y_4) - sin(x_2+x_4)\,sin(y_2+y_4)) -$$
$$-e^{x_1-x_3+y_1-y_3}(\cos(x_2-x_4)\cos(y_2-y_4) - sin(x_2-x_4)\,sin(y_2-y_4)] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}\cos(x_2+x_4+y_2+y_4) - e^{x_1-x_3+y_1-y_3}\cos(x_2-x_4+y_2-y_4)$$

Finally, fourth component can be evaluated as below

$$u_4 \times w_4 = u_4w_1 + u_3w_2 + u_2w_3 + u_1w_4 =$$
$$= \left[\frac{1}{2}\left(e^{x_1+x_3}sin(x_2+x_4) - e^{x_1-x_3}sin(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) + e^{y_1-y_3}\cos(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}(e^{x_1+x_3}\cos(x_2+x_4) - e^{x_1-x_3}\cos(x_2-x_4))\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) + e^{y_1-y_3}\sin(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}(e^{x_1+x_3}\sin(x_2+x_4) + e^{x_1-x_3}\sin(x_2-x_4))\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\cos(y_2+y_4) - e^{y_1-y_3}\cos(y_2-y_4)\right)\right] +$$
$$+\left[\frac{1}{2}\left(e^{x_1+x_3}\cos(x_2+x_4) + e^{x_1-x_3}\cos(x_2-x_4)\right)\right]$$
$$\times \left[\frac{1}{2}\left(e^{y_1+y_3}\sin(y_2+y_4) - e^{y_1-y_3}\sin(y_2-y_4)\right)\right] =$$
$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3}(\cos(x_2+x_4)sin(y_2+y_4) + sin(x_2+x_4)\cos(y_2+y_4)) -$$
$$-e^{x_1-x_3+y_1-y_3}(\cos(x_2-x_4)sin(y_2-y_4) + sin(x_2-x_4)\cos(y_2-y_4)] =$$

Journal of Physics: Conference Series **1988** (2021) 012027 doi:10.1088/1742-6596/1988/1/012027

$$= \frac{1}{2}[e^{x_1+x_3+y_1+y_3} sin(x_2 + x_4 + y_2 + y_4) - e^{x_1-x_3+y_1-y_3} sin(x_2 - x_4 + y_2 - y_4)$$

If finalize the right hand side then we obtain

$$e^X \cdot e^Y = \begin{pmatrix} \frac{1}{2}[e^{x_1+x_3+y_1+y_3}\cos(x_2 + x_4 + y_2 + y_4) + e^{x_1-x_3+y_1-y_3}\cos(x_2 - x_4 + y_2 - y_4) \\ \frac{1}{2}[e^{x_1+x_3+y_1+y_3} sin(x_2 + x_4 + y_2 + y_4) + e^{x_1-x_3+y_1-y_3} sin(x_2 - x_4 + y_2 - y_4) \\ \frac{1}{2}[e^{x_1+x_3+y_1+y_3}\cos(x_2 + x_4 + y_2 + y_4) - e^{x_1-x_3+y_1-y_3}\cos(x_2 - x_4 + y_2 - y_4) \\ \frac{1}{2}[e^{x_1+x_3+y_1+y_3} sin(x_2 + x_4 + y_2 + y_4) - e^{x_1-x_3+y_1-y_3} sin(x_2 - x_4 + y_2 - y_4) \end{pmatrix}$$

The proof of the formula (3) is completed.
Next, we proceed with the definition of the trigonometric functions in $\mathcal{R}^4$. Let us define the following functions $sin(X)$ and $cos(X)$

$$sin(X) = \frac{e^{J_2X} - e^{-J_2X}}{2J_2}, \quad cos(X) = \frac{e^{J_2X} + e^{-J_2X}}{2}$$

By similar way to one dimensional case we can establish the following formulas

$$sin(X + Y) = \sin(X)\cos(Y) + \cos(X)\sin(Y)$$
$$sin(X - Y) = \sin(X)\cos(Y) - \cos(X)\sin(Y)$$
$$cos(X + Y) = \cos(X)\cos(Y) + \sin(X)\sin(Y)$$
$$cos(X - Y) = \cos(X)\cos(Y) - \sin(X)\sin(Y)$$

Let check validity of the (1) summation formula

$$\sin(X)\cos(Y) + \cos(X)\sin(Y) = \frac{e^{J_2X} - e^{-J_2X}}{2J_2} \cdot \frac{e^{J_2Y} + e^{-J_2Y}}{2} + \frac{e^{J_2Y} - e^{-J_2Y}}{2J_2} \cdot \frac{e^{J_2X} + e^{-J_2X}}{2}$$

$$= \frac{e^{J_2(X+Y)} + e^{J_2(X-Y)} - e^{J_2(Y-X)} - e^{-J_2(X+Y)} + e^{J_2(X+Y)} - e^{J_2(X-Y)} + e^{J_2(Y-X)} - e^{-J_2(X+Y)}}{4J_2}$$

$$= \frac{e^{J_2(X+Y)} - e^{-J_2(X+Y)}}{2J_2} = sin(X + Y)$$

The rest can be verified by similar way.

3. Main Results

The main result of this paper is the following

Theorem 3. For any positive integer n the following formulas hold:

$$\frac{1}{2} + \cos(X) + \cos(2X) + \cdots + \cos(nX) = \frac{\sin(n+\frac{1}{2})X}{2\sin\left(\frac{X}{2}\right)} \tag{4}$$

$$\sin(X) + \sin(2X) + \cdots + \sin(nX) = \frac{\cos\left(\frac{1}{2}J_2X\right) - \cos\left(n+\frac{1}{2}\right)J_2X}{2\sin\left(\frac{1}{2}J_2X\right)} \tag{5}$$

Proof. Let consider the sum

$$T = \sum_{k=-n}^{n} e^{kJ_2X} = e^{-nJ_2X} + e^{-(n-1)J_2X} + \cdots + e_2^{-J_2X} + 1 + e^{J_2X} + \cdots + e^{nJ_2X}$$

Multiplying both sides by e^{J_2X} we have

$$e^{J_2X} \cdot T = e^{J_2X} \cdot \sum_{k=-n}^{n} e^{kJ_2X} = e^{-(n-1)J_2X} + \cdots + e_2^{-J_2X} + 1 + e^{J_2X} + \cdots + e^{(n+1)J_2X}$$

By subtracting the latter equations

$$(1 - e^{J_2X})T = e^{-nJ_2X} - e^{(n+1)J_2X}$$

Multiplying by $e^{-\frac{1}{2}J_2X}$ we obtain

$$(e^{\frac{1}{2}J_2X} - e^{-\frac{1}{2}J_2X})T = e^{(n+\frac{1}{2})J_2X} - e^{-(n+\frac{1}{2})J_2X}$$

$$(\frac{e^{\frac{1}{2}J_2X} - e^{-\frac{1}{2}J_2X}}{2J_2})T = \frac{e^{(n+\frac{1}{2})J_2X} - e^{-(n+\frac{1}{2})J_2X}}{2J_2}$$

By using the formula $sin\left(\frac{X}{2}\right) = \frac{e^{\frac{1}{2}J_2X} - e^{-\frac{1}{2}J_2X}}{2J_2}$ and $sin\frac{2n+1}{2}X = \frac{e^{(n+\frac{1}{2})J_2X} - e^{-(n+\frac{1}{2})J_2X}}{2J_2}$

$$T\, sin\left(\frac{X}{2}\right) = sin\frac{2n+1}{2}X$$

$$T = \frac{sin\frac{2n+1}{2}X}{sin\left(\frac{X}{2}\right)}$$

Thus, we have proved

$$\sum_{k=-n}^{n} e^{kJ_2X} = \frac{sin(n+\frac{1}{2})X}{sin\left(\frac{X}{2}\right)}$$

But taking into account

$$T = e^{-nJ_2X} + \cdots + e^{nJ_2X} = 1 + 2\cos(X) + 2\cos(2X) + \cdots + 2\cos(nX)$$

After all calculations, we obtain the following result for summation of trigonometric functions in four-dimensional space

$$\frac{1}{2} + cos(X) + cos(2X) + \cdots + cos(nX) = \frac{sin\frac{2n+1}{2}X}{2\, sin\left(\frac{X}{2}\right)}$$

Using the just established we derive formula for the following summation

$$\mathrm{S} = \sin(X) + \sin(2 \cdot X) + \cdots + \sin(n \cdot X)$$

First let prove the following

$$1 + e^{J_2X} + e^{2\cdot J_2X} + \cdots + e^{n\cdot J_2X} = \frac{1 - e^{(n+1)\cdot J_2X}}{1 - e^{J_2X}}$$

Which is easy consequence of

$$1 + p + p^2 + \cdots + p^n = \frac{1 - p^{n+1}}{1 - p}, p \in \mathcal{R}^4.$$

By induction we can prove

$$(\cos(X) + J_2 \cdot \sin(X))^k = \cos(k \cdot X) + J_2 \cdot \sin(k \cdot X) \tag{6}$$

Let

$$k = 1: \quad \cos(X) + J_2 \cdot \sin(X) = \cos(X) + J_2 \cdot \sin(X)$$

Assume that (6) is established for all $1 \le k \le n$. Then for $k = n + 1$ we have

$$(cos(X) + J_2 \cdot sin(X))^{n+1} = (cos(X) + J_2 \cdot sin(X))(cos(X) + J_2 \cdot sin(X))^n =$$
$$= (cos(X) + J_2 \cdot sin(X)) \cdot (cos(nX) + J_2 \cdot sin(n \cdot X)) =$$
$$= (cos(X)\, cos(n \cdot X) + (J_2)^2\, sin(X)\, sin(nX)) + J_2(sin(nX)\, cos(X) + sin(X)\, cos(nX))$$
$$= cos(n+1)X + J_2\, sin(n+1)X$$

$$1 + e^{J_2 X} + e^{2J_2 X} + \cdots + e^{nJ_2 X} = \frac{1}{2} + T + J_2 S = \frac{1 - e^{(n+1)\cdot J_2 X}}{1 - e^{J_2 X}}$$

Making S as a subject we obtain

$$J_2 S = \frac{1 - e^{(n+1)J_2 X}}{1 - e^{J_2 X}} - \frac{1}{2} - T = \frac{e^{-\frac{1}{2}J_2 X} + e^{\left(n+\frac{1}{2}\right)J_2 X}}{e^{-\frac{1}{2}J_2 X} - e^{\frac{1}{2}J_2 X}} - \frac{1}{2} - T =$$

$$= \frac{1}{-2J_2\, sin\left(\frac{1}{2}J_2 X\right)}\left[cos\left(\frac{1}{2}J_2 X\right) - J_2\, sin\left(\frac{1}{2}J_2 X\right) - cos\left(n + \frac{1}{2}\right)J_2 X - 2J_2\, sin\left(n + \frac{1}{2}\right)J_2 X\right] - \frac{1}{2} - T$$

$$= J_2 \frac{cos\left(\frac{1}{2}J_2 X\right) - cos\left(n + \frac{1}{2}\right)J_2 X}{2\, sin\left(\frac{1}{2}J_2 X\right)} + \frac{sin\left(\frac{1}{2}J_2 X\right) + sin\left(n + \frac{1}{2}\right)J_2 X}{2\, sin\left(\frac{1}{2}J_2 X\right)} - \frac{1}{2} - T$$

Finally, we obtain

$$S = \frac{cos\left(\frac{1}{2}J_2 X\right) - cos\left(n + \frac{1}{2}\right)J_2 X}{2\, sin\left(\frac{1}{2}J_2 X\right)}$$

Theorem 4.

$$cos(X) + 2cos(2X) + \cdots + n\, cos(nX) = \frac{2n\, cos\left((n + \frac{1}{2})X\right) sin\left(\frac{X}{2}\right) - sin\, nX}{4\, sin^2\left(\frac{X}{2}\right)}$$

$$sin(X) + 2sin(2X) + \cdots + n\, sin(nX) = J_2 \frac{cos\, nJ_2 X + 2nsin\left(n + \frac{1}{2}\right)J_2 X\, sin\left(\frac{1}{2}J_2 X\right) - 1}{4\, sin^2\left(\frac{1}{2}J_2 X\right)}$$

Journal of Physics: Conference Series **1988** (2021) 012027 doi:10.1088/1742-6596/1988/1/012027

Proof. By differentiating the T and S we derive

$$\frac{d}{dX}\left(\frac{1}{2}+\cos(X)+\cos(2X)+\cdots+\cos(nX)\right)=\frac{d}{dX}\left(\frac{\sin\left(n+\frac{1}{2}\right)X}{\sin\left(\frac{X}{2}\right)}\right)$$

and

$$\frac{d}{dX}(\sin(X)+\sin(2X)+\cdots+\sin(nX))=\frac{d}{dX}\left(\frac{\cos\left(\frac{1}{2}J_2X\right)-\cos\left(n+\frac{1}{2}\right)J_2X}{2\sin\left(\frac{1}{2}J_2X\right)}\right)$$

By differentiating the T and S we derive

$$-(\sin(X)+2\sin(2X)+\cdots+n\sin(nX))=$$

$$=\frac{2\left(n+\frac{1}{2}\right)\cos\left(\left(n+\frac{1}{2}\right)X\right)\sin\left(\frac{X}{2}\right)-\sin\left(\left(n+\frac{1}{2}\right)X\right)\cos\left(\frac{X}{2}\right)}{4\,sin^2\left(\frac{X}{2}\right)}=$$

$$=\frac{2n\cos\left(\left(n+\frac{1}{2}\right)X\right)\sin\left(\frac{X}{2}\right)+\cos\left(\left(n+\frac{1}{2}\right)X\right)\sin\left(\frac{X}{2}\right)-\sin\left(\left(n+\frac{1}{2}\right)X\right)\cos\left(\frac{X}{2}\right)}{4\,sin^2\left(\frac{X}{2}\right)}=$$

$$=\frac{2n\cos\frac{2n+1}{2}X\sin\left(\frac{X}{2}\right)-\sin nX}{4\,sin^2\left(\frac{X}{2}\right)}$$

and

$$\cos(X)+\cos(2X)+\cdots+\cos(nX)=$$

$$=\frac{1}{4\,sin^2\left(\frac{1}{2}J_2X\right)}\left\{\left[-\frac{1}{2}J_2sin\left(\frac{1}{2}J_2X\right)+\left(n+\frac{1}{2}\right)J_2sin\left(n+\frac{1}{2}\right)J_2X\right]2\sin\left(\frac{1}{2}J_2X\right)-\right.$$

$$\left.-J_2\left[\cos\left(\frac{1}{2}J_2X\right)-\cos\left(n+\frac{1}{2}\right)J_2X\right]\cos\left(\frac{1}{2}J_2X\right)\right\}=$$

$$=\frac{1}{4\,sin^2\left(\frac{1}{2}J_2X\right)}\left\{-J_2\,sin^2\left(\frac{1}{2}J_2X\right)+2nJ_2\sin\left(n+\frac{1}{2}\right)J_2X\sin\left(\frac{1}{2}J_2X\right)+\right.$$

$$\left.+J_2\sin\left(n+\frac{1}{2}\right)J_2X\sin\left(\frac{1}{2}J_2X\right)-J_2\,cos^2\left(\frac{1}{2}J_2X\right)+J_2\cos\left(n+\frac{1}{2}\right)J_2X\cos\left(\frac{1}{2}J_2X\right)\right\}=$$

$$=\frac{2nJ_2\sin\left(n+\frac{1}{2}\right)J_2X\sin\left(\frac{1}{2}J_2X\right)+J_2\cos nJ_2X-J_2}{4\,sin^2\left(\frac{1}{2}J_2X\right)}$$

$$=J_2\frac{\cos nJ_2X+2n\sin\left(n+\frac{1}{2}\right)J_2X\sin\left(\frac{1}{2}J_2X\right)-1}{4\,sin^2\left(\frac{1}{2}J_2X\right)}.$$

4. Conclusion

Using the new multiplication rule of the vectors in $\mathcal{R}^4$ we introduce four-dimensional trigonometric functions. By applying the properties of the $\mathcal{R}^4$ we extended the formula for trigonometric sum to the case of $\mathcal{R}^4$. Obtained formula will be useful in the theory of four-dimensional trigonometric series. Further investigations related to the convergence of the four-dimensional trigonometric series will another paper.

References

[1] Abenov M M 2015 On the continuum of exact solutions of the general continuity equation *Abstracts of reports of the international scientific conference "Actual problems of mathematics and mathematical modeling* (Almaty) p 215

[2] Abenov M M 2019 *Four-dimensional mathematics: Methods and applications* (Scientific monograph. Almaty.: Kazakh University Publishing House)

[3] Abenov M M, Gabbassov, M B, Ismagulova F Y2018 Movement of fluid inside the sphere *International Journal of Engineering & Technology* (7, no. 4) pp 42-44

[4] Abenov M M, Gabbassov M B 2020 Anisotropic four-dimensional spaces or new quaternions (*Preprint Nur-Sultan)*

[5] Rakhymova A T, Gabbassov M B, Shapen K M 2020 On one space of four-dimensional numbers *Journal of Mathematics, Mechanics and Computer Science* (Vol. 4) pp 199-225