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Well posedness for one class of elliptic equations with drift

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Abstract

We studied one class of second-order elliptic equations with intermediate coefficient and proved that the semi-periodic problem on a strip is unique solvable in Hilbert space. We assume that the intermediate coefficient of the equation is continuously differentiable and grows rapidly near infinity, for example, it grows faster than $(|x| + 1)\ln(|x| + 3)$. However, we do not impose bounds on its derivatives. We believe that the lower-order coefficient is continuous, can be unlimited and change sign.

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1 Introduction and formulation of the result

Linear second-order elliptic equations and systems defined in unbounded domains have received considerable progress thanks to important applications in stochastic analysis, biology, and financial mathematics (see [1–6] and the references therein). Solvability and properties of solutions of this system are significantly influenced by growth and properties of coefficients near infinity. Therefore, they are quite different from those elliptic equations and systems defined in a bounded domain. For the following equation

$$-\Delta u + F \cdot \nabla u + Vu = f(x), \quad x \in R^n, \tag{1}$$

the solvability, regularity, and other related issues were discussed in [7–12] in the case when the intermediate coefficient (drift) F at the infinity grows, but not faster than $|x|\ln(1 + |x|)$, and its growth is not always controlled by the potential V (for example, in [7] the authors considered the case that the intermediate coefficient has a linear growth, [9] and [11] considered the case that the intermediate coefficient has a growth as $|x|\ln(1 + |x|)$). At the same time, there are correctly solvable elliptic equations with intermediate coefficients, the growths of this intermediate coefficients are different orders. For example, the following second-order elliptic equation:

$$T\omega = \omega_{z\bar{z}} - \frac{B_z}{B}\omega_{\bar{z}} - |B|^2\omega = F, \tag{2}$$

where $z = x + iy \in E$ (E is the complex plane)) $\omega_{z\bar{z}} = 1/4\triangle\omega$, $\omega_{\bar{z}} = 1/2(\omega_x + i\omega_y)$ and $B_z = 1/2(B_x - iB_y)$. If the coefficient B is continuously differentiable and satisfies the following conditions:

$$|\operatorname{Re} B| \geq 1, \qquad \sup_{z,\theta\in E:|z-\theta|\leq 1} \frac{\operatorname{Re} B(z)}{\operatorname{Re} B(\theta)} < +\infty, \tag{3}$$

then (2) is uniquely solvable for each F in $L_2(E)$. In fact, (2) is reduced to the following system:

$$\omega_{\bar{z}} - B\overline{\omega} = 2B\overline{p}, \tag{4}$$

where

$$p_{\bar{z}} + B\overline{p} = -\frac{1}{2\overline{B}}F. \tag{5}$$

If (3) holds, then first-order systems (4) and (5) are correctly solvable (see [13, 14]), and for their solutions ω and p, respectively, the following estimates hold:

$$\|\omega_x\|_{2,E} + \|\omega_y\|_{2,E} + \|B\overline{\omega}\|_{2,E} \leq C_2 \left\| \frac{1}{\overline{B}}F \right\|_{2,E},$$

$$\|p_x\|_{2,E} + \|p_y\|_{2,E} + \|B\overline{p}\|_{2,E} \leq C_1 \left\| \frac{1}{2\overline{B}}F \right\|_{2,E},$$

where $\|\cdot\|_{2,E}$ is the norm in $L_2(E)$.

In general, this naturally leads to the following question: does there exist a more or less general class of correct elliptic equations of the form (1) such that the intermediate coefficient F has a higher growth than $|x|\ln(1 + |x|)$ and not controlled by the potential V?

In this paper, we discuss this question for equation (1) in an infinite strip $\Omega = \{(x, y) : -\infty < x < \infty, -\pi < y < \pi\}$. We consider the following problem:

$$Lu = -u_{xx} - u_{yy} + a(x)u_x + b(x)u = f(x, y), \tag{6}$$

$$u(x, -\pi) = u(x, \pi), \qquad u_y(x, -\pi) = u_y(x, \pi), \tag{7}$$

where $a(x)$ is continuously differentiable, $b(x)$ is continuous, and $f \in L_2(\Omega)$. We assume that the growth of the intermediate coefficient a at infinity does not depend on the behavior of the lower term b. Our goal is to find sufficient conditions for the correct solvability of problem (6), (7). We found conditions for the coefficients of equation (6), these conditions are also applicable to the case that the coefficients are growing at infinity and quickly fluctuate (see Examples 1.1 and 1.2).

The forms of equation (6), where the coefficients depend only on x, and of periodic conditions (7) are motivated by application of the Fourier method. First, we consider problem (6), (7) with $b = 0$ and reduce it to a one-dimensional differential equation in $L_2(R)$ (see equation (15) below). We show that the latter is correctly solvable under our conditions. We use the well-known perturbation theorem for a linear operator to prove the well-posedness of problem (6), (7) in the case of $b \neq 0$.

For equation (1) with $F = 0$, the solvability conditions and regularity estimates were established in a number of works (see [15–17]). In [18], the regularity estimate was applied to the study of nonlinear Schrödinger equation. In the case $n = 1$, the correctness of equation (1) with a rapidly growing drift was shown in [19] (see also [20–22]).

Let $\Theta_m = \{(x,y) : -\infty < x < +\infty, -m < y < m\}$ for $m > 0$. We denote by $C_0(R)$ the set of continuous functions $v(x)$ $(x \in R)$ with compact support, i.e., there is a constant $n_v > 0$ such that for any $x \in (-\infty, n_v) \cup (n_v, +\infty)$, $v(x) = 0$. Set

$$C_{0,m}(\Theta_m) = \left\{ u(x,y) \in C(\overline{\Theta_m}) : \begin{array}{c} u(\cdot,y) \in C_0(R) \text{ for each } y \in [-m,m] \text{ and } u(x,\cdot) \text{ is} \\ \text{a periodic function with period } 2m \text{ for each } x \in R \end{array} \right\}.$$

Let

$$C^{(2)}_{0,m}(\Theta_m) = \left\{ u(x,y) \in C_{0,m}(\Theta_m) : \begin{array}{c} \text{differentials of } u \text{ of first and second orders} \\ \text{belong to } C_{0,m}(\Theta_m) \end{array} \right\}.$$

Definition 1.1 The function $u \in L_2(\Omega)$ is called a solution of problem (6), (7) if there exists a sequence $\{u_n\}_{n=-\infty}^{\infty} \subseteq C^{(2)}_{0,\pi}(\Omega)$ such that $\|u_n - u\|_{2,\Omega} \to 0$, $\|Lu_n - f\|_{2,\Omega} \to 0$ as $n \to \infty$.

We introduce the notations

$$\alpha_{g,h}(t) = \|g\|_{L_2(0,t)} \|h^{-1}\|_{L_2(t,+\infty)} \quad (t > 0),$$

$$\beta_{g,h}(\tau) = \|g\|_{L_2(\tau,0)} \|h^{-1}\|_{L_2(-\infty,\tau)} \quad (\tau < 0),$$

$$\gamma_{g,h} = \max\Big(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \beta_{g,h}(\tau)\Big),$$

where g and $h \neq 0$ are given continuous functions.

The following statement is a special case of Lemma 2.1 [23].

Lemma 1.1 *If g and $h \neq 0$ are continuous functions with $\gamma_{g,h} < \infty$, then*

$$\int_R \big|g(x)v(x)\big|^2\,dx \le C \int_R \big|h(x)v'(x)\big|^2\,dx, \quad \forall v \in C_0^{(1)}(R).$$

Moreover, if C is the smallest constant for which this inequality holds, then

$$4\Big[\min\Big(\sup_{t>0} \alpha_{g,h}(t), \sup_{\tau<0} \alpha_{g,h}(\tau)\Big)\Big]^2 \le C \le 4(\gamma_{g,h})^2.$$

Theorem 1.1 *Let $a(x)$ be a continuously differentiable function, $b(x)$ be a continuous function, and the following conditions be fulfilled*:

(a) $|a(x)| \ge 1$, $\gamma_{1,\sqrt{|a|}} < \infty$;

(b) $\gamma_{b,\sqrt{|a|}} < \infty$.

Then, for each $f \in L_2(\Omega)$, there exists a unique solution u of problem (6), (7) and the following estimate holds:

$$\big\|\sqrt{|a|}u_x\big\|_{2,\Omega} + \big\|\big(|b|+1\big)u\big\|_{2,\Omega} \le C_3\|f\|_{2,\Omega}. \tag{8}$$

We will prove Theorem 1.1 in Sect. 3. Further, we will prove all our statements for the case $a(x) \geq 1$. The case $a(x) \leq -1$ is reduced from this case by replacement of independent variable.

Example 1.1 In Ω, we consider

$$-u_{xx} - u_{yy} + (x^2+3)^\alpha u_x + x^3 \cos e^{9x} u = f_1(x,y) \tag{9}$$

with boundary conditions (7). It is easy to show that if $\alpha \geq 4$, then the conditions of Theorem 1.1 are satisfied. Thus, for any $f_1 \in L_2(\Omega)$, problem (9), (7) has a unique solution u and

$$\left\| (x^2+3)^{\frac{\alpha}{2}} u_x \right\|_{2,\Omega} + \left\| \left(|x|^3 \left|\cos e^{9x}\right| + 1\right) u \right\|_{2,\Omega} \leq C_4 \|f_1\|_{2,\Omega}.$$

Example 1.2 Let

$$-u_{xx} - u_{yy} - \left[1 + 20 e^{\sqrt{1+x^2}} (3 + 2\sin 4x)\right] u_x + x^{2n} \cos^2 5x u = f_2(x,y), \tag{10}$$

where $(x,y) \in \Omega$, $n \in N$, and $f_2 \in L_2(\Omega)$. Then the conditions of Theorem 1.1 hold. So, problem (10), (7) has a unique solution u and

$$\left\| \left[1 + 20 e^{\sqrt{1+x^2}} (3 + 2\sin 4x)\right]^{1/2} u_x \right\|_{2,\Omega} + \left\| \left(x^{2n} \cos^2 5x + 1\right) u \right\|_{2,\Omega} \leq C_5 \|f_2\|_{2,\Omega}.$$

2 The case $b = 0$

In $\Theta_m = \{(x,y) : x \in (-\infty, +\infty), -m < y < m\}$ $(m > 0)$, we consider the following problem:

$$-u_{xx} - u_{yy} + \lambda a(x) u_x = g(x,y), \tag{11}$$

$$u(x,-m) = u(x,m), \qquad u_y(x,-m) = u_y(x,m), \tag{12}$$

where $g \in L_2(\Theta_m)$, $\lambda \geq 1$.

Let $l_\lambda u = -u_{xx} - u_{yy} + \lambda a(x) u_x$ for $u \in C_{0,m}^{(2)}(\Theta_m)$. It is easy to show that l_λ is a closable operator in the norm of $L_2(\Theta_m)$. We still denote by l_λ its closure.

Definition 2.1 The function $u \in L_2(\Theta_m)$ is called a solution of problem (11), (12) if $u \in D(l_\lambda)$ and $l_\lambda u = g$.

Lemma 2.1 *Let $a(x) \geq 1$ be continuously differentiable and satisfy the condition $\gamma_{1,\sqrt{a}} < \infty$. If there exists the solution $u(x,y)$ to problem* (11), (12), *then u is unique and*

$$\|\sqrt{\lambda a} u_x\|_{2,\Theta_m} + \|u\|_{2,\Theta_m} \leq (2\gamma_{1,\sqrt{a}} + 1) \|g\|_{2,\Theta_m} \tag{13}$$

holds.

Proof Let $u(x,y) \in C_{0,m}^{(2)}(\Theta_m)$. Integrating by parts and using the boundary conditions, we obtain that

$$(l_\lambda u, u_x) = \int_{\Theta_m} \lambda a(x) u_x^2(x,y) \, dx \, dy.$$

By the Hölder inequality, we get

$$\|\sqrt{\lambda a}u_x\|_{2,\Theta_m} \le \|l_\lambda u\|_{2,\Theta_m}.$$

It is easy to check that $\gamma_{1,\sqrt{\lambda a}} \le \gamma_{1,\sqrt{a}}$ for $\lambda \ge 1$. By Lemma 1.1,

$$\int_{-\infty}^{\infty} |u(x,y)|^2\,dx \le [2\gamma_{1,\sqrt{a}}]^2 \int_{-\infty}^{\infty} \lambda a(x)|u_x(x,y)|^2\,dx, \quad y \in (-m,m),$$

therefore we obtain that

$$\|\sqrt{\lambda a}u_x\|_{2,\Theta_m} + \|u\|_{2,\Theta_m} \le (1 + 2\gamma_{1,\sqrt{a}})\|l_\lambda u\|_{2,\Theta_m}.$$

If u is a solution of problem (11), (12), then there exists a sequence $\{u_n\}_{n=-\infty}^{\infty}$ in $C_{0,m}^{(2)}(\Theta_m)$ such that $\|u_n - u\|_{2,\Theta_m} \to 0$, $\|l_\lambda u_n - g\|_{2,\Theta_m} \to 0$ as $n \to \infty$. Then we have

$$\lim_{n\to\infty} \|u_n\|_{2,\Theta_m} = \|u\|_{2,\Theta_m}, \qquad \lim_{n\to\infty} \|l_\lambda u_n\|_{2,\Theta_m} = \|g\|_{2,\Theta_m}. \tag{14}$$

Since

$$\|\sqrt{\lambda a}(u_n)_x\|_{2,\Theta_m} + \|u_n\|_{2,\Theta_m} \le (2\gamma_{1,\sqrt{a}} + 1)\|l_\lambda u_n\|_{2,\Theta_m},$$

taking limit as $n \to \infty$ and using (14) and the closedness of the operator of generalized differentiation, we obtain (13). It is clear that (13) implies the uniqueness of the solution. □

Remark 2.1 We note that if the condition $\gamma_{1,\sqrt{|a|}} < \infty$ in Lemma 2.1 is not satisfied, then Lemma 1.1 implies that estimate (13), generally speaking, does not hold.

Remark 2.2 Lemma 2.1 is also true if $a(x) \ge \delta > 0$, $\delta < 1$. In fact, it suffices to prove (13) for $u(x,y) \in C_{0,m}^{(2)}(\Theta_m)$. If we denote $x = st$, $y = s\tau$ $(0 < s \le \delta)$, $\tilde{u}(t,\tau) = u(st,s\tau)$, $\tilde{a}(t) = a(st)$, $\tilde{g}(t,\tau) = g(st,s\tau)$, then instead of (11) and (12) we have

$$\tilde{l}_{s,\lambda}\tilde{u} := -s^{-2}\tilde{u}_{tt} - s^{-2}\tilde{u}_{\tau\tau} + \lambda s^{-1}\tilde{a}(t)\tilde{u}_t = \tilde{g},$$
$$\tilde{u}(t,-s^{-1}m) = \tilde{u}(t,s^{-1}m), \qquad \tilde{u}_\tau(t,-s^{-1}m) = \tilde{u}_\tau(t,s^{-1}m),$$

where $s^{-1}\tilde{a} \ge 1$. As in the proof of Lemma 2.1, we get

$$\|\sqrt{\lambda s^{-1}\tilde{a}}\tilde{u}_t\|_{2,\Theta_{s^{-1}m}} \le \|\sqrt{\lambda s^{-1}\tilde{a}}\tilde{l}_{s,\lambda}\tilde{u}\|_{2,\Theta_{s^{-1}m}}, \quad \tilde{u} \in C_{0,s^{-1}m}^{(2)}(\Theta_{s^{-1}m}).$$

Since $\lambda s^{-1}\tilde{a} \ge 1$, $\|\sqrt{\lambda s^{-1}\tilde{a}}\tilde{u}_t\|_{2,\Theta_{s^{-1}m}} = s^{-3/2}\|\sqrt{\lambda a}u_x\|_{2,\Theta_m}$, we have that $s^{-3/2}\|\sqrt{\lambda a}u_x\|_{2,\Theta_m} \le \|l_\lambda u\|_{2,\Theta_m}$. For $\lambda \ge 1$, $\gamma_{1,\sqrt{\lambda a}} \le \gamma_{1,\sqrt{a}}$. Using Lemma 1.1, we obtain (13).

Next, we prove the existence of a solution to problem (11), (12). Let the right-hand side g of equation (11) be represented as follows:

$$g = \sum_{\sigma=-\infty}^{\infty} g_\sigma(x)e^{-i\frac{\sigma\pi}{m}y}.$$

It is known that $g_\sigma \in L_2(R)$, $R = (-\infty, \infty)$. We denote $\sigma_m = \frac{\sigma\pi}{m}$ and consider the following equation:

$$-v'' + \lambda a(x)v' + \sigma_m^2 v = g_\sigma(x). \tag{15}$$

Let $C_0^{(2)}(R)$ be the set of twice continuously differentiable functions with compact support. Since a is a smooth function, the differential operator $l_{0,\lambda}^{(\sigma)} v = -v'' + \lambda a(x)v' + \sigma_m^2 v$ is defined on $C_0^{(2)}(R)$. Clearly, this operator is closable in $L_2(R)$. We denote its closure by $l_\lambda^{(\sigma)}$.

Definition 2.2 The function $v \in L_2(R)$ is called a solution of equation (15) if $v \in D(l_\lambda^{(\sigma)})$ and $l_\lambda^{(\sigma)} v = g_\sigma$.

The following statement is true.

Lemma 2.2 *If v_σ is a solution of equation (15) for each $\sigma \in Z$, then $v = \sum_{\sigma=-\infty}^{\infty} v_\sigma(x)e^{-i\sigma_m y}$ is a solution of problem (11), (12).*

Proof Let $G^{(k)} = \sum_{\sigma=-k}^{k} g_\sigma(x)e^{-i\sigma_m y}$. It is obvious that $\|G^{(k)} - g\|_{2,\Theta_m} \to 0$ $(k \to \infty)$. If g is replaced by $G^{(k)}$ in (11), then $v^{(k)} = \sum_{\sigma=-k}^{k} v_\sigma(x)e^{-i\sigma_m y}$ is a solution of problem (11), (12). To verify this, we multiply (15) by $e^{-i\sigma_m y}$ and sum up the obtained equality from $\sigma = -k$ to $\sigma = k$. Then we get equation (11) with respect to the unknown function $v^{(k)}$. It is clear that $v^{(k)}$ satisfies condition (12). By Definition 2.2, there exists a sequence $\{w_{s\sigma}\}_{s=1}^{\infty}$ in $C_0^{(2)}(R)$ such that $\|w_{s\sigma} - v_\sigma\|_{2,R} \to 0$, $\|l_{0,\lambda}^{(\sigma)} w_{s\sigma} - g_\sigma\|_{2,R} \to 0$ as $s \to +\infty$, where $\|\cdot\|_{2,R}$ is the norm in $L_2(R)$. Then

$$\left\| \sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma_m y} - v^{(k)} \right\|_{2,\Theta_m}^2 = \sum_{\sigma=-k}^{k} \|w_{s\sigma} - v_\sigma\|_{2,R}^2 \to 0$$

and

$$\left\| l_\lambda \left(\sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma_m y} \right) - G^{(k)} \right\|_{2,\Theta_m}^2 = \sum_{\sigma=-k}^{k} \left\| l_{0,\lambda}^{(\sigma)} w_{s\sigma} - g_\sigma \right\|_{2,R}^2 \to 0$$

as $s \to +\infty$. Therefore, the function $v^{(k)} = \sum_{\sigma=-k}^{k} v_\sigma(x)e^{-i\sigma_m y}$ is a solution to problem (11), (12), where $g = G^{(k)}$.

Further,

$$\begin{aligned} &\left\| l_\lambda \left(\sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma_m y} \right) - g \right\|_{2,\Theta_m} \\ &\quad \le \left\| l_\lambda \left(\sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma_m y} \right) - G^{(k)} \right\|_{2,\Theta_m} \\ &\qquad + \left\| G^{(k)} - g \right\|_{2,\Theta_m} \to 0 \quad (s \to +\infty, k \to +\infty). \end{aligned} \tag{16}$$

By Lemma 2.1,

$$\left\| \sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma m y} \right\|_{2,\Theta_m} \le C_6 \left\| l_\lambda \left(\sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma m y} \right) \right\|_{2,\Theta_m}, \quad s = 1, 2, \ldots .$$

Therefore, functions $\sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma m y}$ ($s \in N$, $k = 0, 1, 2, \ldots$) form a Cauchy sequence, which converges to $v \in L_2(\Theta_m)$:

$$\left\| \sum_{\sigma=-k}^{k} w_{s\sigma}(x)e^{-i\sigma m y} - v \right\|_{2,\Theta_m} \to 0 \tag{17}$$

as $s \to +\infty$ and $k \to +\infty$. By (16) and (17), $v = \sum_{\sigma=-\infty}^{\infty} v_\sigma(x)e^{-i\sigma m y}$ is a solution to problem (11), (12). □

Lemma 2.2 shows that the existence of a solution of (15) for any $g_\sigma \in L_2(R)$ ($\sigma \in Z$) implies the solvability of problem (11), (12). We prove the following auxiliary statement.

Lemma 2.3 *Let $a(x) \ge 1$ be continuously differentiable and $\gamma_{1,\sqrt{a}} < \infty$. Then*

$$\left\| \sqrt{\lambda a} v' \right\|_{2,R} + \|v\|_{2,R} \le C_7 \left\| l_\lambda^{(\sigma)} v \right\|_{2,R}, \quad \forall v \in D\left(l_\lambda^{(\sigma)}\right), \tag{18}$$

where $C_7 = 2\gamma_{1,\sqrt{a}} + 1$.

Proof Let $v(x) \in C_0^{(2)}(R)$. Since v is finite,

$$\left(l_{0,\lambda}^{(\sigma)} v, v'\right) = \int_{-\infty}^{\infty} \lambda a(x) \left(v'\right)^2 dx.$$

Using the Hölder inequality and the condition $a \ge 1$, we get

$$\left\| \sqrt{\lambda a} v' \right\|_{2,R} \le \left\| l_{0,\lambda}^{(\sigma)} v \right\|_{2,R}. \tag{19}$$

By Lemma 1.1, we obtain

$$\|v\|_{2,R} \le 2\gamma_{1,\sqrt{a}} \left\| \sqrt{\lambda a} v' \right\|_{2,R}. \tag{20}$$

From (20) and (19) it follows that

$$\left\| \sqrt{\lambda a} v' \right\|_{2,R} + \|v\|_{2,R} \le C_7 \left\| l_{0,\lambda}^{(\sigma)} v \right\|_{2,R}, \quad \forall v \in C_0^{(2)}(R).$$

Let $v \in D(l_\lambda^{(\sigma)})$. Since $l_\lambda^{(\sigma)}$ is a closed operator, there exists a sequence $\{v_s\}_{s=1}^{\infty}$ in $C_0^{(2)}(R)$ such that

$$\|v_s - v\|_{2,R} \to 0, \qquad \left\| l_{0,\lambda}^{(\sigma)} v_s - l_\lambda^{(\sigma)} v \right\|_{2,R} \to 0 \quad \text{as } s \to \infty. \tag{21}$$

According to (19) and (20), we have

$$\left\| \sqrt{\lambda a} v_s' \right\|_{2,R} + \|v_s\|_{2,R} \le C_7 \left\| l_{0,\lambda}^{(\sigma)} v_s \right\|_{2,R}. \tag{22}$$

We denote by $W^1_{2,\sqrt{\lambda a}}(R)$ the completion of $C_0^{(1)}(R)$ with respect to the norm $\|\theta\|_W = \|\sqrt{\lambda a}\theta'\|_{2,R} + \|\theta\|_{2,R}$. (21) and (22) imply that the sequence $\{v_s\}_{s=1}^{\infty}$ is a Cauchy sequence in $W^1_{2,\sqrt{\lambda a}}(R)$. Hence, there exists $v \in W^1_{2,\sqrt{\lambda a}}(R)$ such that $\|v_s - v\|_W \to 0$ $(s \to \infty)$. Then, passing to the limit in (22) and using (21), we obtain (18). □

Remark 2.3 Lemma 2.3 remains true if $a \geq \delta > 0$, where $\delta < 1$. This fact is verified similarly to Remark 2.2.

It follows from Lemma 2.3 that the solution of equation (15) belongs to the space $W^1_{2,\sqrt{\lambda a}}(R)$.

Let $Q \subseteq R$. A complex number μ is called a regular-type point of the linear operator $L : L_2(Q) \to L_2(Q)$ if there exists a constant $\varepsilon > 0$ such that

$$\|(L - \mu E)u\|_{L_2(Q)} \geq \varepsilon \|u\|_{L_2(Q)}$$

for each $u \in D(L)$, where E is the identity operator.

The next result is known (see, for example, [24], Ch. 8).

Lemma 2.4 *Let M be a connected subset of the set of complex numbers C, and let $\mu \in M$ be a regular type point of a linear operator L. Then the dimensions of the spaces $L_2(Q) \ominus (L - \mu E)D(L)$ (this is the orthogonal complement of the range of $L - \mu E$ to $L_2(Q)$) are the same for all values of μ.*

The following is our main result in this section.

Lemma 2.5 *Let $a(x) \geq \delta > 0$ be a continuously differentiable function and satisfy the condition $\gamma_{1,\sqrt{a}} < \infty$. Then, for any $g \in L_2(\Theta_m)$, there exists a unique solution u to problem (11), (12) and (13) holds.*

Proof By Remarks 2.2 and 2.3, we may assume that $a(x) \geq 1$. If u is a solution to problem (11), (12), then by Lemma 2.1 u is unique and for it inequality (13) holds. By (18), the range $R(l_\lambda^{(\sigma)})$ of $l_\lambda^{(\sigma)}$ is a closed set since $l_\lambda^{(\sigma)}$ is a closed operator. By Lemma 2.2, it suffices to show that $R(l_\lambda^{(\sigma)}) = L_2(R)$. For any $\mu \in C$ similar to Lemma 2.3, it is easy to prove that

$$\|\sqrt{\lambda a}u'\|_{2,R} + \|u\|_{2,R} \leq C_8 \|(l_\lambda^{(\sigma)} - \mu E)u\|_{2,R}, \quad u \in D(l_\lambda^{(\sigma)})\ (\sigma \in Z), \tag{23}$$

where $C_8 = \gamma_{1,\sqrt{a}} + 1$ does not depend on μ. This means that each point $\mu \in C$ is a regular type point for the operator $l_\lambda^{(\sigma)}u = -u'' + \lambda a(x)u' + \sigma_m^2 u$ $(\sigma \in Z)$. In particular, the point $\mu = 0$ is a regular type point of the operator $l_\lambda^{(0)} = l_\lambda^{(\sigma)} - \sigma_m^2 E$. Let us prove that

$$R(l_\lambda^{(0)}) = L_2(R). \tag{24}$$

If this does not hold, then there exists a nonzero element $v \in L_2(R) \ominus R(l_\lambda^{(0)})$ such that

$$(l_{0,\lambda}^{(0)}u, v) = 0, \quad \forall u \in D(l_{0,\lambda}^{(0)}).$$

Since $D(l_{0,\lambda}^{(0)}) = C_0^{(2)}(R)$ is dense in $L_2(R)$, we have that

$$\left(l_{0,\lambda}^{(0)}\right)^* v = -\left(v' + \lambda a v\right)' = 0.$$

Then

$$\left(v(x)\exp\int_\theta^x \lambda a(t)\,dt\right)' = c_9 \exp\int_\theta^x \lambda a(t)\,dt, \quad \theta \in R.$$

If $C_9 = 0$, then $v = C_{10}\exp[-\int_\theta^x \lambda a(t)\,dt]$. Since $a(x) \geq 1$, it follows that $v \notin L_2(R)$. If $C_9 \neq 0$, then without loss of generality, we assume that $C_9 = -1$. So,

$$\left(v(x)\exp\int_\theta^x \lambda a(t)\,dt\right)' = -\exp\int_\theta^x \lambda a(t)\,dt \leq -1$$

for $x \geq \theta$. We consider functions $w_1(x) = v(x)\exp\int_\theta^x \lambda a(t)\,dt$ and $w_2(x) = -x + v(\theta) + \theta$. We note that $w_1(\theta) = w_2(\theta) = v(\theta)$, and by the last inequality, $v(x) \leq w_1(x) \leq w_2(x)$ for $x \geq \theta$. However, $w_2(x) \leq -1$ for $x \geq \max\{v(\theta) + \theta + 1, \theta + 1\}$. Consequently, $v \notin L_2(R)$. This is a contradiction. Hence, $R(l_\lambda^{(0)}) = L_2(R)$. Using (23) and Lemma 2.4, we get $R(l_\lambda^{(\sigma)}) = L_2(R)$ holds for any σ. □

3 Proof of the main result

Proof Without loss of generality, we assume that $a \geq 1$. Let $x = kt, y = k\tau$. We denote $\tilde{a}(t) = a(kt), \tilde{b}(t) = b(kt), w(t,\tau) = u(kt,k\tau), \tilde{f}(t,\tau) = k^2 f(kt,k\tau)$. Then (6) takes the following form:

$$-w_{tt} - w_{\tau\tau} + k\lambda\tilde{a}(t)w_t + k^2\tilde{b}(t)w = \tilde{f}(t,\tau), \tag{25}$$

where

$$(t,\tau) \in \Theta_{\pi/k} = \left\{(t,\tau) : -\infty < t < \infty, -\frac{\pi}{k} < \tau < \frac{\pi}{k}\right\}.$$

Conditions (7) pass to the following:

$$w\left(t, -\frac{\pi}{k}\right) = w\left(t, \frac{\pi}{k}\right), \qquad w_y\left(t, -\frac{\pi}{k}\right) = w_y\left(t, \frac{\pi}{k}\right). \tag{26}$$

Let $A_{k,\lambda}$ be the closure in $L_2(\Theta_{\pi/k})$ of the differential operator $A_{0,k,\lambda}w = -w_{tt} - w_{\tau\tau} + k\lambda\tilde{a}(t)w_t$ defined on $C_{0,\frac{\pi}{k}}^{(2)}(\Theta_{\pi/k})$. By Lemmas 2.1 and 2.5, we obtain that the operator $A_{k,\lambda}$ is boundedly invertible in $L_2(\Theta_{\pi/k})$ and

$$\|\sqrt{k\lambda\tilde{a}}w_t\|_{2,\Theta_{\pi/k}} \leq \|A_{k,\lambda}w\|_{2,\Theta_{\pi/k}}, \quad \forall w \in D(A_{k,\lambda}). \tag{27}$$

It is easy to see that $\gamma_{k^2\tilde{b},\sqrt{k\lambda\tilde{a}}} = \sqrt{\frac{k}{\lambda}}\gamma_{b,\sqrt{a}}$. By Lemma 1.1, condition (b) of Theorem 1.1 and (27), we have the inequalities

$$\|k^2\tilde{b}w\|_{2,\Theta_{\pi/k}} \leq 2\gamma_{k^2\tilde{b},\sqrt{k\lambda\tilde{a}}}\|\sqrt{k\lambda\tilde{a}}w_t\|_{2,\Theta_{\pi/k}} \leq 2\sqrt{\frac{k}{\lambda}}\gamma_{b,\sqrt{a}}\|A_{k,\lambda}w\|_{2,\Theta_{\pi/k}}. \tag{28}$$

We choose k such that $k \leq \frac{\lambda}{16\gamma_{b,\sqrt{a}}^2}$. By (28), we obtain

$$\left\|k^2\tilde{b}w\right\|_{2,\Theta_{\pi/k}} \leq \frac{1}{2}\|A_{k,\lambda}w\|_{2,\Theta_{\pi/k}}. \tag{29}$$

Hence, by perturbation theorems (see, for example, [25], Chap. 4, Theorem 1.16), we obtain that the operator $G_{k,\lambda} = A_{k,\lambda} + k^2\tilde{b}(t)E$ corresponding to problem (25), (26) is closed and boundedly invertible in $L_2(\Theta_{\pi/k})$. Using inequality (29), we have

$$\|A_{k,\lambda}w\|_{2,\Theta_{\pi/k}} \leq \|G_{k,\lambda}w\|_{2,\Theta_{\pi/k}} + \frac{1}{2}\|A_{k,\lambda}w\|_{2,\Theta_{\pi/k}}.$$

Therefore,

$$\|A_{k,\lambda}w\|_{2,\Theta_{\pi/k}} \leq 2\|G_{k,\lambda}w\|_{2,\Theta_{\pi/k}}.$$

By (27) and (29), we obtain

$$\|\sqrt{k\lambda\tilde{a}}w_t\|_{2,\Theta_{\pi/k}} + \left\|k^2\tilde{b}w\right\|_{2,\Theta_{\pi/k}} \leq 3\|G_{k,\lambda}w\|_{2,\Theta_{\pi/k}}. \tag{30}$$

Let $w_k(t,\tau) = (G_{k,\lambda}^{-1}\tilde{f})(t,\tau)$ be a solution to problem (25), (26). Then $u(x,y) = w_k(kt,k\tau)$ is a solution of problem (6), (7). And (30) implies the inequality

$$\|\sqrt{a}u_x\|_{2,\Omega} + \|bu\|_{2,\Omega} \leq C_{11}\|f\|_{2,\Omega}. \tag{31}$$

By condition (a) of Theorem 1.1,

$$\|u\|_{2,\Omega} \leq 2\gamma_{1,\sqrt{a}}\|\sqrt{a}u_x\|_{2,\Omega}.$$

Therefore, for a solution u of problem (6), (7), estimate (8) holds. □

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