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TAME AND WILD AUTOMORPHISMS OF DIFFERENTIAL POLYNOMIAL ALGEBRAS OF RANK 2

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Abstract. It is proved that the tame automorphism group of a differential polynomial algebra $k\{x,y\}$ over a field k of characteristic 0 in two variables x, y with m commuting derivations $\delta_1,\dots,\delta_m$ is a free product with amalgamation. An example of a wild automorphism of the algebra $k\{x,y\}$ in the case of $m \geq 2$ derivations is constructed.

1. Introduction

It is well known [3,4,8,12] that every automorphism of a polynomial algebra $k[x,y]$ and a free associative algebra $k\langle x,y\rangle$ in two variables over an arbitrary field k is tame. Moreover [3,12], the automorphism groups of algebras $k[x,y]$ and $k\langle x,y\rangle$ are isomorphic, i.e.,

$$\mathrm{Aut}_k\, k[x,y] \cong \mathrm{Aut}_k\, k\langle x,y\rangle.$$

It is also known that automorphisms of two-generated free Poisson algebras over fields of characteristic zero [13] and automorphisms of two-generated free right-symmetric algebras over arbitrary fields [7] are tame. P. Cohn [1] proved that automorphisms of free Lie algebras of finite rank are tame. An analog of this theorem is true for free algebras of any homogeneous Schreier variety of algebras [10]. We recall that the varieties of all nonassociative algebras [9], commutative and anticommutative algebras [20], Lie algebras [19,26], and Lie superalgebras [14,22] are Schreier varieties.

The automorphism groups of polynomial algebras [17,18,25] and free associative algebras [23,24] in three variables over a field of characteristic zero cannot be generated by all elementary automorphisms, i.e., there exist wild automorphisms. U. U. Umirbaev proved [23,24] that the Anick automorphism

$$\delta = (x + z(xz - zy), y + (xz - zy)z, z)$$

of the free associative algebra $k\langle x,y,z\rangle$ over a field of characteristic 0 is wild.

The main notions of differential algebras can be found in [5,6,16]. We will consider differential algebras with the set of commuting derivations $\Delta = \{\delta_1,\delta_2,\dots,\delta_m\}$. Let k be a differential field of characteristic 0 and $k\{x,y\}$ be the differential polynomial algebra over k in two variables x,y. If $|\Delta| = 0$, then $k\{x,y\}$ becomes the usual polynomial algebra $k[x,y]$ over the field k. W. van der Kulk [8] and M. Nagata [15] proved that the group $\mathrm{Aut}(k[x,y])$ can be represented as an amalgamated free product

$$\mathrm{Aut}(k[x,y]) = A *_C B,$$

where A is the affine automorphism subgroup, B is the triangular automorphism subgroup, and $C = A \cap B$.

In this paper, we prove that the tame automorphism group of the algebra $k\{x,y\}$ admits a similar structure of an amalgamated free product for any set of derivations Δ. Moreover, using this structure we construct an example of a wild automorphism of the algebra $k\{x,y\}$ for $|\Delta| \geq 2$. This example is an analog of the well-known Anick automorphism [2, p. 398].

Thus, the automorphisms of the algebra $k\{x,y\}$ are tame for $|\Delta| = 0$ and $k\{x,y\}$ has wild automorphisms for $|\Delta| \geq 2$. The problem of tame and wild automorphisms of the algebra $k\{x,y\}$ remains open for $|\Delta| = 1$.

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The paper is organized as follows. In Sec. 2, some necessary definitions are given and some well-known statements are formulated. Section 3 is devoted to the representation of the tame automorphism group of the algebra $k\{x,y\}$ in the form of an amalgamated free product. In Sec. 4, we prove the reducibility of any non-affine tame automorphism of the algebra $k\{x,y\}$. An example of a wild automorphism is given in Sec. 5.

2. Definitions and Preliminary Facts

Let R be an arbitrary commutative ring with unity. A mapping $d\colon R \to R$ is called a *derivation* if

$$d(s+t) = d(s) + d(t), \quad d(st) = d(s)t + sd(t)$$

holds for all $s, t \in R$.

Let $\Delta = \{\delta_1, \dots, \delta_m\}$ be a basic set of derivation operators.

A ring R is called a *differential ring* or Δ*-ring* if $\delta_1, \dots, \delta_m$ are commuting derivations of the ring R, i.e., the derivations $\delta_i\colon R \to R$ are defined for all i and $\delta_i\delta_j = \delta_j\delta_i$ for all i and j.

Let Θ be the free commutative monoid on the set $\Delta = \{\delta_1, \dots, \delta_m\}$ of derivation operators. The elements

$$\theta = \delta_1^{i_1} \dots \delta_m^{i_m}$$

of the monoid Θ are called *derivative operators*. The *order* of θ is defined as $|\theta| = i_1 + \dots + i_m$. We also put $\gamma(\theta) = (i_1, \dots, i_m) \in \mathbb{Z}_+^m$, where $\mathbb{Z}_+$ is the set of all non-negative integers.

Let R be an arbitrary differential ring and let $X = \{x_1, \dots, x_n\}$ be a set of symbols. Consider the set of symbols $X^\Theta = \{x_i^\theta \mid 1 \le i \le n,\ \theta \in \Theta\}$ and the polynomial algebra $R[X^\Theta]$ on the set of symbols X^Θ. We turn $R[X^\Theta]$ into a differential algebra by

$$\delta_i(x_j^\theta) = x_j^{\theta\delta_i}$$

for all $1 \le i \le m$, $1 \le j \le n$, $\theta \in \Theta$. The differential algebra $R[X^\Theta]$ is denoted by $R\{X\}$ and is called the *differential polynomial algebra* over R on the set of variables X [5].

Let M be the free commutative monoid on the set of variables x_i^θ, where $1 \le i \le n$ and $\theta \in \Theta$. The elements of M are called *monomials* of the algebra $R\{x_1, x_2, \dots, x_n\}$. Every element $a \in R\{x_1, x_2, \dots, x_n\}$ can be uniquely written in the form

$$a = \sum_{m \in M} r_m m$$

with a finite number of nonzero $r_m \in R$.

For any $x_i^\theta \in X^\Theta$ we put $\alpha(x_i^\theta) = \big(\varepsilon_i, \gamma(\theta)\big) \in \mathbb{Z}_+^{n+m}$, where $\varepsilon_1, \dots, \varepsilon_n$ is the standard basis of $\mathbb{Z}_+^n$. If $m = a_1 \dots a_s \in M$, where $a_1, \dots, a_s \in X^\Theta$, then put $\alpha(m) = \alpha(a_1) + \dots + \alpha(a_s)$. Then $\alpha(m)$ is the vector of multilinear degree of the monomial m with respect to the variables $x_1, \dots, x_n$ and the derivation operators $\delta_1, \dots, \delta_m$. The sum of the components of the vector $\alpha(m)$ is called the *degree* of the monomial m and is denoted by $\deg(m)$.

Moreover, for any $w \in \mathbb{Z}^{n+m}$ we can define a w-degree function $\deg_w$ as $\deg_w(m) = w \cdot \alpha(m)$, where $\cdot$ denotes the usual scalar product. Obviously, $\deg_w$ coincides with $\deg$ if all components of w are equal to 1. If the first n components of w are equal to 1 and the other components are equal to 0, then $\deg_w$ is a general degree of w in the variables $x_1, \dots, x_n$. Any $w \in \mathbb{Z}^{n+m}$ defines a graduation

$$C = \bigoplus_{i \in \mathbb{Z}} C_i$$

of algebra $C = R\{x_1, x_2, \dots, x_n\}$, where C_i is the R-span of monomials of w-degree i. Each nonzero element $c \in C$ is uniquely represented in the form

$$c = c_{i_1} + c_{i_2} + \dots + c_{i_s}, \quad i_1 < i_2 < \dots < i_s, \quad 0 \ne c_{i_j} \in C_{i_j}.$$

The element c_{i_s} is called the *highest homogeneous part* of the element c with respect to the w-degree $\deg_w$. We denote by $\bar{c}$ the highest homogeneous part of c with respect to the degree function deg.

Let k be an arbitrary differential field of characteristic 0 and $B = k\{X\} = k\{x_1, \dots, x_n\}$ the differential polynomial algebra over k on the set of variables X. For any $0 \neq f, g \in B$, we have

$$\alpha(fg) = \alpha(f) + \alpha(g), \quad \deg(fg) = \deg(f) + \deg(g), \quad \overline{fg} = \bar{f}\bar{g}.$$

An element $f \in B$ is called *differentially algebraic* over k if there exists a nonzero element $g \in k\{z\}$ such that $g(f) = 0$. Otherwise $f \in B$ is called *differentially-transcendental* over k. Elements $f_1, f_2, \dots, f_s \in B$ are called *differentially algebraically dependent* over k if there exists a nonzero element $g \in k\{z_1, \dots, z_s\}$ such that $g(f_1, f_2, \dots, f_s) = 0$. If $f_1, f_2, \dots, f_s$ are differentially algebraically independent, then the homomorphism $k\{z_1, \dots, z_s\} \to k\{f_1, \dots, f_s\}$ defined by $z_i \mapsto f_i$ is an isomorphism.

Lemma 1. *Every element of the algebra $B = k\{x_1, \dots, x_n\}$ that does not belong to the field k is differentially transcendental over k.*

Proof. The statement of the lemma is an easy consequence of the well-known theorems on the differential transcendence degree [5, Chap. 2]. Here we propose a direct proof, using the usual algebraic dependence of the elements.

For any $u, v \in X^\Theta$, we put $u < v$ if $\deg(u) < \deg(v)$ or $\deg(u) = \deg(v)$ and $\alpha(u) < \alpha(v)$ with respect to the lexicographic order in $\mathbb{Z}_+^{n+m}$.

Let $0 \neq f \in B$. Let u be the largest element of X^Θ that present in f. Such an element u is called the *leader* of f with respect to the order $\leq$ on X^Θ [5, Chap. 1]. It is easy to understand that the leader of the element f^θ is u^θ, i.e., u^Θ is the set of leaders of the set of elements of f^Θ.

We put $W = X^\Theta \setminus u^\Theta$. Then the set of all elements of u^Θ is algebraically independent over $k[W]$, since u^Θ and W define a partition of the set X^Θ, which is algebraically independent over k.

Note that f is differentially algebraic over k if and only if the set of elements of f^Θ is algebraically dependent over k. Any algebraic dependence of elements of f^Θ over k leads to an algebraic dependence of u^Θ over $k[W]$, but it is impossible. □

If $f_1, f_2, \dots, f_r \in B$, then we denote by $k\{f_1, f_2, \dots, f_r\}$ the subalgebra of B generated by the elements $f_1, f_2, \dots, f_r$. Note that this type of designation does not mean the differentially algebraic independence of the elements $f_1, f_2, \dots, f_r$, i.e., $k\{f_1, f_2, \dots, f_r\}$ is not necessarily isomorphic to a differential polynomial algebra. A similar designation is often used to denote the subalgebras of polynomial algebras in affine algebraic geometry. The statement of the following lemma is true for any homogeneous free algebras (see, for example, [21]).

Lemma 2. *Let $f_1, f_2, \dots, f_r \in B$ and $u \in k\{f_1, f_2, \dots, f_r\}$. If $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r$ are differentially algebraically independent, then $\bar{u} \in k\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r\}$.*

Proof. Let $u = u(z_1, \dots, z_r) \in k\{z_1, \dots, z_r\}$ and let also $\deg(f_i) = n_i$, where $1 \leq i \leq r$. Put $w = (n_1, n_2, \dots, n_r, 1, \dots, 1)$ and consider the degree function $\deg_w$ in the algebra $k\{z_1, \dots, z_r\}$. Then $u = u' + \tilde{u}$, where $\tilde{u}$ is the highest homogeneous part of u with respect to $\deg_w$ and $\deg_w(u') < \deg_w(\tilde{u})$. Let $\deg_w(u) = k$. Note that $f_i = f_i' + \bar{f}_i$ for all i. Then

$$u(f_1, \dots, f_r) = u'(f_1, \dots, f_r) + \tilde{u}(f_1, \dots, f_r) = w' + \tilde{u}(\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r),$$

where $\deg(w') < k$. Since $\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r$ are differentially algebraically independent, it follows that $\tilde{u}(\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r)$ is not zero and has degree k by the choice of w. Consequently, $\bar{u} = \tilde{u}(\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r) \in k\{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_r\}$. □

Corollary 1. *Let $0 \neq f \in B$. If $a \in k\{f\}$, then $\bar{a} \in k\{\bar{f}\}$.*

Proof. It follows immediately from Lemmas 1 and 2. □

3. Amalgamated Free Product

Let $A = k\{x, y\}$ be the differential polynomial algebra in two variables x, y and let $\mathrm{Aut}(A)$ be the group of automorphisms of the algebra A. We denote by $\varphi = (f_1, f_2)$ the automorphism of A such that $\varphi(x) = f_1$, $\varphi(y) = f_2$. Automorphisms of the form

$$\sigma(1, a, f) = (ax + f(y), y), \quad \sigma(2, a, g) = \big(x, ay + g(x)\big),$$

where $0 \neq a \in k$, $f(y) \in k\{y\}$, $g(x) \in k\{x\}$, are called *elementary*. The subgroup $T(A)$ of the group $\mathrm{Aut}(A)$ generated by all elementary automorphisms is called the *tame automorphism subgroup*. Non tame automorphisms are called *wild*.

We define a degree of an automorphism $\theta = (f_1, f_2) \in \mathrm{Aut}(A)$ by

$$\deg(\theta) = \deg(f_1) + \deg(f_2).$$

If

$$\theta = (f_1, f_2), \quad \varphi = (g_1, g_2),$$

then the product in $\mathrm{Aut}(A)$ is defined by

$$\theta \circ \varphi = \big(g_1(f_1, f_2), g_2(f_1, f_2)\big).$$

Let $\mathrm{Af}_2(A)$ be the affine automorphism group of the algebra A, i.e., the group of automorphisms of the form $(a_1x + b_1y + c_1, a_2x + b_2y + c_2)$, where $a_i, b_i, c_i \in k$, $a_1b_2 \neq a_2b_1$; $\mathrm{Tr}_2(A)$ be the triangular automorphism group of the algebra A, i.e., the group of automorphisms of the form $(ax + f(y), by + c)$, where $0 \neq a, b \in k$, $c \in k$, $f(y) \in k\{y\}$; and let $C = \mathrm{Af}_2(A) \cap \mathrm{Tr}_2(A)$.

Let G be an arbitrary group, G_0, G_1, and G_2 be subgroups of the group G, and $G_0 = G_1 \cap G_2$. The group G is called the *free product of the subgroups G_1 and G_2 with the amalgamated subgroup G_0* and is denoted by $G = G_1 *_{G_0} G_2$ if

(a) G is generated by the subgroups G_1 and G_2;
(b) the defining relations of the group G consist only of the defining relations of the subgroups G_1 and G_2.

If S_1 is a complete system of representatives of the left cosets of G_0 in G_1 and S_2 is a complete system of representatives of the left cosets of G_0 in G_2, then the group G is a free product of the subgroups G_1 and G_2 with the amalgamated G_0 (see, for example, [11]) if and only if each $g \in G$ is uniquely represented in the form

$$g = g_1 \dots g_k c,$$

where $g_i \in S_1 \cup S_2$, $i = 1, \dots, k$, g_i and g_{i+1} are neither both in S_1, nor both in S_2, and $c \in G_0$.

The notation $h_i(y)$ in the proofs of the following several lemmas means that $h_i(y) \in k\{y\}$ is a homogeneous differential polynomial of degree i with respect to the degree function deg in $k\{y\}$. It is clear that $h_0(y) \in k$.

Lemma 3.

(a) *The system of elements*

$$A_0 = \{\mathrm{id} = (x, y),\ \gamma = (y, x + ay) \mid a \in k\}$$

is a left coset representative system for $\mathrm{Af}_2(A)$ *modulo* C.

(b) *The system of elements*

$$B_0 = \{\beta = (x + q(y), y) \mid q(y) = h_n(y) + \dots + h_2(y)\}$$

is a left coset representative system for $\mathrm{Tr}_2(A)$ *modulo* C.

Proof. We verify the condition (a). Let $l \in \mathrm{Af}_2(A)$. We must show that for any l there exist $\gamma \in A_0$, $\eta \in C$ such that $l = \gamma \circ \eta$.

If $l = (a_1x + b_1y + c_1, a_2x + b_2y + c_2)$, where $a_2 \neq 0$, then we put

$$\gamma = \left(y, x + \frac{b_2}{a_2}y\right), \quad \eta = \left(\left(b_1 - \frac{a_1b_2}{a_2}\right)x + a_1y + c_1, a_2y + c_2\right).$$

Then l is represented in the form

$$l = \left(y, x + \frac{b_2}{a_2}y\right) \circ \left(\left(b_1 - \frac{a_1b_2}{a_2}\right)x + a_1y + c_1, a_2y + c_2\right) = \gamma \circ \eta.$$

If $a_2 = 0$, then $\gamma = \mathrm{id}$, $\eta = l$, i.e., $l = \mathrm{id} \circ l$.

Assume that $\gamma_1 = (y, x + a_1y)$, $\gamma_2 = (y, x + a_2y)$, and $\gamma_1C = \gamma_2C$. Then

$$\gamma_1^{-1} \circ \gamma_2 = (-a_1x + y, x) \circ (y, x + a_2y) = (x, (-a_1 + a_2)x + y).$$

Hence it follows that $\gamma_1^{-1} \circ \gamma_2 \in C$ if and only if $a_1 = a_2$. Consequently, $\gamma_1 = \gamma_2$.

Now we verify the condition (b). Let $\psi = (ax + h(y), by + c) \in \mathrm{Tr}_2(A)$ and let $h(y) = h_n(y) + \cdots + h_1(y) + h_0(y)$. We must show that for any ψ there exist $\beta \in B_0$ and $\mu \in C$ such that $\psi = \beta \circ \mu$. Put $\beta = (x + q(y), y)$, $\mu = (ax + h_1(y) + h_0(y), by + c)$, where $q(y) = h_n(y) + \cdots + h_2(y)$. Then ψ is represented in form

$$\psi = \left(x + \frac{1}{a}\,q(y), y\right) \circ (ax + h_1(y) + h_0(y), by + c) = \beta \circ \mu.$$

Assume that $\beta_1 = (x + q(y), y)$, $\beta_2 = \big(x + q^{(1)}(y), y\big)$, and $\beta_1C = \beta_2C$. Then we have

$$\beta_1^{-1} \circ \beta_2 = (x - q(y), y) \circ \big(x + q^{(1)}(y), y\big) = \big(x - q(y) + q^{(1)}(y), y\big).$$

Hence, $\beta_1^{-1} \circ \beta_2 \in C$ if and only if $q(y) = q^{(1)}(y)$. Consequently, $\beta_1 = \beta_2$. □

Lemma 4. *Let A_0 and B_0 be the sets defined in Lemma 3. Then any tame automorphism φ of the algebra A decomposes into a product of the form*

$$\varphi = \gamma_1 \circ \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k \circ \gamma_{k+1} \circ \lambda, \tag{1}$$

where $\gamma_i \in A_0$, $\gamma_2, \ldots, \gamma_k \neq \mathrm{id}$, $\beta_i \in B_0$, $\beta_1, \ldots, \beta_k \neq \mathrm{id}$, and $\lambda \in C$.

Proof. We have

$$(ax + h(y), y) = \left(x + \frac{1}{a}\,q(y), y\right) \circ (ax + h_1(y) + h_0(y), y),$$

where $h(y) = h_n(y) + \cdots + h_2(y) + h_1(y) + h_0(y)$, $q(y) = h_n(y) + \cdots + h_2(y)$, and

$$\big(x, by + h^{(1)}(x)\big) = (y, x) \circ \left(x + \frac{1}{b}\,q^{(1)}(y), y\right) \circ \big(y, bx + h_1^{(1)}(y) + h_0^{(1)}(y)\big),$$

where $h^{(1)}(y) = h_m^{(1)}(y) + \cdots + h_2^{(1)}(y) + h_1^{(1)}(y) + h_0^{(1)}(y)$, $q^{(1)}(y) = h_m^{(1)}(y) + \cdots + h_2^{(1)}(y)$. Consequently, every elementary automorphism has the form

$$l_1 \circ \beta \circ l_2,$$

where $\beta \in B_0$, $l_1, l_2 \in \mathrm{Af}_2(A)$.

Any tame automorphism φ is represented as a composition of elementary automorphisms $\varphi_1, \varphi_2, \ldots, \varphi_n$, i.e.,

$$\varphi = \varphi_1 \circ \varphi_2 \circ \cdots \circ \varphi_n.$$

Consequently, we have

$$\varphi = l_1 \circ \beta_1 \circ l_2 \circ \beta_2 \circ \cdots \circ l_n \circ \beta_n \circ l_{n+1}, \tag{2}$$

where $\beta_i \in B_0$, $l_i \in \mathrm{Af}_2(A)$.

We prove by induction on n that φ is represented as a product of the form (1), with $k \leq n$.

By Lemma 3, the automorphism l_1 is written as $\gamma_1 \circ \lambda_1$, where $\gamma_1 \in A_0$, $\lambda_1 \in C$. Then

$$l_1 \circ \beta_1 = \gamma_1 \circ \lambda_1 \circ \beta_1.$$

Let $\lambda_1 = (ax + by + c, b_1 y + c_1)$, $\beta_1 = (x + q(y), y)$. Then

$$\lambda_1 \circ \beta_1 \circ \lambda_1^{-1} = \left(x + \frac{1}{a}\, q(b_1 y + c_1), y\right).$$

We denote by $q_{<2}(b_1 y + c_1)$ the linear part of the differential polynomial $q(b_1 y + c_1)$. Let

$$\lambda = \left(x - \frac{1}{a}\, q_{<2}(b_1 y + c_1), y\right).$$

It is clear that $\lambda \in C$ and $\lambda_1^{-1} \circ \lambda \in C$. We denote $\lambda_1^{-1} \circ \lambda$ by λ_2^{-1}. Then

$$l_1 \circ \beta_1 = \gamma_1 \circ \lambda_1 \circ \beta_1 = \gamma_1 \circ \beta_1' \circ \lambda_2,$$

where

$$\beta_1' = \lambda_1 \circ \beta_1 \circ \lambda_2^{-1} = \left(x + \frac{1}{a}\, q(b_1 y + c_1) - \frac{1}{a}\, q_{<2}(b_1 y + c_1), y\right) \in B_0.$$

We have

$$\varphi = \gamma_1 \circ \beta_1' \circ (\lambda_2 \circ l_2) \circ \beta_2 \circ \cdots \circ l_n \circ \beta_n \circ l_{n+1}.$$

By the induction hypothesis, the product

$$(\lambda_2 \circ l_2) \circ \beta_2 \circ \cdots \circ l_n \circ \beta_n \circ l_{n+1}$$

is written as

$$\gamma_2 \circ \beta_2' \circ \gamma_3 \circ \cdots \circ \gamma_k \circ \beta_k' \circ \gamma_{k+1} \circ \lambda, \quad k \le n.$$

Consequently,

$$\varphi = \gamma_1 \circ \beta_1' \circ \gamma_2 \circ \beta_2' \circ \cdots \circ \gamma_k \circ \beta_k' \circ \gamma_{k+1} \circ \lambda.$$

If $\gamma_2 \ne \mathrm{id}$, then this representation is of the form (1). Now consider the case where $\gamma_2 = \mathrm{id}$. Since $\beta_1' \circ \beta_2' = \beta_2'' \in B_0$, it follows that

$$\varphi = \gamma_1 \circ \beta_1' \circ \beta_2' \circ \gamma_3 \circ \cdots \circ \gamma_k \circ \beta_k' \circ \gamma_{k+1} \circ \lambda = \gamma_1 \circ \beta_2'' \circ \gamma_3 \circ \cdots \circ \gamma_k \circ \beta_k' \circ \gamma_{k+1} \circ \lambda.$$

Since $k - 1 < n$, by the induction hypothesis φ is written as (1). □

Lemma 5. *Let $\varphi = (f_1, f_2)$ be an automorphism of the algebra A, representable as the product*

$$\varphi = (f_1, f_2) = \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k,$$

where $\mathrm{id} \ne \gamma_i \in A_0$, $\mathrm{id} \ne \beta_i \in B_0$ for all i. If $\beta_i = (x + q_i(y), y)$, $\deg\big(q_i(y)\big) = n_i$, and s_i is the function degree of $q_i(y)$ on the variable y for all $1 \le i \le k$, then

$$\begin{aligned}
&\deg(f_1) = n_k + (n_{k-1} - 1)s_k + \cdots + (n_1 - 1)s_k s_{k-1} \ldots s_2,\\
&\deg(f_2) = n_{k-1} + (n_{k-2} - 1)s_{k-1} + \cdots + (n_1 - 1)s_{k-1} s_{k-2} \ldots s_2 \quad \text{if} \quad k > 1\\
&\deg(f_2) = 1 \quad \text{if} \quad k = 1.
\end{aligned}$$

Proof. We prove the lemma by induction on k. If $k = 1$, then $\varphi = \beta_1$ and

$$\begin{aligned}
\deg(f_1) &= \deg\big(q_1(y)\big) = n_1,\\
\deg(f_2) &= 1.
\end{aligned}$$

Suppose that the statement of the lemma holds for $k - 1$. Assume that

$$\varphi_1 = \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_{k-1} \circ \beta_{k-1} = (g_1, g_2).$$

By the induction hypothesis, we have

$$\begin{aligned}
\deg(g_1) &= n_{k-1} + (n_{k-2} - 1)s_{k-1} + \cdots + (n_1 - 1)s_{k-1} s_{k-2} \ldots s_2,\\
\deg(g_2) &= n_{k-2} + (n_{k-3} - 1)s_{k-2} + \cdots + (n_1 - 1)s_{k-2} s_{k-3} \ldots s_2.
\end{aligned}$$

Then

$$\varphi = (f_1, f_2) = \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k = \varphi_1 \circ \gamma_k \circ \beta_k = (g_1, g_2) \circ \gamma_k \circ \beta_k.$$

Applying $\gamma_k = (y, x + ay)$ to (g_1, g_2), we obtain

$$(u_1, u_2) = (g_1, g_2) \circ \gamma_k = (g_2, g_1 + ag_2).$$

Then

$$\begin{aligned} \deg(u_1) &= \deg(g_2) = n_{k-2} + (n_{k-3} - 1)s_{k-2} + \cdots + (n_1 - 1)s_{k-2}s_{k-3} \ldots s_2, \\ \deg(u_2) &= \max\{\deg(g_1), \deg(g_2)\} = n_{k-1} + (n_{k-2} - 1)s_{k-1} + \cdots + (n_1 - 1)s_{k-1}s_{k-2} \ldots s_2. \end{aligned}$$

Further,

$$\varphi = (f_1, f_2) = (u_1, u_2) \circ \beta_k = (u_1, u_2) \circ (x + q_k(y), y) = (u_1 + q_k(u_2), u_2).$$

Consequently,

$$\begin{aligned} \deg(f_1) &= \max\{\deg(u_1), \deg\big(q_k(u_2)\big)\}, \\ \deg(f_2) &= \deg(u_2). \end{aligned}$$

Recall that $\deg(q_k) = n_k$ and

$$\deg(u_2) = n_{k-1} + (n_{k-2} - 1)s_{k-1} + \cdots + (n_1 - 1)s_{k-1}s_{k-2} \ldots s_2.$$

Note that

$$\overline{q_k(u_2)} = \tilde{q}_k(\bar{u}_2),$$

where $\tilde{q}_k$ is the highest homogeneous part of q_k with respect to $\deg_w$, $w = (t, \underbrace{1, 1, \ldots, 1}_{m})$, and $t = \deg(u_2)$. Then

$$\begin{aligned} \deg\big(q_k(u_2)\big) &= \deg\big(\overline{q_k(u_2)}\big) = \deg\big(\tilde{q}_k(\bar{u}_2)\big) = \deg_w(q_k) = (t, 1, 1, \ldots, 1) \cdot \alpha(q_k) \\ &= \deg(q_k) + (t - 1)s_k = n_k + (n_{k-1} - 1)s_k + (n_{k-2} - 1)s_k s_{k-1} + \cdots + (n_1 - 1)s_k s_{k-1} \ldots s_2. \end{aligned}$$

Consequently,

$$\begin{aligned} \deg(f_1) &= n_k + (n_{k-1} - 1)s_k + \cdots + (n_1 - 1)s_k s_{k-1} \ldots s_2, \\ \deg(f_2) &= n_{k-1} + (n_{k-2} - 1)s_{k-1} + \cdots + (n_1 - 1)s_{k-1}s_{k-2} \ldots s_2. \end{aligned}$$

□

Lemma 6. *The decomposition* (1) *of an automorphism* φ *from Lemma* 4 *is unique.*

Proof. It suffices to show that

$$\gamma_1 \circ \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k \circ \gamma_{k+1} \circ \lambda \neq \mathrm{id},$$

where $k \geq 1$, $\gamma_i \in A_0$, $\gamma_2, \ldots, \gamma_k \neq \mathrm{id}$, $\beta_i \in B_0$, $\beta_1, \ldots, \beta_k \neq \mathrm{id}$, $\lambda \in C$.

Let us prove this by contradiction. Assume that

$$\gamma_1 \circ \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k \circ \gamma_{k+1} \circ \lambda = \mathrm{id}.$$

Then

$$\beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k = \gamma_1^{-1} \circ \lambda^{-1} \circ \gamma_{k+1}^{-1}. \tag{3}$$

By Lemma 5, the automorphism

$$\varphi = (f_1, f_2) = \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k$$

has degree

$$\begin{aligned} \deg(\varphi) = \deg(f_1) + \deg(f_2) = n_k &+ (n_{k-1} - 1)s_k + \cdots + (n_1 - 1)s_k s_{k-1} \ldots s_2 \\ &+ n_{k-1} + (n_{k-2} - 1)s_{k-1} + \cdots + (n_1 - 1)s_{k-1}s_{k-2} \ldots s_2. \end{aligned}$$

We denote the right-hand side of the equality (3) by ρ, i.e.,

$$\rho = \gamma_1^{-1} \circ \lambda^{-1} \circ \gamma_{k+1}^{-1}.$$

It is clear that $\rho \in \mathrm{Af}_2(A)$ and $\deg(\rho) = 2$. Consequently, $\deg(\varphi) \neq \deg(\rho)$, which contradicts the equality (3). $\square$

Theorem 1. *The tame automorphism group of the algebra $A = k\{x, y\}$ is a free product of the affine automorphism subgroup $\mathrm{Af}_2(A)$ and the triangular automorphism subgroup $\mathrm{Tr}_2(A)$ with an amalgamated subgroup $C = \mathrm{Af}_2(A) \cap \mathrm{Tr}_2(A)$, i.e.,*

$$T(A) = \mathrm{Af}_2(A) *_C \mathrm{Tr}_2(A).$$

Proof. Since A_0 and B_0 are, respectively, left coset representative systems for $\mathrm{Af}_2(A)$ and $\mathrm{Tr}_2(A)$ modulo subgroup C, by Lemma 4 and by Lemma 6 any automorphism is uniquely represented in the form (1). According to [11],

$$T(A) = \mathrm{Af}_2(A) *_C \mathrm{Tr}_2(A). \qquad \square$$

4. Reducibility of Tame Automorphisms

Recall that $\bar{f}$ is the highest homogeneous part of f with respect to the degree function deg and the degree of an automorphism $\theta = (f_1, f_2)$ is defined as

$$\deg(\theta) = \deg(f_1) + \deg(f_2).$$

A transformation (f_1, f_2) that changes only one element f_i $(i = 1, 2)$ to an element of the form $\alpha f_i + g$, where $0 \neq \alpha \in k$, $g \in k\{f_j \mid j \neq i\}$, is called *elementary.*

The notation $\theta \to \varphi$ means that φ is obtained from θ by a single elementary transformation. An automorphism θ is called *elementary reducible* if there exists an automorphism φ such that $\theta \to \varphi$ and $\deg(\varphi) < \deg(\theta)$.

Lemma 7. *Let $\theta = (f_1, f_2)$ be a non-affine tame automorphism of the algebra $A = k\{x, y\}$. If $\bar{f}_1$ and $\bar{f}_2$ are linearly dependent, then the automorphism π is elementary reducible.*

Proof. Let $\bar{f}_1 = \gamma \bar{f}_2$. Consider the elementary transformation

$$\theta = (f_1, f_2) \to (f_1 - \gamma f_2, f_2) = \sigma,$$

where $\gamma \in k^*$. We have $\deg(f_1) > \deg(f_1 - \gamma f_2)$. It follows that $\deg(\theta) > \deg(\sigma)$ and the automorphism π is elementary reducible. $\square$

Theorem 2. *Any non-affine tame automorphism of the algebra $A = k\{x, y\}$ is elementary reducible.*

Proof. Let $\theta = (f_1, f_2)$ be non-affine tame automorphism of the algebra A. By Lemma 4 θ is written as (1). If $\gamma_{k+1} \circ \lambda = \mathrm{id}$, then

$$\theta = \gamma_1 \circ \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k = (f_1, f_2).$$

Put

$$\tau = \gamma_1 \circ \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k = (g_1, g_2).$$

If $\beta_k = (x + q_k(y), y)$, then

$$\theta = (g_1 + q_k(g_2), g_2).$$

By Lemma 5, we have

$$\deg(\tau) = \deg(g_1) + \deg(g_2) < \deg(\theta) = \deg\big(g_1 + q_k(g_2)\big) + \deg(g_2).$$

Since $\theta \to \tau$, it follows that the automorphism θ is elementary reducible. Assume that

$$\gamma_{k+1} \circ \lambda = (a_1 x + b_1 y + c_1, a_2 x + b_2 y + c_2) \neq \mathrm{id}.$$

Put

$$\pi = \gamma_1 \circ \beta_1 \circ \gamma_2 \circ \beta_2 \circ \cdots \circ \gamma_k \circ \beta_k = (g_1 + q_k(g_2), g_2) = (u_1, u_2).$$

By Lemma 5, $\deg(u_1) > \deg(u_2)$.

Consequently,

$$\theta = \pi \circ \gamma_{k+1} \circ \lambda = (a_1 u_1 + b_1 u_2 + c_1, a_2 u_1 + b_2 u_2 + c_2) = (f_1, f_2).$$

If $a_1, a_2 \neq 0$, then $\bar{f}_1$ and $\bar{f}_2$ are linearly dependent and, by Lemma 7, the automorphism θ is elementary reducible.

If $a_1 = 0$, then $\bar{f}_1 = \bar{u}_2$ and $\bar{f}_2 = \bar{u}_1 = \overline{q_k(u_2)}$. In this case the automorphism θ is elementary reducible by using the automorphism $\psi = \big(f_1, f_2 - q_k(f_1)\big)$.

The case where $a_2 = 0$ is similar to the previous one. □

Corollary 2. *Let (f_1, f_2) be a non-affine tame automorphism of the algebra $A = k\{x, y\}$. Then there exist i and $g \in k\{f_j \mid j \neq i\}$ such that $\bar{f}_i = \bar{g}$.*

Proof. By Theorem 2, the automorphism (f_1, f_2) is elementary reducible. Assume that f_1 is a reducible element of this automorphism. Then there exist $g \in k\{f_2\}$ such that $\deg\big(f_1 - g(f_2)\big) < \deg(f_1)$. This means that $\bar{f}_1 = \overline{g(f_2)}$. □

5. An Analog of the Anick Automorphism

Lemma 8. *Let $|\Delta| \geq 2$. The endomorphism δ of the algebra $A = k\{x, y\}$ given as*

$$\delta(x) = x + w^{\delta_2}, \quad \delta(y) = y + w^{\delta_1},$$

where $w = x^{\delta_1} - y^{\delta_2}$, is an automorphism.

Proof. Assume that

$$f_1 = x + w^{\delta_2}, \quad f_2 = y + w^{\delta_1}.$$

We show that $k\{x, y\} = k\{f_1, f_2\}$. It is obvious that $k\{f_1, f_2\} \subseteq k\{x, y\}$. We have

$$x = f_1 - w^{\delta_2}, \quad y = f_2 - w^{\delta_1}.$$

Consequently,

$$w = x^{\delta_1} - y^{\delta_2} = \big(f_1 - w^{\delta_2}\big)^{\delta_1} - \big(f_2 - w^{\delta_1}\big)^{\delta_2} = f_1^{\delta_1} - f_2^{\delta_2} \in k\{f_1, f_2\}$$

and

$$x = f_1 - w^{\delta_2} \in k\{f_1, f_2\}, \quad y = f_2 - w^{\delta_1} \in k\{f_1, f_2\}.$$

This means that $k\{x, y\} \subseteq k\{f_1, f_2\}$. It follows that δ is a surjective homomorphism.

The linear parts of f_1 and f_2 are equal to x and y, respectively. Consequently, f_1 and f_2 are differentially algebraically independent. This shows that δ is injective homomorphism. □

Theorem 3. *The automorphism δ of the algebra $A = k\{x, y\}$ is wild.*

Proof. We have

$$\bar{f}_1 = \overline{x + x^{\delta_1\delta_2} - y^{\delta_2^2}} = x^{\delta_1\delta_2} - y^{\delta_2^2}, \quad \bar{f}_2 = \overline{y + x^{\delta_1^2} - y^{\delta_1\delta_2}} = x^{\delta_1^2} - y^{\delta_1\delta_2}.$$

Consequently, $\deg(x^{\delta_1^2} - y^{\delta_1\delta_2}) = 3$ and $\deg(x^{\delta_1\delta_2} - y^{\delta_2^2}) = 3$. Note that any homogeneous element of degree 3 of the algebra $k\{x^{\delta_1\delta_2} - y^{\delta_2^2}\}$ has the form $a(x^{\delta_1\delta_2} - y^{\delta_2^2})$ for some $a \in k^*$. Therefore, $x^{\delta_1^2} - y^{\delta_1\delta_2} \notin k\{x^{\delta_1\delta_2} - y^{\delta_2^2}\}$, since $x^{\delta_1^2} - y^{\delta_1\delta_2} = a(x^{\delta_1\delta_2} - y^{\delta_2^2})$ is impossible.

Similarly, $x^{\delta_1\delta_2} - y^{\delta_2^2} \notin k\{x^{\delta_1^2} - y^{\delta_1\delta_2}\}$.

Consequently, the automorphism δ does not satisfy the statement of Corollary 2, i.e., it is wild. □

REFERENCES

1. P. M. Cohn, "Subalgebras of free associative algebras," *Proc. London Math. Soc.*, **56**, 618–632 (1964).
2. P. M. Cohn, *Free Ideal Rings and Localization in General Rings*, New Math. Monogr., Vol. 3, Cambridge Univ. Press, Cambridge (2006).
3. A. G. Czerniakiewicz, "Automorphisms of a free associative algebra of rank 2. I, II," *Trans. Am. Math. Soc.*, **160**, 393–401 (1971); **171**, 309–315 (1972).
4. H. W. E. Jung, "Über ganze birationale Transformationen der Ebene," *J. Reine Angew. Math.*, **184**, 161–174 (1942).
5. E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Pure Appl. Math., Vol. 54, Academic Press, New York (1973).
6. M. V. Kondratieva, A. B. Levin, A. V. Mikhalev, and E. V. Pankratiev, *Differential and Difference Dimension Polynomials*, Math. Its Appl., Vol. 461, Kluwer Academic, Dordrecht (1999).
7. D. Kozybaev, L. Makar-Limanov, and U. Umirbaev, "The Freiheitssatz and the automorphisms of free right-symmetric algebras," *Asian-Eur. J. Math.*, **1**, 243–254 (2008).
8. W. van der Kulk, "On polynomial rings in two variables," *Nieuw Arch. Wisk.*, **3**, No. 1, 33–41 (1953).
9. A. G. Kurosh, "Nonassociative free algebras and free products of algebras," *Mat. Sb.*, **20**, 239–262 (1947).
10. J. Lewin, "On Schreier varieties of linear algebras," *Trans. Am. Math. Soc.*, **132**, 553–562 (1968).
11. W. Magnus, A. Karras, and D. Solitar, *Combinatorial Group Theory*, Dover, New York (1966).
12. L. Makar-Limanov, "Automorphisms of a free algebra with two generators," *Funkts. Anal. Pril.*, **4**, 107–108 (1970).
13. L. Makar-Limanov, U. Turusbekova, and U. U. Umirbaev, "Automorphisms and derivations of free Poisson algebras in two variables," *J. Algebra*, **322**, No. 9, 3318–3330 (2009).
14. A. A. Mikhalev, "Subalgebras of free colored Lie superalgebras," *Mat. Zametki*, **37**, No. 5, 653–661 (1985).
15. M. Nagata, *On Automorphism Group of* $k[x, y]$, Lect. Math., Kyoto Univ., No. 5, Kinokuniya, Tokyo (1972).
16. J. F. Ritt, *Differential Algebra*, Dover, New York (1966).
17. I. P. Shestakov and U. U. Umirbaev, "The Nagata automorphism is wild," *Proc. Natl. Acad. Sci. USA*, **100**, No. 22, 12561–12563 (2003).
18. I. P. Shestakov and U. U. Umirbaev, "Tame and wild automorphisms of rings of polynomials in three variables," *J. Am. Math. Soc.*, **17**, 197–227 (2004).
19. A. I. Shirshov, "Subalgebras of free Lie algebras," *Mat. Sb.*, **33**, No. 75, 441–452 (1953).
20. A. I. Shirshov, "Subalgebras of free commutative and free anticommutative algebras," *Mat. Sb.*, **34**, No. 76, 81–88 (1954).
21. A. I. Shirshov, *Rings and Algebras* [in Russian], Nauka, Moscow (1984).
22. A. S. Shtern, "Free Lie superalgebras," *Sib. Mat. Zh.*, **27**, 170–174 (1986).
23. U. U. Umirbaev, "Defining relations for tame automorphism groups of polynomial rings and wild automorphisms of free associative algebras," *Dokl. Math.*, **73**, No. 2, 229–233 (2006).
24. U. U. Umirbaev, "The Anick automorphism of free associative algebras," *J. Reine Angew. Math.*, **605**, 165–178 (2007).
25. U. U. Umirbaev and I. P. Shestakov, "Subalgebras and automorphisms of polynomial rings," *Dokl. Ross. Akad. Nauk*, **386**, No. 6, 745–748 (2002).
26. E. Witt, "Die Unterringe der freien Lieschen Ringe," *Math. Z.*, **64**, 195–216 (1956).

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