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ON ITERATED DISCRETE HARDY TYPE OPERATORS

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Abstract. The paper discusses a new iterated discrete inequality of Hardy type involving an operator with some matrix kernel. Under certain conditions on this matrix kernel, the given inequality is characterized.

1. Introduction

In the manuscript [7], V. Burenkov and R. Oinarov studied the problem of boundedness of the multidimensional Hardy operator from a Lebesgue space to a local Morrey-type space. They proved that this problem of boundedness is equivalent to the validity of the following inequality

$$\left(\int_0^\infty u^q(x)\left(\int_0^x\left|w(t)\int_0^t f(s)\,ds\right|^r dt\right)^{\frac{q}{r}} dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty |v(x)f(x)|^p\,dx\right)^{\frac{1}{p}}, \qquad (1)$$

for $1 \leqslant p < \infty$ and $0 < q, r < \infty$, where $u(\cdot)$, $v(\cdot)$ and $w(\cdot)$ are positive functions locally summable on the interval $(0;\infty)$. This result provided a strong impetus to study inequalities of type (1), so that in the last decade numerous works have been focused on them (see, e.g., [9], [10], [11], [14], [21], [22], [24] and references given therein). Since the operator

$$Kf(x) = \left(\int_0^x\left|w(t)\int_0^t f(s)\,ds\right|^r dt\right)^{\frac{1}{r}}$$

is quasilinear and contains iteration and, in addition, inequality (1) is a generalization of the famous weighted Hardy inequality

$$\left(\int_0^\infty\left|u(x)\int_0^x f(t)\,dt\right|^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty |v(x)f(x)|^p\,dx\right)^{\frac{1}{p}},$$

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in all these works, inequalities of type (1) are referred to as iterated Hardy-type inequalities or Hardy-type inequalities with quasilinear operators.

Inequalities of type (1) have one more interesting application: characterizations of bilinear Hardy-type inequalities can easily be derived from characterisations of inequalities of type (1) (see, e.g., [5], [12], [13], [21], [24] and references given therein). Due to these applications, inequalities of type (1) are currently one of the main objects of research in the theory of Hardy inequalities.

In this work, we consider a discrete analogue of inequality (1)

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} \left| w_k \sum_{i=1}^{k} a_{k,i} f_i \right|^r \right)^{\frac{q}{r}}\right)^{\frac{1}{q}} \leqslant C \left(\sum_{j=1}^{\infty} |v_j f_j|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,v}, \tag{2}$$

with the operator

$$(Kf)_n = \left(\sum_{k=1}^{n} \left| w_k \sum_{i=1}^{k} a_{k,i} f_i \right|^r \right)^{\frac{1}{r}}$$

that contains a matrix $(a_{k,i})$, $k \geqslant i \geqslant 1$, whose entries $a_{k,i} \geqslant 0$ are non-decreasing in k and non-increasing in i, satisfying the discrete Oinarov condition stating that there exists a constant $d \geqslant 1$ such that

$$\frac{1}{d}(a_{k,j} + a_{j,i}) \leqslant a_{k,i} \leqslant d(a_{k,j} + a_{j,i}) \tag{3}$$

for $k \geqslant j \geqslant i \geqslant 1$. Here $u = \{u_j\}_{j=1}^{\infty}$, $v = \{v_j\}_{j=1}^{\infty}$ and $w = \{w_j\}_{j=1}^{\infty}$ are weight sequences, i.e., positive sequences of real numbers. The aim of this paper is to characterize inequality (2) for $0 < r < p \leqslant q < \infty$ and $p > 1$. The case when $a_{k,i} = 1$ for all $k \geqslant i \geqslant 1$ was studied in the work [20] for the following relations between p, q and r: (1) $1 < p \leqslant q < \infty$, $0 < r < \infty$ and (2) $0 < r < q < p < \infty$, $p > 1$. The paper [8] also focuses on iterated discrete Hardy-type inequalities involving supremum operators.

If we denote by $l_{p,v}$ the space of sequences $f = \{f_j\}_{j=1}^{\infty}$ of real numbers with the finite norm

$$\|vf\|_p = \left(\sum_{j=1}^{\infty} |v_j f_j|^p\right)^{\frac{1}{p}}$$

and by $A^{\pm} f$ the matrix operator

$$(A^+ f)_k = \sum_{i=1}^{k} a_{k,i} f_i, \quad k \geqslant 1, \tag{4}$$

or

$$(A^- f)_i = \sum_{k=i}^{\infty} a_{k,i} f_k, \quad i \geqslant 1, \tag{5}$$

then, as in the integral case, inequality (2) is a generalization of the following weighted discrete Hardy-type inequality

$$\|uA^{\pm} f\|_q \leqslant C \|vf\|_p, \quad \forall f \in l_{p,v}. \tag{6}$$

Note that the validity of inequality (6) is equivalent to the problem of boundedness of the operator $A^{\pm}f$ from $l_{p,v}$ to $l_{q,u}$. It is still a well-known open problem to characterize inequality (6) without any restrictions on the matrix $(a_{k,i})$. The problem has been completely solved for the classical case $a_{k,i}=1$ in the works [2], [3], [4] and [6] (see Theorem 8 in [16]). The case where entries of the matrix $(a_{k,i})$ satisfy the discrete Oinarov condition has been studied in [18] for $1<p\leqslant q<\infty$, in [17] for $1<q<p<\infty$ and in [23] for $0<p\leqslant 1$, $p\leqslant q<\infty$. The case where the matrix operators $(a_{k,i})$ satisfy a certain modified discrete Oinarov condition has been recently investigated in [15] for $1<p,q<\infty$. In the paper [19], the expanding classes of matrix operators $\mathscr{O}_n^{\pm}$, $n\geqslant 0$, have been introduced, matrices $(a_{k,i})$ from which satisfy conditions weaker than condition (3). Characterizations of inequality (6) for operators from these expanding classes have been found in [19] for $1<p\leqslant q<\infty$ and in [25] for $1<q<p<\infty$ but for operators from the classes $\mathscr{O}_1^{\pm}$ only. The method presented in the proof of the main result is based on the reduction of the given problem to characterizations of inequality (6) for operators from the class $\mathscr{O}_2^{-}$. Since in [19] the range of parameters is $1<p\leqslant q<\infty$, here we are able to characterize inequality (2) in the case $0<r<p\leqslant q<\infty$ and $p>1$. As soon as inequality (6) for operators from the class $\mathscr{O}_2^{-}$ are established for $1<q<p<\infty$, inequality (2) can be established for $0<r<q<p<\infty$ and $q>1$ by the same method as presented here.

The paper is organized as follows. Section 2 contains all the auxiliary statements and definitions necessary to characterize inequality (2). Section 3 presents the main result on inequality (2) in the case when the involved matrices satisfy the discrete Oinarov condition. In Section 4, there is a comparison with the results obtained earlier in [20] for the case when $a_{k,i}=1$ for all $k\geqslant i\geqslant 1$.

2. Auxiliary statements and definitions

Let $\frac{1}{p}+\frac{1}{p'}=1$. The symbol $E\ll F$ means $E\leqslant CF$ with some constant C, depending on the parameters p, q and r. Moreover, the notation $E\approx F$ means $E\ll F\ll E$.

As pointed out in the Introduction, there are two expanding classes $\mathscr{O}_n^{+}$ and $\mathscr{O}_n^{-}$, $n\geqslant 0$, of matrices $(a_{k,i})$ in the paper [19]. To prove the main result we need the class $\mathscr{O}_n^{-}$, $n\geqslant 0$. Let us present its definition. Assume that $a_{k,i}\equiv a_{k,i}^{(n)}$ if $(a_{k,i})\in\mathscr{O}_n^{-}$. Define the classes $\mathscr{O}_n^{-}$ by induction. Let $(a_{k,i})$ be a matrix, whose entries $a_{k,i}\geqslant 0$ are non-increasing in i for all $k\geqslant i\geqslant 1$. Suppose that the class $\mathscr{O}_0^{-}$ is the set of matrices of the type $a_{k,i}^{(0)}=\alpha_k$ for all $k\geqslant i\geqslant 1$. Next, we assume that the classes $\mathscr{O}_\gamma^{-}$ have already been defined for $\gamma=0,1,\ldots,n-1$, $n\geqslant 1$. A matrix $(a_{k,i})\equiv(a_{k,i}^{(n)})$ belongs to the class $\mathscr{O}_n^{-}$ if and only if there exist matrices $(a_{k,i}^{(\gamma)})\in\mathscr{O}_\gamma^{-}$, $\gamma=0,1,\ldots,n-1$, such that

$$a_{k,i}^{(n)}\approx\sum_{\gamma=0}^{n}a_{k,j}^{(\gamma)}b_{j,i}^{\gamma,n} \tag{7}$$

for all $k\geqslant j\geqslant i\geqslant 1$, where $b_{j,i}^{\gamma,n}=\inf\limits_{j\leqslant k\leqslant\infty}\frac{a_{k,i}^{(n)}}{a_{k,j}^{(\gamma)}}$, $\gamma=0,1,\ldots,n-1$, and $b_{j,i}^{n,n}\equiv 1$.

REMARK 1. ([19, Remark 2]) We may assume that the matrices $b^{\gamma,n}_{j,i}$, $\gamma = 0,1,\dots,n$, $n \geqslant 1$, are arbitrary non-negative matrices which satisfy (7).

To prove the main result we need characterizations of inequality (6) found in [19].

THEOREM A. [19, Theorem 4.2] *Let* $1 < p \leqslant q < \infty$. *Let the matrix* $(a_{k,i})$ *in* (5) *belong to the class* $\mathscr{O}_n^-$, $n \geqslant 0$. *Then estimate* (6) *for the operator* (5) *holds if and only if one of the conditions*

$$E_1^- = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} u_i^q \left(\sum_{k=j}^{\infty} a_{k,i}^{p'} v_k^{-p'}\right)^{\frac{q}{p'}}\right)^{\frac{1}{q}} < \infty$$

and

$$E_2^- = \sup_{j\geqslant 1}\left(\sum_{k=j}^{\infty} v_k^{-p'} \left(\sum_{i=1}^{j} a_{k,i}^{q} u_i^{q}\right)^{\frac{p'}{q}}\right)^{\frac{1}{p'}} < \infty$$

holds. Moreover, $C \approx E_1^- \approx E_2^-$, *where* C *is the best constant in* (6).

In order to highlight that the introduced expanding classes include almost all classes of matrix operators considered earlier to characterize inequality (6), let us present two theorems, which are particular cases of Theorem A. Since a matrix $(a_{k,i})$, $k \geqslant i \geqslant 1$, whose entries $a_{k,i} = 1$, belongs to the class $\mathscr{O}_0^-$, we have the following well-known result on the weighted discrete Hardy inequality (see, e.g., [2] or [16]):

THEOREM A_1. *Let* $1 < p \leqslant q < \infty$. *The estimate*

$$\left(\sum_{i=1}^{\infty}\left|u_i \sum_{k=i}^{\infty} f_k\right|^q\right)^{\frac{1}{q}} \leqslant C\left(\sum_{j=1}^{\infty}|v_j f_j|^p\right)^{\frac{1}{p}}, \quad \forall f \in l_{p,v}, \tag{8}$$

holds if and only if

$$H^- = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=j}^{\infty} v_k^{-p'}\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $C \approx H^-$, *where* C *is the best constant in* (8).

Since the matrices of the class $\mathscr{O}_1^-$ are characterized by the relation

$$a_{k,i}^{(1)} \approx a_{k,j}^{(1)} + a_{k,j}^{(0)} b_{j,i}^{0,1} = a_{k,j}^{(1)} + \alpha_k b_{j,i}^{0,1}, \;\; \forall\, k \geqslant j \geqslant i \geqslant 1, \tag{9}$$

in view of Remark 1, it is obvious that a matrix satisfying condition (3) belongs to $\mathscr{O}_1^-$. Thus, we again get the well-known result on the boundedness of the matrix operator (5), whose entries satisfy condition (3), from $l_{p,v}$ to $l_{q,u}$ found in [18]:

THEOREM A_2. *Let $1 < p \leqslant q < \infty$ and the entries of the matrix $(a_{k,i})$ in (5) satisfy condition (3). Then estimate (6) for the operator (5) holds if and only if $M^- = \max\{M_1^-, M_2^-\} < \infty$, where*

$$M_1^- = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=j}^{\infty} a_{k,j}^{p'} v_k^{-p'}\right)^{\frac{1}{p'}},$$

$$M_2^- = \sup_{j\geqslant 1}\left(\sum_{i=1}^{j} a_{j,i}^q u_i^q\right)^{\frac{1}{q}}\left(\sum_{k=j}^{\infty} v_k^{-p'}\right)^{\frac{1}{p'}}.$$

Moreover, $C \approx M^-$, where C is the best constant in (6).

It is easy to see that due to condition (3) we can combine the values M_1^- and M_2^- so that this combination is equivalent to only one of the values either E_1^- or E_2^-.

The matrices of the class $\mathscr{O}_2^-$ are described by the relation

$$a_{k,i}^{(2)} \approx a_{k,j}^{(2)} + a_{k,j}^{(1)} b_{j,i}^{1,2} + a_{k,j}^{(0)} b_{j,i}^{0,2} = a_{k,j}^{(2)} + a_{k,j}^{(1)} b_{j,i}^{1,2} + \alpha_k b_{j,i}^{0,2}, \ \forall\, k \geqslant j \geqslant i \geqslant 1, \tag{10}$$

where $a_{k,j}^{(1)}$ belongs to $\mathscr{O}_1^-$. We have already mentioned in the Introduction that in the proof of the main result we meet operators from the class $\mathscr{O}_2^-$. Moreover, we have also pointed out that expanding the range of values of the parameters p and q in Theorem A can expand the range of them in the presented main result.

We also need the result of the work [17] for the operator (4) with entries satisfying the discrete Oinarov condition.

THEOREM B. *Let $1 < q < p < \infty$. Let the entries of the matrix $(a_{k,i})$ in (4) satisfy condition (3). Then inequality (6) for the operator (4) holds if and only if $M^+ = \max\{M_1^+, M_2^+\} < \infty$, where*

$$M_1^+ = \left(\sum_{j=1}^{\infty}\left(\sum_{k=j}^{\infty} a_{k,j}^q u_k^q\right)^{\frac{p}{p-q}}\left(\sum_{i=1}^{j} v_i^{-p'}\right)^{\frac{p(q-1)}{p-q}} v_j^{-p'}\right)^{\frac{p-q}{pq}},$$

$$M_2^+ = \left(\sum_{j=1}^{\infty}\left(\sum_{i=1}^{j} a_{j,i}^{p'} v_i^{-p'}\right)^{\frac{q(p-1)}{p-q}}\left(\sum_{k=j}^{\infty} u_k^q\right)^{\frac{q}{p-q}} u_j^q\right)^{\frac{p-q}{pq}}.$$

Moreover, $C \approx M^+$, where C is the best constant in (6).

For the proofs we use the following statement.

LEMMA A. [1, p. 844] *Let $\gamma > 0$ and $\{\beta_k\}_k$ be a nonnegative sequence. Then*

$$\left(\sum_{k=1}^{j}\beta_k\right)^{\gamma} \approx \sum_{k=1}^{j}\beta_k\left(\sum_{i=1}^{k}\beta_i\right)^{\gamma-1}, \quad j \geqslant 1. \tag{11}$$

If $\sum_k \beta_k < \infty$ *and* $1 \leqslant j \leqslant m \leqslant \infty$*, then*

$$\left(\sum_{k=j}^{m} \beta_k\right)^{\gamma} \approx \sum_{k=j}^{m} \beta_k \left(\sum_{i=k}^{m} \beta_i\right)^{\gamma-1}. \tag{12}$$

3. Main result

Let

$$E_1 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_n^{-p'}\left(\sum_{i=1}^{n} v_i^{-p'}\right)^{\frac{p(r-1)}{p-r}}\left(\sum_{j=k}^{\infty}\left(\sum_{i=n}^{j} a_{i,n}^r w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{pr}{q(p-r)}}\right)^{\frac{p-r}{pr}},$$

$$E_2 = \sup_{k\geqslant 1}\left(\sum_{j=k}^{\infty} u_j^q\left(\sum_{n=1}^{k}\left(\sum_{i=n}^{j} a_{i,n}^r w_i^r\right)^{\frac{p}{p-r}} v_n^{-p'}\left(\sum_{i=1}^{n} v_i^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{q(p-r)}{pr}}\right)^{\frac{1}{q}},$$

$$M_1 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} a_{k,n}^{p'} v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=k}^{\infty}\left(\sum_{i=k}^{j} w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{1}{q}},$$

$$M_2 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} w_n^r\left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}}\left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'}\right)^{\frac{r(p-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}.$$

THEOREM 1. *Let* $0 < r < p \leqslant q < \infty$ *and* $p > 1$*. Let* $(a_{k,i})$ *be a matrix that is non-decreasing in* k*, non-increasing in* i*, and that satisfies condition* (3)*. Then inequality* (2) *holds if and only if one of the conditions* $E_1M = \max\{E_1, M_1, M_2\} < \infty$ *or* $E_2M = \max\{E_2, M_1, M_2\} < \infty$ *holds. Moreover,* $C \approx E_1M \approx E_2M$*, where* C *is the best constant in* (2)*.*

Proof. We estimate

$$C = \sup_{f\geqslant 0}\frac{\left(\sum\limits_{n=1}^{\infty} u_n^q\left(\sum\limits_{k=1}^{n}\left|w_k\sum\limits_{i=1}^{k} a_{k,i} f_i\right|^r\right)^{\frac{q}{r}}\right)^{\frac{1}{q}}}{\|vf\|_p},$$

where C is the best constant in (2).

Define $\theta = \frac{q}{r}$, $u_n^q = \tilde{u}_n^\theta$ and $G_n = \sum_{k=1}^{n} \left| w_k \sum_{i=1}^{k} a_{k,i} f_i \right|^r$. Using the Hölder's inequality we have

$$C^r = \sup_{f \geqslant 0} \frac{\left(\sum_{n=1}^{\infty} (\tilde{u}_n G_n)^\theta \right)^{\frac{1}{\theta}}}{\|vf\|_p^r} = \sup_{\varphi \geqslant 0} \sup_{f \geqslant 0} \frac{\sum_{n=1}^{\infty} \varphi_n G_n}{\|\tilde{u}^{-1}\varphi\|_{\theta'} \|vf\|_p^r},$$

where $\frac{1}{\theta} + \frac{1}{\theta'} = 1$. Replacing back instead of G_n its sum representation into the latter expression and changing the order of sums, we obtain

$$C^r = \sup_{\varphi \geqslant 0} \sup_{f \geqslant 0} \frac{\sum_{k=1}^{\infty} \left| w_k \sum_{i=1}^{k} a_{k,i} f_i \right|^r \sum_{n=k}^{\infty} \varphi_n}{\|\tilde{u}^{-1}\varphi\|_{\theta'} \|vf\|_p^r}$$

$$= \sup_{\varphi \geqslant 0} \frac{1}{\|\tilde{u}^{-1}\varphi\|_{\theta'}} \sup_{f \geqslant 0} \left(\frac{\left(\sum_{k=1}^{\infty} w_k^r \sum_{n=k}^{\infty} \varphi_n \left| \sum_{i=1}^{k} a_{k,i} f_i \right|^r \right)^{\frac{1}{r}}}{\|vf\|_p} \right)^r.$$

Since $0 < r < p < \infty$ and $p > 1$, by Theorem B we have

$$C^r \approx \sup_{\varphi \geqslant 0} \frac{\left(\sum_{n=1}^{\infty} \left(\sum_{j=n}^{\infty} a_{j,n}^r w_j^r \sum_{i=j}^{\infty} \varphi_i \right)^{\frac{p}{p-r}} \left(\sum_{i=1}^{n} v_i^{-p'} \right)^{\frac{p(r-1)}{p-r}} v_n^{-p'} \right)^{\frac{p-r}{p}}}{\|\tilde{u}^{-1}\varphi\|_{\theta'}}$$

$$+ \sup_{\varphi \geqslant 0} \frac{\left(\sum_{n=1}^{\infty} \left(\sum_{j=n}^{\infty} w_j^r \sum_{i=j}^{\infty} \varphi_i \right)^{\frac{r}{p-r}} \left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'} \right)^{\frac{r(p-1)}{p-r}} w_n^r \sum_{k=n}^{\infty} \varphi_k \right)^{\frac{p-r}{p}}}{\|\tilde{u}^{-1}\varphi\|_{\theta'}}$$

$$= J_1 + J_2. \quad (13)$$

Let us estimate J_1. Changing the order of sums, we get the matrix operator

$$\sum_{j=n}^{\infty} a_{j,n}^r w_j^r \sum_{i=j}^{\infty} \varphi_i = \sum_{i=n}^{\infty} \varphi_i \sum_{j=n}^{i} a_{j,n}^r w_j^r = \sum_{i=n}^{\infty} \widetilde{a}_{i,n} \varphi_i$$

with the matrix kernel $\widetilde{a}_{i,n} = \sum_{j=n}^{i} a_{j,n}^r w_j^r$. Applying condition (3), for $\widetilde{a}_{i,n}$ we find that

$$\widetilde{a}_{i,n} \approx \sum_{j=n}^{k} a_{j,n}^r w_j^r + \sum_{j=k}^{i} a_{j,n}^r w_j^r \approx \widetilde{a}_{k,n} + \sum_{j=k}^{i} a_{j,k}^r w_j^r + \sum_{j=k}^{i} a_{k,n}^r w_j^r$$

$$= \widetilde{a}_{k,n} + \widetilde{a}_{i,k} + a_{k,n}^r \sum_{j=k}^{i} w_j^r = \widetilde{a}_{i,k} + \overline{a}_{i,k} \cdot a_{k,n}^r + 1 \cdot \widetilde{a}_{k,n}, \quad i \geqslant k \geqslant n \geqslant 1, \quad (14)$$

where $\overline{a}_{i,k}=\sum\limits_{j=k}^{i}w_j^r$. Since for $i\geqslant s\geqslant k\geqslant 1$

$$\overline{a}_{i,k}=\sum_{j=k}^{i}w_j^r\leqslant\sum_{j=k}^{s}w_j^r+\sum_{j=s}^{i}w_j^r=\overline{a}_{i,s}+\overline{a}_{s,k}$$

and

$$\overline{a}_{i,k}\geqslant\sum_{j=k}^{s}w_j^r \text{ and } \overline{a}_{i,k}\geqslant\sum_{j=s}^{i}w_j^r \text{ so that } \overline{a}_{i,k}\geqslant\frac{1}{2}\left(\overline{a}_{i,s}+\overline{a}_{s,k}\right),$$

we conclude that

$$\overline{a}_{i,k}\approx\overline{a}_{i,s}+\overline{a}_{s,k},\quad i\geqslant s\geqslant k\geqslant 1. \tag{15}$$

Therefore, the entries of the matrix $(\overline{a}_{i,k})$, satisfying condition (3), belong to $\mathscr{O}_1^-$ (see (9)). Moreover, by Remark 1, assuming that $b_{k,n}^{1,2}=a_{k,n}^r$ and $b_{k,n}^{0,2}=\widetilde{a}_{k,n}$, from (14) we have that the entries of the matrix $(\widetilde{a}_{i,n})$ satisfy the condition (10). Thus, they belong to the class $\mathscr{O}_2^-$.

Since by the assumption $\frac{1}{\theta'}=1-\frac{1}{\theta}=1-\frac{r}{q}=\frac{q-r}{q}$ and by the condition $0<r<p\leqslant q<\infty$ and $p>1$, we have that $\theta'=\frac{q}{q-r}\leqslant\frac{p}{p-r}$. Hence, taking into account that $\tilde{u}_n=u_n^r$, by Theorem A we get

$$J_1=\sup_{\varphi\geqslant 0}\frac{\left(\sum\limits_{n=1}^{\infty}\left(\sum\limits_{i=1}^{n}v_i^{-p'}\right)^{\frac{p(r-1)}{p-r}}v_n^{-p'}\left(\sum\limits_{i=n}^{\infty}\widetilde{a}_{i,n}\varphi_i\right)^{\frac{p}{p-r}}\right)^{\frac{p-r}{p}}}{\|u^{-r}\varphi\|_{\frac{q}{q-r}}}\approx E_1^r\approx E_2^r. \tag{16}$$

Let us estimate J_2. Setting $B_n=w_n^r\sum\limits_{k=n}^{\infty}\varphi_k$, $A_n=\left(\sum\limits_{i=1}^{n}a_{n,i}^{p'}v_i^{-p'}\right)^{\frac{r(p-1)}{p-r}}$ and $D^{\frac{p-r}{p}}$ as the numerator of J_2, we obtain

$$D=\sum_{n=1}^{\infty}\left(\sum_{j=n}^{\infty}w_j^r\sum_{i=j}^{\infty}\varphi_i\right)^{\frac{r}{p-r}}\left(\sum_{i=1}^{n}a_{n,i}^{p'}v_i^{-p'}\right)^{\frac{r(p-1)}{p-r}}w_n^r\sum_{k=n}^{\infty}\varphi_k=\sum_{n=1}^{\infty}B_n\left(\sum_{j=n}^{\infty}B_j\right)^{\frac{r}{p-r}}A_n.$$

Let

$$D_m=\sum_{n=1}^{m}B_n\left(\sum_{j=n}^{m}B_j\right)^{\frac{r}{p-r}}A_n.$$

Since the entries $a_{n,i}\geqslant 0$ are non-decreasing in n, the sequence $\{A_n\}_n$ is non-decreasing. Using this fact, we define $\Delta A_n=A_n-A_{n-1}$, where $A_0=0$, and find

$$D_m=\sum_{n=1}^{m}B_n\left(\sum_{j=n}^{m}B_j\right)^{\frac{r}{p-r}}A_n=\sum_{n=1}^{m}\left(\sum_{k=n}^{m}B_k\left(\sum_{j=k}^{m}B_j\right)^{\frac{r}{p-r}}\right)\Delta A_n.$$

Applying inequality (12) to the last expression, we get

$$D_m \approx \sum_{n=1}^{m} \left(\sum_{k=n}^{m} B_k \right)^{\frac{p}{p-r}} \Delta A_n .$$

The latter gives that

$$D = \lim_{m\to\infty} D_m \approx \sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} B_k \right)^{\frac{p}{p-r}} \Delta A_n .$$

Changing the order of sums, we find that

$$\sum_{k=n}^{\infty} B_k = \sum_{k=n}^{\infty} w_k^r \sum_{j=k}^{\infty} \varphi_j = \sum_{j=n}^{\infty} \varphi_j \sum_{k=n}^{j} w_k^r = \sum_{j=n}^{\infty} \overline{a}_{j,n} \varphi_j ,$$

where the $\overline{a}_{j,n} = \sum_{k=n}^{j} w_k^r$ satisfy the discrete Oinarov condition (see (15)). Therefore, since $\theta' = \frac{q}{q-r} \leqslant \frac{p}{p-r}$, by Theorem A_2 we get

$$J_2 = \sup_{\varphi \geqslant 0} \frac{\left(\sum_{n=1}^{\infty} \Delta A_n \left(\sum_{j=n}^{\infty} \overline{a}_{j,n} \varphi_j \right)^{\frac{p}{p-r}} \right)^{\frac{p-r}{p}}}{\|u^{-r}\varphi\|_{\frac{q}{q-r}}} \approx \max\{\overline{M}_1, \overline{M}_2\}, \tag{17}$$

where

$$\overline{M}_1 = \sup_{k\geqslant 1} \left(\sum_{n=1}^{k} \Delta A_n \right)^{\frac{p-r}{p}} \left(\sum_{j=k}^{\infty} \left(\sum_{i=k}^{j} w_i^r \right)^{\frac{q}{r}} u_j^q \right)^{\frac{r}{q}},$$

$$\overline{M}_2 = \sup_{k\geqslant 1} \left(\sum_{n=1}^{k} \left(\sum_{i=n}^{k} w_i^r \right)^{\frac{p}{p-r}} \Delta A_n \right)^{\frac{p-r}{p}} \left(\sum_{j=k}^{\infty} u_j^q \right)^{\frac{r}{q}} .$$

Since $\sum_{n=1}^{k} \Delta A_n = A_k = \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{r(p-1)}{p-r}}$, we obtain

$$\overline{M}_1 = \sup_{k\geqslant 1} \left(\left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{r(p-1)}{p-r}} \right)^{\frac{p-r}{p}} \left(\sum_{j=k}^{\infty} \left(\sum_{i=k}^{j} w_i^r \right)^{\frac{q}{r}} u_j^q \right)^{\frac{r}{q}}$$

$$= \sup_{k\geqslant 1} \left(\sum_{i=1}^{k} a_{k,i}^{p'} v_i^{-p'} \right)^{\frac{r}{p'}} \left(\sum_{j=k}^{\infty} \left(\sum_{i=k}^{j} w_i^r \right)^{\frac{q}{r}} u_j^q \right)^{\frac{r}{q}} = M_1^r . \tag{18}$$

Using inequality (12), we get

$$\sum_{n=1}^{k}\left(\sum_{i=n}^{k} w_i^r\right)^{\frac{p}{p-r}} \Delta A_n \approx \sum_{n=1}^{k}\left(\sum_{i=n}^{k} w_i^r \left(\sum_{j=i}^{k} w_j^r\right)^{\frac{r}{p-r}}\right) \Delta A_n$$

$$= \sum_{n=1}^{k} w_n^r \left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}} A_n = \sum_{n=1}^{k} w_n^r \left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}} \left(\sum_{i=1}^{n} a_{n,i}^{p'} v_i^{-p'}\right)^{\frac{r(p-1)}{p-r}}.$$

Replacing the last expression in $\overline{M}_2$, we find that $\overline{M}_2 \approx M_2^r$. This equivalence and (18), together with (17), give that

$$J_2 \approx \max\{M_1^r, M_2^r\}. \tag{19}$$

Combining (16) and (19) with (13), we complete the proof. □

4. Corollaries

Suppose that

$$F_1 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_n^{-p'} \left(\sum_{i=1}^{n} v_i^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}} \left(\sum_{j=k}^{\infty}\left(\sum_{i=k}^{j} a_{i,k}^r w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{1}{q}},$$

$$F_2 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} a_{k,n}^{\frac{pr}{p-r}} v_n^{-p'} \left(\sum_{i=1}^{n} v_i^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}} \left(\sum_{j=k}^{\infty}\left(\sum_{i=k}^{j} w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{1}{q}},$$

$$F_3 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k}\left(\sum_{i=n}^{k} a_{i,n}^r w_i^r\right)^{\frac{p}{p-r}} v_n^{-p'} \left(\sum_{i=1}^{n} v_i^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}} \left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}.$$

Using (14), each of the expressions E_1 and E_2 can be equivalently splitted into the sum of three terms $E_1 \approx E_2 \approx F_1 + F_2 + F_3$. Therefore, Theorem 1 can be rewritten as follows.

COROLLARY 1. *Let $0 < r < p \leqslant q < \infty$ and $p > 1$. Let $(a_{k,i})$ be a matrix that is non-decreasing in k, non-increasing in i, and that satisfies condition (3). Then inequality (2) holds if and only if $FM = \max\{F_1, F_2, F_3,\ M_1, M_2\} < \infty$ holds. Moreover, $C \approx FM$, where C is the best constant in (2).*

Applying (11) to F_1, it can be modified

$$F_1 \approx \sup_{k\geqslant 1}\left(\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{r(p-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty}\left(\sum_{i=k}^{j} a_{i,k}^r w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{1}{q}}$$

$$= \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=k}^{\infty}\left(\sum_{i=k}^{j} a_{i,k}^r w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{1}{q}}.$$

In the case when $a_{n,i}=1$, $n\geqslant i\geqslant 1$, we denote F_1, F_2, F_3, M_1 and M_2 by $\widetilde{F}_1$, $\widetilde{F}_2$, $\widetilde{F}_3$, $\widetilde{M}_1$ and $\widetilde{M}_2$, respectively. It is easy to see that

$$\widetilde{M}_1 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} v_n^{-p'}\right)^{\frac{1}{p'}}\left(\sum_{j=k}^{\infty}\left(\sum_{i=k}^{j} w_i^r\right)^{\frac{q}{r}} u_j^q\right)^{\frac{1}{q}} \approx \widetilde{F}_1 = \widetilde{F}_2.$$

Applying (11), changing the order of sums and then applying (12), we obtain

$$\widetilde{M}_2 = \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} w_n^r\left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}}\left(\sum_{i=1}^{n} v_i^{-p'}\right)^{\frac{r(p-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}$$

$$\approx \sup_{k\geqslant 1}\left(\sum_{n=1}^{k} w_n^r\left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}}\sum_{i=1}^{n} v_i^{-p'}\left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}$$

$$= \sup_{k\geqslant 1}\left(\sum_{n=1}^{k}\sum_{i=1}^{n} w_n^r\left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}} v_i^{-p'}\left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}$$

$$= \sup_{k\geqslant 1}\left(\sum_{i=1}^{k}\sum_{n=i}^{k} w_n^r\left(\sum_{j=n}^{k} w_j^r\right)^{\frac{r}{p-r}} v_i^{-p'}\left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}}$$

$$\approx \sup_{k\geqslant 1}\left(\sum_{i=1}^{k}\left(\sum_{n=i}^{k} w_n^r\right)^{\frac{p}{p-r}} v_i^{-p'}\left(\sum_{j=1}^{i} v_j^{-p'}\right)^{\frac{p(r-1)}{p-r}}\right)^{\frac{p-r}{pr}}\left(\sum_{j=k}^{\infty} u_j^q\right)^{\frac{1}{q}} = \widetilde{F}_3.$$

Thus, from Corollary 1 we get one more statement.

COROLLARY 2. *Let* $0 < r < p \leqslant q < \infty$ *and* $p > 1$. *Then the inequality*

$$\left(\sum_{n=1}^{\infty} u_n^q \left(\sum_{k=1}^{n} \left| w_k \sum_{i=1}^{k} f_i \right|^r \right)^{\frac{q}{r}} \right)^{\frac{1}{q}} \leqslant C \left(\sum_{j=1}^{\infty} |v_j f_j|^p \right)^{\frac{1}{p}}, \quad \forall f \in l_{p,v}, \tag{20}$$

holds if and only if $\widetilde{M} = \max\{\widetilde{M}_1, \widetilde{M}_2\} < \infty$ *holds. Moreover,* $C \approx \widetilde{M}$, *where* C *is the best constant in* (20).

Corollary 2 completely coincides with the statement of [20, Theorem 2].

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