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OSCILLATORY AND SPECTRAL PROPERTIES OF A CLASS OF FOURTH–ORDER DIFFERENTIAL OPERATORS VIA A NEW HARDY–TYPE INEQUALITY

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Abstract. In this paper, we study oscillatory properties of a fourth-order differential equation and spectral properties of a corresponding differential operator. These properties are established by first proving a new second-order Hardy-type inequality, where the weights are the coefficients of the equation and the operator. This new inequality, in its turn, is established for functions satisfying certain boundary conditions that depend on the boundary behavior of one of its weights at infinity and at zero.

1. Introduction

Let $I=(0,\infty)$, $1<p,q<\infty$, and $\frac{1}{p}+\frac{1}{p'}=1$. Let u, v, and $v^{1-p'}$ be locally summable and positive weight functions on I. Moreover, suppose that v is twice continuously differentiable on the interval I.

Denote by $W^2_{p,v}\equiv W^2_{p,v}(I)$ the space of functions $f:I\to\mathbb{R}$ having generalized derivatives on I, for which $\|f''\|_{p,v}<\infty$, where $\|g\|_{p,v}=\left(\int\limits_0^\infty v(t)|g(t)|^p dt\right)^{\frac{1}{p}}$ is the norm of the Lebesgue space $L_{p,v}\equiv L_{p,v}(I)$. By the conditions on the function v it follows that for any $f\in W^2_{p,v}$ there exist the finite limits $\lim\limits_{t\to 1}f(t)=f(1)$ and $\lim\limits_{t\to 1}f'(t)=f'(1)$. Therefore, the space $W^2_{p,v}$ has the norm

$$\|f\|_{W^2_{p,v}}=\|f''\|_{p,v}+|f'(1)|+|f(1)|. \tag{1}$$

Let $C_0^\infty(I)$ be the set of finitely supported functions, which are infinitely differentiable on the interval I. By the conditions on the function v we have that $C_0^\infty(I)\subset W^2_{p,v}(I)$. Denote by $\mathring{W}^2_{p,v}\equiv\mathring{W}^2_{p,v}(I)$ the closure of the set $C_0^\infty(I)$ with respect to the norm defined by (1).

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We investigate the oscillatory properties of the following fourth-order differential equation
$$(v(t)y''(t))'' - \lambda u(t)y(t) = 0,\ t \in I, \tag{2}$$
where $\lambda > 0$ and the spectral properties of the corresponding differential operator L generated by the expression

$$Ly = \frac{1}{u(t)}(v(t)y''(t))'', \tag{3}$$

in the space $L_{2,u}(I)$ with inner product $(f,g)_{2,u} = \int_0^\infty f(t)g(t)u(t)dt$. Note that (2) can be interpreted as that λ is the eigenvalue of the differential operator L defined by (3).

The oscillatory properties of the second-order equation

$$(v(t)y'(t))' - u(t)y(t) = 0,\ t\ \in I, \tag{4}$$

have been well studied by known methods in the qualitative theory of differential equations (see [3] and the references therein). One of these methods transforms equation (4) into a Hamiltonian system, while the second method treats the equation (4) as a perturbation of an Euler-type equation. These methods have been much less developed for the fourth-order equation (2) (see, e.g., [5], [6], [22], and [23]). In this paper, we study the oscillatory properties of the equation (2) and the spectral properties of the operator (3) by the variational method, using their connections with the following second-order Hardy-type inequality:

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty v(t)|f''(t)|^p dt\right)^{\frac{1}{p}},\ f \in \mathring{W}^2_{p,v},\ \ 1 < p \leqslant q < \infty. \tag{5}$$

Moreover, the required oscillation and spectral conditions are obtained explicitly in terms of the coefficients u and v of the equation (2) and the operator (3).

Concerning the current knowledge of higher-order Hardy-type inequalities we refer to [13, Chapter 4].

The inequality (5) can be investigated depending on a "type of singularity" of the weight function v at the endpoints of I. The concept "type of singularity" follows from the combined results of the works [15] and [18] (see Theorems R and L below). For $f \in W^2_{p,v}$ we assume that $\lim\limits_{t\to 0^+} f(t) = f(0)$, $\lim\limits_{t\to 0^+} f'(t) = f'(0)$, $\lim\limits_{t\to\infty} f(t) = f(\infty)$, and $\lim\limits_{t\to\infty} f'(t) = f'(\infty)$ if these limits are finite.

THEOREM R. *Let* $1 < p < \infty$.
(i) If $v^{1-p'} \notin L_1(1,\infty)$, *then*

$$\mathring{W}^2_{p,v}(1,\infty) = W^2_{p,v}(1,\infty)$$

(in this case, for all $f \in W^2_{p,v}$ *there do not exist both* $f(\infty)$ *and* $f'(\infty)$*).*

(ii) If $v^{1-p'} \in L_1(1,\infty)$ *and* $t^{p'}v^{1-p'}(t) \notin L_1(1,\infty)$*, then*

$$\mathring{W}^2_{p,v}(1,\infty) = \{f \in W^2_{p,v}(1,\infty) : f'(\infty) = 0\}$$

(in this case, for all $f \in W^2_{p,v}$ *there exists only* $f'(\infty)$*).*

(iii) If $t^{p'}v^{1-p'}(t) \in L_1(1,\infty)$*, then*

$$\mathring{W}^2_{p,v}(1,\infty) = \{f \in W^2_{p,v}(1,\infty) : f(\infty) = f'(\infty) = 0\}$$

(in this case, for all $f \in W^2_{p,v}$ *there exist both* $f(\infty)$ *and* $f'(\infty)$*).*

THEOREM L. *Let* $1 < p < \infty$.
(i) If $t^{p'}v^{1-p'}(t) \notin L_1(0,1)$*, then*

$$\mathring{W}^2_{p,v}(0,1) = W^2_{p,v}(0,1)$$

(in this case, for all $f \in W^2_{p,v}$ *there do not exist both* $f(0)$ *and* $f'(0)$*).*

(ii) If $v^{1-p'} \notin L_1(0,1)$ *and* $t^{p'}v^{1-p'}(t) \in L_1(0,1)$*, then*

$$\mathring{W}^2_{p,v}(0,1) = \{f \in W^2_{p,v}(0,1) : f(0) = 0\}$$

(in this case, for all $f \in W^2_{p,v}$ *there exists only* $f(0)$*).*

(iii) If $v^{1-p'} \in L_1(0,1)$*, then*

$$\mathring{W}^2_{p,v}(0,1) = \{f \in W^2_{p,v}(0,1) : f(0) = f'(0) = 0\}$$

(in this case, for all $f \in W^2_{p,v}$ *there exist both* $f(0)$ *and* $f'(0)$*).*

We say that the weight function v is strong-singular at infinity if there do not exist both $f(\infty)$ and $f'(\infty)$ (see item (i), Theorem R), weak-singular at infinity if there exists only $f'(\infty)$ (see item (ii), Theorem R), and non-singular at infinity if there exist both $f(\infty)$ and $f'(\infty)$ (see item (iii), Theorem R). Similarly, Theorem L defines the concepts of strong singularity, weak singularity, and non-singularity of the function v at zero.

It is obvious that the second-order Hardy-type inequality (5) does not hold if none or only one of the values $f(\infty)$, $f'(\infty)$, $f(0)$, and $f'(0)$ exists in both endpoints of the interval I. If there exist exactly two values, we have "standard" cases, because then the second-order inequality (5) has two boundary conditions. The oscillatory properties when $\mathring{W}^2_{p,v}(I) = \{f \in W^2_{p,v}(I) : f(\infty) = f'(\infty) = 0\}$ and $\mathring{W}^2_{p,v}(I) = \{f \in W^2_{p,v}(I) : f(0) = f'(0) = 0\}$ can be derived from the well-known results on the standard differential Hardy-type inequalities (see, e.g., [12] or [13]) as they are obtained in the paper [9] when $\mathring{W}^2_{p,v}(I) = \{f \in W^2_{p,v}(I) : f(0) = f'(\infty) = 0\}$. The paper [11] considers the case when $\mathring{W}^2_{p,v}(I) = \{f \in W^2_{p,v}(I) : f(0) = f'(0) = f'(\infty) = 0\}$, which can be called "overdetermined" because the second-order inequality (5) has three boundary conditions. Extra boundary condition causes additional difficulties in characterization of the inequality (5). In the present paper, we investigate one more "overdetermined" case, namely when $\mathring{W}^2_{p,v}(I) = \{f \in W^2_{p,v}(I) : f(0) = f(\infty) = f'(\infty) = 0\}$.

Note that the spectral properties of the operator L were also studied earlier in some papers, but under conditions different from those discussed in this paper (see, e.g., [1], [2], [7] (Chapters 29 and 34), [14], [21], and the references therein). Our conditions follow from a new Hardy-type inequality of independent interest.

The paper is organized as follows: Section 2 contains all the auxiliary statements, which are necessary to prove the main results. In Section 3 we investigate the inequality (5) and state and prove our main result concerning Hardy-type inequalities (see Theorem 1). In Section 4, we study the oscillatory properties of the differential equation (2). Our main results are stated in Theorems 2 and 3. Finally, in Section 5, we investigate the spectral properties of the operator (3). Our new results are given in Theorems 4–7.

2. Auxiliary statements

The symbol $A \ll B$ means $A \leqslant CB$ with some constant C. If $A \ll B \ll A$, then we write $A \approx B$. Moreover, denote by χ_M the characteristic function of the set M.

Assume that $\overline{v}(t) = \frac{v(t)}{t^p}$, $t \in I$. Since $t^{p'}v^{1-p'} = t^{p'}\overline{v}^{1-p'}t^{p(1-p')} = \overline{v}^{1-p'}t^{p'+p-pp'} = \overline{v}^{1-p'}$, from $t^{p'}v^{1-p'} \in L_1(I)$ we have that $\overline{v}^{1-p'} \in L_1(I)$. In this case, for any $\tau \in I$ there exists k_τ such that

$$\int_0^\tau \overline{v}^{1-p'}(t)dt = k_\tau \int_\tau^\infty \overline{v}^{1-p'}(t)dt. \tag{6}$$

In addition, k_τ increases in τ, $\lim\limits_{\tau\to 0^+} k_\tau = 0$, and $\lim\limits_{\tau\to\infty} k_\tau = \infty$.

Let $0 \leqslant a < b \leqslant \infty$. We need the following well-known statement (see, e.g., [12] or [13]).

THEOREM A. *Let* $1 < p \leqslant q < \infty$ *and* $0 \leqslant a < b \leqslant \infty$.

(i) *The inequality*

$$\left(\int_a^b u(x)\left|\int_a^x f(t)\,dt\right|^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b v(t)|f(t)|^p dt\right)^{\frac{1}{p}}, \tag{7}$$

holds if and only if

$$A^+ = \sup_{a<z<b}\left(\int_z^b u(x)dx\right)^{\frac{1}{q}}\left(\int_a^z v^{1-p'}(t)\,dt\right)^{\frac{1}{p'}} < \infty.$$

Moreover, $A^+ \leqslant C \leqslant p^{\frac{1}{q}}(p')^{\frac{1}{p'}}A^+$, *where* C *is the best constant in* (7).

(ii) *The inequality*

$$\left(\int_a^b u(x)\left|\int_x^b f(t)\,dt\right|^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b v(t)|f(t)|^p dt\right)^{\frac{1}{p}}, \tag{8}$$

holds if and only if

$$A^- = \sup_{a<z<b} \left(\int_a^z u(x)dx \right)^{\frac{1}{q}} \left(\int_z^b v^{1-p'}(t)\,dt \right)^{\frac{1}{p'}} < \infty.$$

Moreover, $A^- \leqslant C \leqslant p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A^-$, *where* C *is the best constant in* (8).

We also need the combined results from the papers [10] and [20]. Let $0 < \tau < \infty$ and

$$B_1^-(\tau) = \sup_{z>\tau} \left(\int_\tau^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty (s-z)^{p'} v^{1-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

$$B_2^-(\tau) = \sup_{z>\tau} \left(\int_\tau^z (z-t)^q u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty v^{1-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

$$B^-(\tau) = \max\{B_1^-(\tau), B_2^-(\tau)\}.$$

THEOREM B. *Let* $1 < p \leqslant q < \infty$ *and* $0 < \tau < \infty$. *Then the inequality*

$$\left(\int_\tau^\infty u(t) \left| \int_t^\infty (x-t) f(x)\,dx \right|^q dt \right)^{\frac{1}{q}} \leqslant C \left(\int_\tau^\infty v(t) |f(t)|^p dt \right)^{\frac{1}{p}}, \tag{9}$$

holds if and only if $B^-(\tau) < \infty$; *in addition,* $B^-(\tau) \leqslant C \leqslant 8 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} B^-(\tau)$, *where* C *is the best constant in* (9).

3. A characterization of the Hardy-type inequality (5)

We study inequality (5) in the case when v is weak-singular at zero and non-singular at infinity, i.e., when

$$\mathring{W}^2_{p,v}(I) = \{ f \in W^2_{p,v}(I) : f(0) = f(\infty) = f'(\infty) = 0 \}. \tag{10}$$

Let $0 < \tau < \infty$ and

$$B_3^-(\tau) = \frac{1}{\tau} \left(\int_0^\tau t^q u(t)dt \right)^{\frac{1}{q}} \left(\int_\tau^\infty (s-\tau)^{p'} v^{1-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

$$F_1^+(\tau) = \sup_{0<z<\tau} \frac{1}{\tau} \left(\int_0^z t^q u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\tau (\tau-s)^{p'} v^{1-p'}(s)\,ds \right)^{\frac{1}{p'}}, \tag{11}$$

$$F_2^+(\tau) = \sup_{0<z<\tau} \frac{1}{\tau} \left(\int_z^\tau (\tau-t)^q u(t)dt \right)^{\frac{1}{q}} \left(\int_0^z s^{p'} v^{1-p'}(s)\,ds \right)^{\frac{1}{p'}}, \tag{12}$$

$$\mathscr{B}^-(\tau) = \max\{B^-(\tau), B_3^-(\tau)\},\ F^+(\tau) = \max\{F_1^+(\tau), F_2^+(\tau)\},$$

$$\mathscr{B}^- F^+(\tau) = \inf_{\tau\in I} \max\{\mathscr{B}^-(\tau), F^+(\tau)\}.$$

Our main result in this Section reads:

THEOREM 1. *Let* $1 < p \leqslant q < \infty$, $v^{1-p'} \notin L_1(0,1)$, $t^{p'} v^{1-p'}(t) \in L_1(0,1)$, *and* $t^{p'} v^{1-p'}(t) \in L_1(1,\infty)$. *Then the inequality* (5) *holds if and only if* $\mathscr{B}^- F^+(\tau) < \infty$ *for any finite* $\tau > 0$. *Moreover, for the best constant* C *in* (5) *the following estimates*

$$4^{-\frac{1}{p}} \inf_{\tau\in I} \mathscr{B}^- F^+(\tau) \leqslant C \leqslant 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} \inf_{\tau\in I} \mathscr{B}^- F^+(\tau), \tag{13}$$

$$\sup_{\tau>0} (1 + k_\tau^{p-1})^{-\frac{1}{p}} F^+(\tau) \leqslant C \leqslant 11 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} F^+(\tau^+) \tag{14}$$

hold, where

$$\tau^+ = \inf\{\tau > 0 : \mathscr{B}^-(\tau) \leqslant F^+(\tau)\}. \tag{15}$$

Proof. Sufficiency. Let $\mathscr{B}^- F^+(\tau) < \infty$ for any finite $\tau > 0$. By the condition on the weight function v, due to Theorems R (item (iii)) and L (item (ii)), we get that (10) holds. Therefore, for $f \in \mathring{W}^2_{p,v}$ we have that $f(t) = \int_0^t f'(x)dx$ for $0 < t < \tau$, $f(t) = -\int_t^\infty f'(x)dx$ for $t > \tau$ and $f'(t) = -\int_t^\infty f''(s)ds$ for all $t > 0$. Then

$$\begin{aligned}
f(t) &= -\int_0^t \int_x^\infty f''(s)ds\,dx = -\int_0^t \int_x^t f''(s)ds\,dx - \int_0^t \int_t^\infty f''(s)ds\,dx \\
&= -\int_0^t f''(s)ds \int_0^s dx - \int_t^\infty f''(s)ds \int_0^t dx = -\int_0^t sf''(s)ds - \int_t^\infty tf''(s)ds \\
&= -\int_0^t sf''(s)ds - \int_t^\tau tf''(s)ds - \int_\tau^\infty tf''(s)ds \qquad (16)\\
&= -\int_0^t sf''(s)ds - \int_t^\tau sf''(s)\frac{t}{s}ds - \int_\tau^\infty sf''(s)\frac{t}{s}ds. \qquad (17)
\end{aligned}$$

Let $g(s) = sf''(s)$. Hence, from (17) it follows that

$$f(t) = -\int_\tau^\infty g(s)\frac{t}{s}ds - \int_t^\tau g(s)\frac{t}{s}ds - \int_0^t g(s)ds. \tag{18}$$

Since $\int_0^\infty f'(x)dx = 0$, we get that

$$f(t) = c_1 \int_0^\infty \int_x^\infty f''(s)ds\,dx = c_2 \int_0^\infty sf''(s)ds = \int_0^\infty g(s)ds = 0.$$

Therefore, for $f \in \mathring{W}^2_{p,v}(I)$, in view of (18), we obtain that

$$\begin{aligned} f(t) &= -\int_\tau^\infty g(s)\frac{t}{s}ds - \int_t^\tau g(s)\frac{t}{s}ds - \int_0^t g(s)ds + \frac{t}{\tau}\int_0^\infty g(s)ds \\ &= \int_\tau^\infty g(s)\frac{t(s-\tau)}{s\tau}ds - \int_t^\tau g(s)\frac{t(\tau-s)}{s\tau}ds - \left(1-\frac{t}{\tau}\right)\int_0^t g(s)ds \end{aligned} \tag{19}$$

for $0 < t < \tau$. Moreover,

$$\begin{aligned} f(t) &= \int_t^\infty \int_x^\infty f''(s)dsdx = \int_t^\infty f''(s)\int_t^s dx\,ds \\ &= \int_t^\infty (s-t)f''(s)ds = \int_t^\infty (s-t)\frac{g(s)}{s}ds \end{aligned} \tag{20}$$

for $t > \tau$. Thus, for $f \in \mathring{W}^2_{p,v}(I)$ from (19) and (20) it follows that

$$\begin{aligned} f(t) = \chi_{(0,\tau)}(t)&\left[\int_\tau^\infty g(s)\frac{t(s-\tau)}{s\tau}ds - \int_t^\tau g(s)\frac{t(\tau-s)}{s\tau}ds - \left(1-\frac{t}{\tau}\right)\int_0^t g(s)ds\right] \\ &+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)\frac{g(s)}{s}ds. \end{aligned} \tag{21}$$

According to (21), the inequality (5) can be written in the form

$$\begin{aligned} &\left(\int_0^\tau u(t)\left|\frac{t}{\tau}\int_\tau^\infty \frac{(s-\tau)}{s}g(s)ds - \frac{(\tau-t)}{\tau}\int_0^t g(s)ds - \frac{t}{\tau}\int_t^\tau \frac{(\tau-s)}{s}g(s)ds\right|^q dt\right. \\ &\qquad \left. + \int_\tau^\infty u(t)\left|\int_t^\infty (s-t)\frac{g(s)}{s}ds\right|^q dt\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty \overline{v}(s)|g(s)|^p ds\right)^{\frac{1}{p}}. \end{aligned} \tag{22}$$

By now, using the Minkowski's inequality for sums and Hölder's inequality, from Theorems A and B the latter gives

$$\left(\int_0^\infty u(t)|f(t)|^q dt\right)^{\frac{1}{q}} \leqslant p^{\frac{1}{q}}(p')^{\frac{1}{p'}}(F_1^+(\tau)+F_2^+(\tau))\left(\int_0^\tau v(t)|f''(t)|^p dt\right)^{\frac{1}{p}}$$
$$+(B_3^-(\tau)+8p^{\frac{1}{q}}(p')^{\frac{1}{p'}}B^-(\tau))\left(\int_\tau^\infty v(t)|f''(t)|^p dt\right)^{\frac{1}{p}}$$
$$\leqslant 11p^{\frac{1}{q}}(p')^{\frac{1}{p'}}\mathscr{B}^-F^+(\tau)\left(\int_0^\infty v(t)|f''(t)|^p dt\right)^{\frac{1}{p}}. \tag{23}$$

Since $\mathscr{B}^-F^+(\tau)<\infty$ for any finite $\tau>0$ and the left-hand side of (23) is independent of $\tau>0$, we get the right-hand side estimate of (13).

For $0<N<\tau$ we have that

$$B_3^-(\tau) < \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$
$$+\left(\int_N^\tau \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$
$$\leqslant \left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$
$$+\left(\int_N^\tau u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}.$$

Since

$$\lim_{\tau\to\infty}\left(\int_0^N \left(\frac{t}{\tau}\right)^q u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}=0,$$

then

$$B_3^-(\tau) \ll \left(\int_N^\tau u(t)dt\right)^{\frac{1}{q}}\left(\int_\tau^\infty s^{p'}v^{1-p'}(s)ds\right)^{\frac{1}{p'}}$$

for a sufficiently large $\tau > N$. Let

$$F_2^+ = \lim_{\tau\to\infty} F_2^+(\tau) = \lim_{\tau\to\infty} \sup_{0<z<\tau} \left(\int_z^\tau \left(1-\frac{t}{\tau}\right)^q u(t)dt \right)^{\frac{1}{q}} \left(\int_0^z s^{p'} v^{1-p'}(s)ds \right)^{\frac{1}{p'}}$$
$$= \sup_{z>0} \left(\int_z^\infty u(t)dt \right)^{\frac{1}{q}} \left(\int_0^z s^{p'} v^{1-p'}(s)ds \right)^{\frac{1}{p'}}.$$

If $F_2^+ = \infty$, then it is obvious that $B_3^-(\tau) < F_2^+(\tau)$ for a sufficiently large τ. If $F_2^+ < \infty$, then $\int_z^\infty u(t)dt < \infty$, which implies that $\lim_{\tau\to\infty} B_3^-(\tau) = 0$. Hence, we again have that $B_3^-(\tau) < F_2^+(\tau)$ for a sufficiently large τ. Since the function $B^-(\tau)$ is decreasing, then we have that $B^-(\tau) < F^+(\tau)$ for a sufficiently large τ. Moreover, the estimate

$$B^-(\tau) < \sup_{z>\tau} \left(\int_\tau^z u(t)dt \right)^{\frac{1}{q}} \left(\int_z^\infty s^{p'} v^{1-p'}(s)\,ds \right)^{\frac{1}{p'}},$$

gives that $B_i^-(\tau) < F^+(\tau)$, $i = 1,2$. Thus, $\mathscr{B}^-(\tau) < F^+(\tau)$ in some neighborhood of infinity. Therefore, in relation (15) there exists $\tau^+ > 0$ such that $\mathscr{B}^-(\tau^+) \leqslant F^+(\tau^+)$. Consequently,

$$\mathscr{B}^- F^+(\tau) = \inf_{\tau\in I} \max\{\mathscr{B}^-(\tau), F^+(\tau)\} \leqslant F^+(\tau^+)$$

and the right-hand side estimate of (14) holds, so the proof of the sufficiency is complete.

Necessity. Let inequality (5) hold with the best constant $C > 0$. We use the methods presented in the paper [17]. Since $g(s) = sf''(s)$ and $\overline{v}(t) = \frac{v(t)}{t^p}$, we have that

$$\int_0^\infty v(s)|f''(s)|^p ds = \int_0^\infty \frac{v(s)}{s^p} |sf''(s)|^p ds = \int_0^\infty \overline{v}(s)|g(s)|^p ds.$$

Therefore, the condition $f \in \mathring{W}^2_{p,v}(I)$ is equivalent to the condition $g \in \widetilde{L}_{p,\overline{v}}(I)$, where $\widetilde{L}_{p,\overline{v}}(I) = \left\{ g \in L_{p,\overline{v}}(I) : \int_0^\infty g(s)ds = 0 \right\}$.

By the conditions $t^{p'} v^{1-p'}(t) \in L_1(0,1)$ and $t^{p'} v^{1-p'}(t) \in L_1(1,\infty)$, we find that $\overline{v}^{1-p'} \in L_1(I)$ and (6) holds. Let the function $\rho(s)$ be such that

$$\int_0^s \overline{v}^{1-p'}(t)dt = k_\tau \int_{\rho(s)}^\infty \overline{v}^{1-p'}(t)dt,\ s \in (0,\tau). \tag{24}$$

On the basis of the positiveness of the function $\overline{v}$, the positive function ρ is strictly decreasing and locally absolutely continuous on I. Moreover, from (24) it follows that $\rho(\tau)=\tau$, $\lim\limits_{s\to 0^+}\rho(s)=\infty$, and

$$\int\limits_0^{\rho^{-1}(s)} \overline{v}^{1-p'}(t)dt = k_\tau \int\limits_s^\infty \overline{v}^{1-p'}(t)dt,\ s\in(\tau,\infty), \tag{25}$$

where ρ^{-1} is the inverse function to the function ρ. Differentiating the both sides of (24) and (25), we get that

$$\frac{1}{k_\tau} = \frac{\overline{v}^{1-p'}(\rho(s))}{\overline{v}^{1-p'}(s)}|\rho'(s)|,\ s\in(0,\tau);\ \ k_\tau = \frac{\overline{v}^{1-p'}(\rho^{-1}(s))}{\overline{v}^{1-p'}(s)}|(\rho^{-1}(s))'|,\ s\in(\tau,\infty). \tag{26}$$

We consider the following two sets:

$$\mathscr{L}_1=\{g\in L_{p,\overline{v}}(0,\tau): g\leqslant 0\} \ \text{ and } \ \mathscr{L}_2=\{g\in L_{p,\overline{v}}(\tau,\infty): g\geqslant 0\}.$$

Next we will prove that for any $g_1\in\mathscr{L}_1$ there exists $g_2\in\mathscr{L}_2$ and inversely, for any $g_2\in\mathscr{L}_2$ there exists $g_1\in\mathscr{L}_1$ such that

$$\int\limits_\tau^\infty \overline{v}(t)|g_2(t)|^p dt = k_\tau^{p-1}\int\limits_0^\tau \overline{v}(t)|g_1(t)|^p dt. \tag{27}$$

For $g_1\in\mathscr{L}_1$ we assume that

$$g_2(t) = -k_\tau g_1(\rho^{-1}(t))\frac{\overline{v}^{1-p'}(t)}{\overline{v}^{1-p'}(\rho^{-1}(t))}. \tag{28}$$

From the first equality in (26) we obtain that

$$\begin{aligned}
\int\limits_\tau^\infty \overline{v}(t)|g_2(t)|^p dt &= k_\tau^p \int\limits_\tau^\infty \overline{v}(t)\left| g_1(\rho^{-1}(t))\frac{\overline{v}^{1-p'}(t)}{\overline{v}^{1-p'}(\rho^{-1}(t))}\right|^p dt \\
&= k_\tau^p \int\limits_0^\tau \overline{v}(\rho(s))\,|g_1(s)|^p\,\frac{\overline{v}^{(1-p')p}(\rho(s))}{\overline{v}^{(1-p')p}(s)}|\rho'(s)|ds \\
&= k_\tau^p \int\limits_0^\tau \overline{v}(s)|g_1(s)|^p\,\frac{\overline{v}^{1-p'}(\rho(s))}{\overline{v}^{1-p'}(s)}|\rho'(s)|ds \\
&= k_\tau^{p-1}\int\limits_0^\tau \overline{v}(s)|g_1(s)|^p ds < \infty,
\end{aligned}$$

i.e., $g_2\in\mathscr{L}_2$ and (27) holds. Similarly, for $g_2\in\mathscr{L}_2$ by assuming that

$$g_1(t) = -\frac{1}{k_\tau}g_2(\rho(t))\frac{\overline{v}^{1-p'}(t)}{\overline{v}^{1-p'}(\rho(t))}, \tag{29}$$

and using the second equality in (26), we get that $g_1 \in \mathscr{L}_1$ and (27) holds. In both cases, assuming that $g(t)=g_1(t)$ for $0<t\leqslant\tau$ and $g(t)=g_2(t)$ for $t>\tau$, we obtain that $g\in L_{p,\overline{v}}(I)$. Moreover, according to (27), we have that

$$\int_0^\infty \overline{v}(t)|g(t)|^p dt = (1+k_\tau^{p-1})\int_0^\tau \overline{v}(t)|g(t)|^p dt = (1+k_\tau^{1-p})\int_\tau^\infty \overline{v}(t)|g(t)|^p dt. \tag{30}$$

From the condition $\overline{v}^{1-p'}\in L_1(I)$ it follows that the constructed function $g\in L_1(I)$. From (28) and (29) it follows that $\int_0^\tau g_1(t)dt = -\int_\tau^\infty g_2(t)dt$, i.e., $\int_0^\infty g(t)dt=0$. Hence, $g\in\widetilde{L}_{p,\overline{v}}(I)$.

Denote by G_τ the set of functions $g\in\widetilde{L}_{p,\overline{v}}(I)$ constructed from the functions $g_1\in\mathscr{L}_1$ and $g_2\in\mathscr{L}_2$. By substituting $g\in G_\tau$ into (22) we get that

$$\left(\int_0^\tau u(t)\left(\frac{(\tau-t)}{\tau}\int_0^t |g_1(s)|ds + \frac{t}{\tau}\int_t^\tau \frac{(\tau-s)}{s}|g_1(s)|ds + \frac{t}{\tau}\int_\tau^\infty \frac{(s-\tau)}{s}g_2(s)ds\right)^q dt\right.$$

$$\left. + \int_\tau^\infty u(t)\left(\int_t^\infty (s-t)\frac{g_2(s)}{s}ds\right)^q dt\right)^{\frac{1}{q}} \leqslant C\left(\int_0^\infty \overline{v}(t)\,|g(t)|^p\,dt\right)^{\frac{1}{p}}, \tag{31}$$

Assuming that the function $g_2\in\mathscr{L}_2$ is constructed from the function $g_1\in\mathscr{L}_1$ by (28) and $g_1\in\mathscr{L}_1$ is constructed from the function $g_2\in\mathscr{L}_2$ by (29), from (30) and (31) we have that

$$\left(\int_0^\tau u(t)\left(\frac{(\tau-t)}{\tau}\int_0^t |g_1(s)|ds\right)^q dt\right)^{\frac{1}{q}} \leqslant C(1+k_\tau^{p-1})^{\frac{1}{p}}\left(\int_0^\tau \overline{v}(t)|g_1(t)|^p dt\right)^{\frac{1}{p}},$$

$$\left(\int_0^\tau u(t)\left(\frac{t}{\tau}\int_t^\tau \frac{(\tau-s)}{s}|g_1(s)|ds\right)^q dt\right)^{\frac{1}{q}} \leqslant C(1+k_\tau^{p-1})^{\frac{1}{p}}\left(\int_0^\tau \overline{v}(t)|g_1(t)|^p dt\right)^{\frac{1}{p}},$$

$$\left(\int_0^\tau \frac{t^q}{\tau^q}u(t)dt\right)^{\frac{1}{q}}\int_\tau^\infty \frac{(s-\tau)}{s}g_2(s)ds \leqslant C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}},$$

$$\left(\int_\tau^\infty u(t)\left(\int_t^\infty (s-t)\frac{g_2(s)}{s}ds\right)^q dt\right)^{\frac{1}{q}} \leqslant C(1+k_\tau^{1-p})^{\frac{1}{p}}\left(\int_\tau^\infty \overline{v}(t)|g_2(t)|^p dt\right)^{\frac{1}{p}}.$$

Due to the arbitrariness of the function $g_1\in L_{p,\overline{v}}(0,\tau)$, by Theorem A, the first two inequalities give that

$$F^+(\tau)\leqslant C(1+k_\tau^{p-1})^{\frac{1}{p}}. \tag{32}$$

Moreover, due to the arbitrariness of the function $g_2 \in L_{p,\overline{v}}(\tau,\infty)$, the third inequality implies that

$$B_3^-(\tau) \leqslant C(1+k_\tau^{1-p})^{\frac{1}{p}},$$

so that, by Theorem B, the last estimate implies that

$$B^-(\tau) \leqslant C(1+k_\tau^{1-p})^{\frac{1}{p}}.$$

Thus,

$$\mathscr{B}^-(\tau) \leqslant C(1+k_\tau^{1-p})^{\frac{1}{p}}. \tag{33}$$

From (32) and (33) it follows the finiteness of $F^+(\tau)$ and $\mathscr{B}^-(\tau)$ for any $\tau > 0$ and

$$\inf_{\tau\in I}\max \mathscr{B}^- F^+(\tau) \leqslant C \inf_{\tau\in I}\max[(1+k_\tau^{p-1})(1+k_\tau^{1-p})]^{\frac{1}{p}} \leqslant 4^{\frac{1}{p}}C.$$

This gives the left-hand side estimate of (13). Moreover, from (32) we also have the left-hand side estimate of (14). The proof is complete. □

4. Oscillatory properties of the equation (2)

First we consider the equation (2) for $\lambda = 1$:

$$(v(t)y''(t))'' - u(t)y(t) = 0,\ t \in I, \tag{34}$$

Two points t_1 and t_2 of the interval I such that $t_1 \neq t_2$ are called conjugate with respect to equation (34), if there exists a nonzero solution y of equation (34) having zeros of multiplicity two $y^{(i)}(t_1) = y^{(i)}(t_2) = 0$, $i = 0,1$, at these points t_1 and t_2.

Equation (34) is called oscillatory at zero if for any $T > 0$ there exist conjugate points with respect to equation (34) to the left of T. Otherwise, equation (34) is called non-oscillatory at zero.

The crucial connection between the oscillatory properties of equation (34) and inequality (5) stated in Introduction is explained by the following well-known variational lemma (see [7]).

LEMMA A. *If for some $T > 0$ the inequality*

$$\int_0^T \left[v(t)|f''(t)|^2 - u(t)|f(t)|^2\right]dt \geqslant 0$$

holds for all non-zero $f \in C_0^\infty(0,T)$, then equation (34) is non-oscillatory at zero.

Let us consider the following second-order Hardy-type inequality

$$\int_0^T u(t)|f(t)|^2 dt \leqslant C_T \int_0^T v(t)|f''(t)|^2 dt,\ \ f \in \mathring{W}_{2,v}^2(0,T). \tag{35}$$

As in [11], from Lemma A we can prove the following lemma:

LEMMA 1. *Let C_T be the best constant in (35). Then*

(i) equation (34) is non-oscillatory at zero if and only if there exists a constant $T>0$ such that $0<C_T\leqslant 1$;

(ii) equation (34) is oscillatory at zero if and only if $C_T>1$ for any $T>0$.

Let us turn to equation (2) with the parameter $\lambda>0$. Equation (2) is called strong oscillatory (non-oscillatory) at zero if it is oscillatory (non-oscillatory) at zero for any $\lambda>0$. From inequality (35) for equation (2) we have the inequality

$$\lambda\int_0^T u(t)|f(t)|^2dt\leqslant\lambda C_T\int_0^T v(t)|f''(t)|^2dt,\quad f\in\mathring{W}^2_{2,v}(0,T), \tag{36}$$

with the best constant λC_T, where C_T is the best constant in (35). Again as in [11], from Lemma 1 we can prove one more useful lemma.

LEMMA 2. *Let C_T be the best constant in (36). Then*

(i) equation (2) is strong non-oscillatory at zero if and only if $\lim\limits_{T\to 0^+}C_T=0$;

(ii) equation (2) is strong oscillatory at zero if and only if $C_T=\infty$ for any $T>0$.

Now, on the basis of Lemma 2 and Theorem 1 we can establish criteria of strong non-oscillation and strong oscillation of the equation (2) in the following main theorem of this Section.

THEOREM 2. *Let $v^{-1}\notin L_1(0,1)$, $t^2v^{-1}(t)\in L_1(0,1)$, and $t^2v^{-1}(t)\in L_1(1,\infty)$. Then*

(i) equation (2) is strong non-oscillatory at zero if and only if

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int_0^z t^2u(t)dt\int_z^\tau v^{-1}(s)\,ds=0, \tag{37}$$

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int_z^\tau u(t)dt\int_0^z s^2v^{-1}(s)\,ds=0; \tag{38}$$

(ii) equation (2) is strong oscillatory at zero if and only if

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int_0^z t^2u(t)dt\int_z^\tau v^{-1}(s)\,ds=\infty \tag{39}$$

or

$$\lim_{\tau\to 0^+}\sup_{0<z<\tau}\int_z^\tau u(t)dt\int_0^z s^2v^{-1}(s)\,ds=\infty. \tag{40}$$

Proof. Let $F_1^+(\tau)$ and $F_2^+(\tau)$ be defined by (11) and (12), respectively. First we rewrite the squared values $F_1^+(\tau)$ and $F_2^+(\tau)$ for $p=q=2$ in the forms:

$$(F_1^+(\tau))^2 = \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau \left(1-\frac{s}{\tau}\right)^2 v^{-1}(s)ds,$$

$$(F_2^+(\tau))^2 = \sup_{0<z<\tau} \int_z^\tau \left(1-\frac{t}{\tau}\right)^2 u(t)dt \int_0^z s^2 v^{-1}(s)ds,$$

and define $F^+(\tau)$ by

$$(F^+(\tau))^2 = \max\{(F_1^+(\tau))^2, (F_2^+(\tau))^2\}.$$

(i) Suppose that equation (2) is strong non-oscillatory at zero. Then, by Lemma 2, we have that $\lim\limits_{T\to 0^+} C_T = 0$. From Theorem 1 (see the left-hand side estimate of (14)) it follows that

$$\sup_{0<\tau<T} (1+k_\tau)^{-1}(F^+(\tau))^2 \leqslant C_T,$$

which gives that

$$\lim_{T\to 0^+} \sup_{0<\tau<T} (1+k_\tau)^{-1}(F^+(\tau))^2 = 0.$$

Hence,

$$\lim_{\tau\to 0^+} (1+k_\tau)^{-1}(F^+(\tau))^2 = \lim_{\tau\to 0^+} (F^+(\tau))^2 = 0,$$

i.e.,

$$\lim_{\tau\to 0^+} (F_1^+(\tau))^2 = \lim_{\tau\to 0^+} (F_2^+(\tau))^2 = 0.$$

Thus,

$$\begin{aligned} 0 = \lim_{\tau\to 0^+} (F_1^+(\tau))^2 &\geqslant \lim_{\tau\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 u(t)dt \int_z^{\frac{\tau}{2}} \left(1-\frac{s}{\tau}\right)^2 v^{-1}(s)ds \\ &\geqslant \frac{1}{4} \lim_{\tau\to 0^+} \sup_{0<z<\frac{\tau}{2}} \int_0^z t^2 u(t)dt \int_z^{\frac{\tau}{2}} v^{-1}(s)ds, \end{aligned}$$

i.e., (37) holds. Similarly, we can prove that also (38) holds.

Inversely, let (37) and (38) hold. Since $1-\frac{t}{\tau} \leqslant 1$ for $0<t<\tau$, we obtain that

$$\begin{aligned} 0 &= \lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau v^{-1}(s)ds \\ &\geqslant \lim_{\tau\to 0^+} \sup_{0<z<\tau} \int_0^z t^2 u(t)dt \int_z^\tau \left(1-\frac{s}{\tau}\right)^2 v^{-1}(s)ds = \lim_{\tau\to 0^+} (F_1^+(\tau))^2, \end{aligned}$$

so we can conclude that $\lim_{\tau\to 0^+}(F_1^+(\tau))^2=0$. Similarly, we find that $\lim_{\tau\to 0^+}(F_2^+(\tau))^2=0$. Hence, $\lim_{\tau\to 0^+}(F^+(\tau))^2=0$. From Theorem 1 (see the right-hand estimate of (14)) it follows that

$$C_T \leqslant 22^2(F^+(\tau^+))^2,\ 0<\tau^+<T. \tag{41}$$

Therefore, we get that

$$0=22^2\lim_{T\to 0^+}(F^+(\tau^+))^2=22^2\lim_{\tau\to 0^+}(F^+(\tau))^2\geqslant \lim_{T\to 0^+}C_T.$$

Thus, $\lim_{T\to 0^+}C_T=0$ and, by Lemma 2, the equation (2) is strong non-oscillatory at zero.

(ii) Let the equation (2) be strong oscillatory at zero. Then, by Lemma 2, we have that $C_T=\infty$ for any $T>0$. Therefore, according to (41), we have that

$$\lim_{T\to 0^+}F^+(\tau^+)=\lim_{\tau\to 0^+}F^+(\tau)=\infty.$$

This means that at least one of the conditions (39) or (40) holds.

Inversely, let (39) hold. Then

$$\begin{aligned}
\infty &= \lim_{\frac{\tau}{2}\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int_0^z t^2u(t)dt\int_z^{\frac{\tau}{2}}v^{-1}(s)ds\\
&= \lim_{\frac{\tau}{2}\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int_0^z t^2u(t)dt\int_z^{\frac{\tau}{2}}4^{-1}v^{-1}(s)ds\\
&\leqslant \lim_{\frac{\tau}{2}\to 0^+}\sup_{0<z<\frac{\tau}{2}}\int_0^z t^2u(t)dt\int_z^{\frac{\tau}{2}}\left(1-\frac{s}{\tau}\right)^2v^{-1}(s)ds\\
&= \lim_{\frac{\tau}{2}\to 0^+}\left(F_1^+\left(\frac{\tau}{2}\right)\right)^2=\lim_{\tau\to 0^+}(F_1^+(\tau))^2.
\end{aligned}$$

Thus, $\lim_{\tau\to 0^+}(F_1^+(\tau))^2=\infty$. Since $\sup_{0<\tau<T}(1+k_\tau)^{-1}(F_1^+(\tau))^2\leqslant C_T$ and

$$\begin{aligned}
\lim_{T\to 0^+}\sup_{0<\tau<T}(1+k_\tau)^{-1}(F_1^+(\tau))^2 &\geqslant \lim_{\tau\to 0^+}(1+k_\tau)^{-1}(F_1^+(\tau))^2\\
&= \lim_{\tau\to 0^+}(F_1^+(\tau))^2=\infty,
\end{aligned}$$

we get that $C_T=\infty$ for any $T>0$. Therefore, by Lemma 2, we can conclude that equation (2) is strong oscillatory at zero. Moreover, by arguing similarly, if (40) holds, then equation (2) is strong oscillatory at zero. The proof is complete. □

Now, we assume that the function u together with the function v is twice continuously differentiable on the interval I. In the theory of oscillatory properties of differential equations, there is a well known reciprocity principle (see [4]), from which it follows that equation (2) and its reciprocal equation

$$(u^{-1}(t)y''(t))'' - \lambda v^{-1}(t)y(t) = 0,\ t \in I, \tag{42}$$

are simultaneously oscillatory or non-oscillatory.

On the basis of this reciprocity principle, from our new second-order Hardy-type inequality (Theorem 1) and Theorem 2 it follows the next main result in this Section:

THEOREM 3. *Let $u \notin L_1(0,1)$, $t^2u(t) \in L_1(0,1)$, and $t^2u(t) \in L_1(1,\infty)$. Then (i) equation (42) is strong non-oscillatory at zero if and only if (37) and (38) hold; (ii) equation (42) is strong oscillatory at zero if and only if (39) or (40) holds.*

5. Spectral characteristics of the differential operator L

Here L is the differential operator defined by (3). Let the minimal differential operator $L_{\min}$ be generated by the differential expression (3), i.e., $L_{\min}$ is an operator with the domain $D(L_{\min}) = C_0^\infty(I)$. It is known that all self-adjoint extensions of the minimal differential operator $L_{\min}$ have the same spectrum (see [7]).

In this Section we find conditions under which any self-adjoint extension L of the operator $L_{\min}$ has a spectrum which is discrete and bounded from below. Motivations to study these spectral properties are completely revealed in [8].

The relationship between the oscillatory properties of equation (2) and the spectral properties of the operator L gives the following statement (see [7]).

LEMMA C. *The operator L has a spectrum, which is discrete and bounded from below, if and only if equation (2) is strong non-oscillatory.*

On the basis of Lemma C, our Theorem 2 implies the following statement.

THEOREM 4. *Let the conditions of Theorem 2 hold. Then the operator L has a spectrum, which is discrete and bounded from below, if and only if (37) and (38) hold.*

The operator L_{min} is nonnegative. Therefore, it has the Friedrich's extension L_F (see, e.g., [16] and [19]). By Theorem 4, the operator L_F has a discrete spectrum if and only if (37) and (38) hold.

Since for $p = q = 2$, inequality (5) can be rewritten as $(f,f)_2 C^{-2} \leqslant (L_F f, f)_{2,u}$, then, in view of Theorem 1, we have the following statement, where $\mathscr{B}^-(\tau)$ and $F^+(\tau)$ are taken for $p = q = 2$.

THEOREM 5. *Let the conditions of Theorem 2 hold. Then the operator L_F is positive definite if and only if $\mathscr{B}^-F^+(\tau) = \inf_{\tau\in I} \max\{\mathscr{B}^-(\tau), F^+(\tau)\} < \infty$. Moreover, from (13) it follows that the estimate $\frac{1}{2}\mathscr{B}^-F^+(\tau) \leqslant \lambda_1^{-\frac{1}{2}} \leqslant 22\mathscr{B}^-F^+(\tau)$ holds for the smallest eigenvalue λ_1 of the operator L_F.*

According to Rellih's lemma (see [16]), the operator L_F^{-1} has a spectrum, which is discrete and bounded from below in $L_{2,u}$, if and only if the space with the norm $(L_F f, f)_{2,u}^{\frac{1}{2}}$ is compactly embedded into the space $L_{2,u}$. Hence, according to Theorem 4, we have one more statement.

THEOREM 6. *Let the conditions of Theorem 2 hold. Then the embedding $\mathring{W}_{2,v}^2(I) \hookrightarrow L_{2,u}$ is compact and the operator L_F^{-1} is uniformly continuous on $L_{2,u}$ if and only if (37) and (38) hold.*

The following statement can be found in [2]:

LEMMA D. *Let $H = H(I)$ be a certain Hilbert function space and $C[0,\infty)\bigcap H$ be dense in it. For any point $t \in I$, we introduce the operator $E_t f = f(t)$ defined on $C[0,\infty)\bigcap H$, which acts in the space of complex numbers. Moreover, assume that E_t is a closure operator. Then, the norm of this operator is equal to the value $\left(\sum\limits_{k=1}^{\infty} |\varphi_k(t)|^2\right)^{\frac{1}{2}}$ (finite or infinite), where $\{\varphi_k(\cdot)\}_{k=1}^{\infty}$ is any complete orthonormal system of continuous functions in H.*

Let

$$D^+(t) = \int_0^t z^2 v^{-1}(z)dz + t^2 \int_t^\infty v^{-1}(z)dz, \; t \in I. \tag{43}$$

Next, we state the following Lemma, which is crucial for the proof of our next main result (Theorem 7) but also of independent interest:

LEMMA 3. *Let the conditions of Theorem 2 hold. Then*

$$\sup_{\tau\in I} D^+(t,\tau) \leqslant \sup_{f\in \mathring{W}_{2,v}^2} \frac{|f(t)|}{\|f''\|_{2,v}} \leqslant \sqrt{2}\inf_{\tau\in I} D^+(t,\tau), \; t \in I, \tag{44}$$

where

$$D^+(t,\tau) = \left(\chi_{(0,\tau)}(t)t^2 \int_\tau^\infty v^{-1}(s)ds + \chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^2 v^{-1}(s)ds \right.$$
$$\left. + \chi_{(0,\tau)}(t)t^2 \int_t^\tau v^{-1}(s)ds + \chi_{(0,\tau)}(t)\int_0^t s^2 v^{-1}(s)ds \right)^{\frac{1}{2}}.$$

Proof. From (16) and (20) for the function $f \in \mathring{W}_{2,v}^2$ it follows that

$$f(t) = \chi_{(0,\tau)}\left[-\int_0^t sf''(s)ds - t\int_t^\tau f''(s)ds - t\int_\tau^\infty f''(s)ds\right]$$
$$+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)f''(s)ds. \tag{45}$$

Moreover, by applying Hölder's inequality, we obtain that

$$
\begin{aligned}
|f(t)| \leqslant & \left[\chi_{(0,\tau)}(t)\left(t^2\int_\tau^\infty v^{-1}(s)ds\right)^{\frac{1}{2}} + \chi_{(\tau,\infty)}(t)\left(\int_t^\infty (s-t)^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right] \\
& \times \left(\int_\tau^\infty v(s)|f''(s)|^2 ds\right)^{\frac{1}{2}} \\
& + \chi_{(0,\tau)}(t)\left[\left(t^2\int_t^\tau v^{-1}(s)ds\right)^{\frac{1}{2}} + \left(\int_0^t s^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right] \\
& \times \left(\int_0^\tau v(s)|f''(s)|^2 ds\right)^{\frac{1}{2}} \\
\leqslant & \left\{\left[\chi_{(0,\tau)}(t)\left(t^2\int_\tau^\infty v^{-1}(s)ds\right)^{\frac{1}{2}} + \chi_{(\tau,\infty)}(t)\left(\int_t^\infty (s-t)^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right]^2\right. \\
& \left. + \chi_{(0,\tau)}(t)\left[\left(t^2\int_t^\tau v^{-1}(s)ds\right)^{\frac{1}{2}} + \left(\int_0^t s^2 v^{-1}(s)ds\right)^{\frac{1}{2}}\right]^2\right\}^{\frac{1}{2}} \|f''\|_{2,v} \\
\leqslant & \left(\chi_{(0,\tau)}(t)t^2\int_\tau^\infty v^{-1}(s)ds + \chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^2 v^{-1}(s)ds\right. \\
& \left. + 2\chi_{(0,\tau)}(t)t^2\int_t^\tau v^{-1}(s)ds + 2\chi_{(0,\tau)}(t)\int_0^t s^2 v^{-1}(s)ds\right)^{\frac{1}{2}} \|f''\|_{2,v} \\
\leqslant & \sqrt{2}\inf_{\tau\in I} D^+(t,\tau)\|f''\|_{2,v}.
\end{aligned}
$$

Therefore, the right-hand side estimate in (44) holds.

Let us prove the left estimate in (44). We fix $t \in I$ in (45) and select a test function $f^{(n)}$ depending on t as follows:

$$
f_t''(s) = \begin{cases}
\chi_{(0,t)}(s)s\,v^{-1}(s) & \text{if } 0 < t < \tau, \\
\chi_{(t,\tau)}(s)t\,v^{-1}(s) & \text{if } 0 < t < \tau, \\
\chi_{(\tau,\infty)}(s)t\,v^{-1}(s) & \text{if } 0 < t < \tau, \\
-\chi_{(t,\infty)}(s)(s-t)v^{-1}(s) & \text{if } t > \tau.
\end{cases}
$$

By putting this function into (45), we get that

$$f_t(t)=\chi_{(0,\tau)}(t)\left(t^2\int_\tau^\infty v^{-1}(s)ds+t^2\int_t^\tau v^{-1}(s)ds+\int_0^t s^2v^{-1}(s)ds\right)$$
$$+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^2v^{-1}(s)ds=\left(D^+(t,\tau)\right)^2. \tag{46}$$

Next, we calculate $\|f_t''\|_{2,v}$:

$$\begin{aligned}\left(\int_0^\infty v(s)|f_t''(s)|^2ds\right)^{\frac12} &= \left(\int_0^\tau v(s)|f_t''(s)|^2ds+\int_\tau^\infty v(s)|f_t''(s)|^2ds\right)^{\frac12}\\ &=\Bigg\{\chi_{(0,\tau)}(t)t^2\int_\tau^\infty v^{-1}(s)ds+\chi_{(0,\tau)}(t)t^2\int_t^\tau v^{-1}(s)ds\\ &\qquad +\chi_{(0,\tau)}(t)\int_0^t s^2v^{-1}(s)ds+\chi_{(\tau,\infty)}(t)\int_t^\infty (s-t)^2v^{-1}(s)ds\Bigg\}^{\frac12}\\ &=D^+(t,\tau).\end{aligned} \tag{47}$$

By combining (46) and (47), we obtain that

$$\sup_{f\in\mathring{W}^2_{2,v}}\frac{|f(t)|}{\|f''\|_{2,v}}\geqslant\frac{|f_t(t)|}{\|f_t''\|_{2,v}}=D^+(t,\tau)$$

for any $\tau\in I$. This relation proves the validity of the left-hand side estimate in (44). The proof is complete. □

Let the operator L_F^{-1} be uniformly continuous on $L_{2,u}$. Let $\{\lambda_k\}_{k=1}^\infty$ be eigenvalues and $\{\varphi_k\}_{k=1}^\infty$ be a corresponding complete orthonormal system of eigenfunctions of the operator L_F^{-1}.

THEOREM 7. *Let the conditions of Theorem 2 and (37) and (38) hold. If $D^+(t)$ is defined by (43), then*

(i) the following two-sided estimates hold:

$$D^+(t)\leqslant\sum_{k=1}^\infty\frac{|\varphi_k(t)|^2}{\lambda_k}\leqslant 2D^+(t),\ t\in I; \tag{48}$$

(ii) the operator L_F^{-1} is nuclear if and only if $\int_0^\infty u(t)D^+(t)dt<\infty$ and for the nuclear norm $\|L_F^{-1}\|_{\sigma_1}$ of the operator L_F^{-1} we have the precise two-sided estimates

$$\int_0^\infty u(t)D^+(t)dt\leqslant\|L_F^{-1}\|_{\sigma_1}=\sum_{k=1}^\infty\frac{1}{\lambda_k}\leqslant 2\int_0^\infty u(t)D^+(t)dt. \tag{49}$$

Proof. Since the operator L_F^{-1} is uniformly continuous on $L_{2,u}$, we can consider $\mathring{W}_{2,v}^2(I)$ with the norm $\|f''\|_{2,v}$ as the space $H(I)$ of Lemma D. Since the system of functions $\{\lambda_k^{-\frac{1}{2}}\varphi_k\}_{k=1}^{\infty}$ is a complete orthonormal system in the space $\mathring{W}_{2,v}^2(I)$, then, by Lemma D, we have that

$$\|E_t\|^2 = \left(\sup_{f\in\mathring{W}_{2,v}^2}\frac{|f(t)|}{\|f''\|_{2,v}}\right)^2 = \sum_{k=1}^{\infty}\frac{|\varphi_k(t)|^2}{\lambda_k},$$

where $E_t f = f(t)$. By using this fact and (44) in Lemma 3 we obtain that

$$\sup_{\tau\in I}\left(D^+(t,\tau)\right)^2 \leqslant \sum_{k=1}^{\infty}\frac{|\varphi_k(t)|^2}{\lambda_k} \leqslant 2\inf_{\tau\in I}\left(D^+(t,\tau)\right)^2. \tag{50}$$

Since

$$\inf_{\tau\in I}\left(D^+(t,\tau)\right)^2 \leqslant \lim_{\tau\to\infty}\left(D^+(t,\tau)\right)^2 = D^+(t) \leqslant \sup_{\tau\in I}\left(D^+(t,\tau)\right)^2,$$

from (50) we have (48), so (i) is proved. Moreover, by multiplying both sides of (48) by u and integrating them from zero to infinity, we get (49), so also (ii) is proved. The proof is complete. □

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