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Homogenization of Attractors to Ginzburg-Landau Equations in Media with Locally Periodic Obstacles: Sub- and Supercritical Cases

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The Ginzburg-Landau equation with rapidly oscillating terms in the equation and boundary conditions in a perforated domain was considered. Proof was given that the trajectory attractors of this equation converge weakly to the trajectory attractors of the homogenized Ginzburg-Landau equation. To do this, we use the approach from the articles and monographs of V.V. Chepyzhov and M.I. Vishik about trajectory attractors of evolutionary equations, and we also use homogenization methods that appeared at the end of the 20th century. First, we use asymptotic methods to construct asymptotics formally, and then we justify the form of the main terms of the asymptotic series using functional analysis and integral estimates. By defining the corresponding auxiliary function spaces with weak topology, we derive a limit (homogenized) equation and prove the existence of a trajectory attractor for this equation. Then, we formulate the main theorems and prove them by using auxiliary lemmas. We prove that the trajectory attractors of this equation tend in a weak sense to the trajectory attractors of the homogenized Ginzburg-Landau equation in the subcritical case, and they disappear in the supercritical case.

Keywords: attractors, homogenization, Ginzburg-Landau equations, nonlinear equations, weak convergence, perforated domain, porous medium.

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Introduction

This work is devoted to investigating boundary value initial problems in the perforated domain. Assuming Robin (Fourier) type of boundary conditions to be set on the boundary of holes, we write down the homogenized (limit) problem and prove the Hausdorff convergence of attractors (Fig.) as the small parameter tends to zero. Thus, we define the homogenized attractor and prove the convergence of the initial attractors to the attractor of the homogenized problem. The asymptotic behaviour of attractors to an initial boundary value problem for complex Ginzburg-Landau equations in perforated domains for the critical case (appearance of additional potential in the homogenized equation) is studied in [1]. In this paper, we investigate subcritical and supercritical cases. For the asymptotic analysis of problems in perforated domains, see, for instance, [2, 3] and [4–7].

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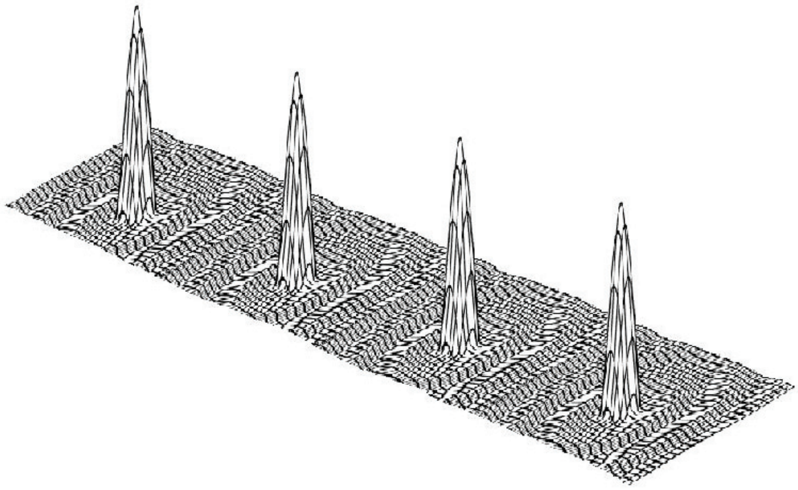

Figure. Attractor of the Ginzburg-Landau equation

About attractors, see, for example, monographs [8–10] and the references therein. Homogenization of attractors were studied in [9, 11–16] (see also [17, 18]).

In the paper, we prove that the trajectory attractor $\mathfrak{A}_\mu$ of the Ginzburg-Landau equation in the perforated domain converges in a weak sense as $\mu \to 0$ to the trajectory attractor $\overline{\mathfrak{A}}$ of the homogenized equation in an appropriate functional space. Here, μ characterizes the diameter of cavities and the distance between them in the perforated medium.

The results are announced in [19].

1 Statement of the problem

First, we define a perforated domain. Let $\Omega \subset \mathbb{R}^d, d \geq 2$ be a smooth bounded domain. Denote

$$\Upsilon_\mu = \left\{ j \in \mathbb{Z}^d \, : \, \mathrm{dist}\,(\mu j, \partial\Omega) \geq \mu\sqrt{d} \right\}, \ \square \equiv \left\{ \xi : -\frac{1}{2} < \xi_j < \frac{1}{2}, \ j = 1, \dots, d \right\}.$$

Given a 1-periodic in ξ smooth function $F(x, \xi)$ such that $F(x,\xi)\Big|_{\xi\in\partial\square} \geq \mathrm{const} > 0$, $F(x, 0) = -1$, $\nabla_\xi F \neq 0$ as $\xi \in \square\backslash\{0\}$, we set

$$D_j^\mu = \left\{ x \in \mu\,(\square + j) \mid F(x, \frac{x}{\mu}) \leq 0 \right\}$$

and introduce the perforated domain as follows:

$$\Omega_\mu = \Omega \backslash \bigcup_{j\in\Upsilon_\mu} D_j^\mu.$$

Denote by ω the set $\left\{ \xi \in \mathbb{R}^d \mid F(x, \xi) < 0 \right\}$, and by S the set $\left\{ \xi \in \mathbb{R}^d \mid F(x, \xi) = 0 \right\}$.

Afterwards, we will often interprete 1-periodic in ξ functions as functions defined on d-dimensional torus $\mathbb{T}^d \equiv \left\{ \xi : \xi \in \mathbb{R}^d/\mathbb{Z}^d \right\}$.

According to the above construction, the boundary $\partial\Omega_\mu$ consists of $\partial\Omega$ and the boundary of the cavities $S_\mu \subset \Omega$, $S_\mu = (\partial\Omega_\mu) \cap \Omega$.

We study the asymptotic behaviour of attractors to the problem

$$\begin{cases} \dfrac{\partial u_\mu}{\partial t} = (1+\alpha \mathrm{i})\Delta u_\mu + R(x,\dfrac{x}{\mu})\, u_\mu - \left(1+\beta(x,\dfrac{x}{\mu})\mathrm{i}\right)|u_\mu|^2 u_\mu + g(x), & x\in\Omega_\mu, \\ (1+\alpha \mathrm{i})\dfrac{\partial u_\mu}{\partial \nu} + \mu^\theta q(x,\dfrac{x}{\mu})u_\mu = 0, & x\in S_\mu, t>0, \\ u_\mu = 0, & x\in\partial\Omega, \\ u_\mu = U(x), & x\in\Omega_\mu, t=0, \end{cases} \tag{1}$$

where $\theta > 1$ (subcritical case) and $0 < \theta < 1$ (supercritical case). Here α is a real constant, ν is the outward unit vector to the boundary, $u = u_1 + \mathrm{i}u_2 \in \mathbb{C}$, $g(x) \in C^1(\Omega;\mathbb{C})$, $q(x,\xi) \in C^1(\Omega;\mathbb{R}^d)$ and $q(x,\xi)$ is a nonnegative 1-periodic in ξ function. We assume that

$$-R_1 \le R(x,\xi) \le R_2,\ -\beta_1 \le \beta(x,\xi) \le \beta_2 \ \ (R_0, R_1, \beta_1, \beta_2 > 0), \tag{2}$$

for $x\in\Omega$, $\xi\in\mathbb{R}^d$ and the functions $R\,(x,\xi)$ and $\beta\,(x,\xi)$ have the averages $\bar{R}(x)$ and $\bar{\beta}(x)$ in $L_{\infty,*w}(\Omega)$ respectively, i.e.,

$$\int_\Omega R\,(x,\xi)\,\varphi_1(x)dx \ \to \ \int_\Omega \bar{R}(x)\varphi_1(x)dx, \int_\Omega \beta\,(x,\xi)\,\varphi_1(x)dx \ \to \ \int_\Omega \bar{\beta}(x)\varphi_1(x)dx$$

as $\mu \to 0+$ for any function $\varphi_1(x) \in L_1(\Omega)$, where $\xi = \dfrac{x}{\mu}$.

We denote the spaces $\mathbf{H} := L_2(\Omega;\mathbb{C})$, $\mathbf{H}_\mu := L_2(\Omega_\mu;\mathbb{C})$, $\mathbf{V} := H_0^1(\Omega;\mathbb{C})$, $\mathbf{V}_\mu := H^1(\Omega_\mu;\mathbb{C};\partial\Omega)$ – set of functions from $H^1(\Omega_\mu;\mathbb{C})$ with zero trace on $\partial\Omega$, and $\mathbf{L}_p := L_p(\Omega;\mathbb{C})$, $\mathbf{L}_{p,\mu} := L_p(\Omega_\mu;\mathbb{C})$. The norms in these spaces are denoted, respectively, by

$$\|v\|^2 := \int_\Omega |v(x)|^2 dx,\ \|v\|_\mu^2 := \int_{\Omega_\mu} |v(x)|^2 dx,\ \|v\|_1^2 := \int_\Omega |\nabla v(x)|^2 dx,$$
$$\|v\|_{1\mu}^2 := \int_{\Omega_\mu} |\nabla v(x)|^2 dx,\ \|v\|_{\mathbf{L}_p}^p := \int_\Omega |v(x)|^p dx,\ \|v\|_{\mathbf{L}_p\,\mu}^p := \int_{\Omega_\mu} |v(x)|^p dx.$$

Recall that $\mathbf{V}' := H^{-1}(\Omega;\mathbb{C})$ and $\mathbf{L}_q$ are the dual spaces of $\mathbf{V}$ and $\mathbf{L}_p$ respectively, where $q = p/(p-1)$, moreover, $\mathbf{V}'_\mu$ and $\mathbf{L}_{q,\mu}$ are the dual spaces for $\mathbf{V}_\mu$ and $\mathbf{L}_{p,\mu}$.

As in [9], we study weak solutions of the initial boundary value problem (1), that is, the functions

$$u_\mu(x,s) \in L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\mu) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\mu})$$

which satisfy the problem (1) in the distributional sense, i.e.

$$-\int_0^\infty \int_{\Omega_\mu} u_\mu \frac{\partial\psi}{\partial t}\, dxdt + (1+\alpha\mathrm{i})\int_0^\infty \int_{\Omega_\mu} \nabla u_\mu \nabla\psi\, dxdt -$$
$$-\int_0^\infty \int_{\Omega_\mu} \left(\left(R\left(x,\frac{x}{\mu}\right)u_\mu - \left(1+\beta\left(x,\frac{x}{\mu}\right)\mathrm{i}\right)|u_\mu|^2 u_\mu\right)\right)\psi\, dxdt +$$
$$+\mu^\theta \int_0^{+\infty}\int_{S_\mu} q\left(x,\frac{x}{\mu}\right)u_\mu\psi\, d\sigma dt = \int_0^\infty \int_{\Omega_\mu} g(x)\psi\, dxdt \tag{3}$$

for any function $\psi \in C_0^\infty(\mathbb{R}_+;\mathbf{V}_\mu \cap \mathbf{L}_{4,\mu})$.

If $u_\mu(x,t) \in L_4(0,M;\mathbf{L}_{4,\mu})$, then it follows that

$$R\left(x,\frac{x}{\mu}\right)u_\mu(x,t) - \left(1+\beta\left(x,\frac{x}{\mu}\right)\mathrm{i}\right)|u_\mu(x,t)|^2 u_\mu(x,t) \in L_{4/3}(0,M;\mathbf{L}_{4/3,\mu}).$$

At the same time, if $u_\mu(x,t) \in L_2(0,M;\mathbf{V}_\mu)$, then $(1+\alpha\mathrm{i})\Delta u_\mu(x,t)+g(x) \in L_2(0,M;\mathbf{V}'_\mu)$. Therefore, for an arbitrary weak solution $u_\mu(x,s)$ of the problem (1) we have

$$\frac{\partial u_\mu(x,t)}{\partial t} \in L_{4/3}(0,M;\mathbf{L}_{4/3,\mu}) + L_2(0,M;\mathbf{V}'_\mu).$$

The Sobolev embedding theorem implies that

$$L_{4/3}(0,M;\mathbf{L}_{4/3,\mu}) + L_2(0,M;\mathbf{V}'_\mu) \subset L_{4/3}\left(0,M;\mathbf{H}_\mu^{-r}\right),$$

where the space $\mathbf{H}_\mu^{-r} := H^{-r}(\Omega_\mu;\mathbb{C})$ and $r = \max\{1, d/4\}$. Hence, for any weak solution $u_\mu(x,t)$ of (1) we have $\frac{\partial u_\mu(x,t)}{\partial t} \in L_{4/3}\left(0,M;\mathbf{H}_\mu^{-r}\right)$.

Remark 1.1. The existence of weak solution $u(x,s)$ to the problem (1) for every $U \in \mathbf{H}_\mu$ and fixed μ, such that $u(x,0) = U(x)$ can be proved by standard approach (see for instance [8]).

The following key Lemma can be proved similar to Proposition 3 from [17].

Lemma 1.1. Let $u_\mu(x,t) \in L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\mu})$ be a weak solution to the problem (1). Then

(i) $u \in C(\mathbb{R}_+;\mathbf{H}_\mu)$;

(ii) the function $\|u_\mu(\cdot,t)\|_\mu^2$ is absolutely continuous on $\mathbb{R}_+$ and, moreover,

$$\frac{1}{2}\frac{d}{dt}\|u_\mu(\cdot,t)\|_\mu^2 + \|\nabla u_\mu(\cdot,t)\|_\mu^2 + \|u_\mu(\cdot,t)\|_{\mathbf{L}_{4,\mu}}^4 - \int_{\Omega_\mu} R\left(x,\frac{x}{\mu}\right)|u_\mu(x,t)|^2 dx +$$
$$+ \mu^\theta \int_{S_\mu} q\left(x,\frac{x}{\mu}\right)|u_\mu(x,t)|^2 d\sigma = \int_{\Omega_\mu} Re\left(g(x)\bar{u}_\mu(x,t)\right)dx,$$

for almost every $t \in \mathbb{R}_+$.

Let us fix μ. In further analysis, we shall omit the index μ in the notation of the spaces, where it is natural. We now apply the scheme described in [1; Section 2] to construct the trajectory attractor for the problem (1), which has the form from the scheme, if we set $E_1 = \mathbf{L}_p \cap \mathbf{V}$, $E_0 = \mathbf{H}^{-r}$, $E = \mathbf{H}$ and $A(u) = (1+\alpha\mathrm{i})\Delta u + R(\cdot)\,u - (1+\beta(\cdot)\mathrm{i})\,|u|^2 u + g(\cdot)$.

To describe the trajectory space $\mathcal{K}_\mu^+$ for the problem (1), we follow the general framework of [1; Section 2] and define the Banach spaces for every $[t_1,t_2] \in \mathbb{R}$

$$\mathcal{F}_{t_1,t_2} := L_4(t_1,t_2;\mathbf{L}_4) \cap L_2(t_1,t_2;\mathbf{V}) \cap L_\infty(t_1,t_2;\mathbf{H}) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}\left(t_1,t_2;\mathbf{H}^{-r}\right)\right\}$$

with norm

$$\|v\|_{\mathcal{F}_{t_1,t_2}} := \|v\|_{L_4(t_1,t_2;\mathbf{L}_4)} + \|v\|_{L_2(t_1,t_2;\mathbf{V})} + \|v\|_{L_\infty(0,M;\mathbf{H})} + \left\|\frac{\partial v}{\partial t}\right\|_{L_{4/3}(t_1,t_2;\mathbf{H}^{-r})}. \qquad (4)$$

According to the scheme, we use the norm (4); in this case, the translation semigroup $\{S(h)\}$ satisfies the conditions from the scheme.

Setting $\mathcal{D}_{t_1,t_2} = L_2\left(t_1,t_2;\mathbf{V}\right)$ we have that $\mathcal{F}_{t_1,t_2} \subseteq \mathcal{D}_{t_1,t_2}$ and if $u(s) \in \mathcal{F}_{t_1,t_2}$, then $A(u(s)) \in \mathcal{D}_{t_1,t_2}$. We can consider a weak solutions of the problem (1) as a solution of an equation in the general scheme from [1; Section 2].

Define the spaces

$$\mathcal{F}_+^{loc} = L_4^{loc}(\mathbb{R}_+;\mathbf{L}_4) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}) \cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+;\mathbf{H}^{-r})\right\},$$
$$\mathcal{F}_{\mu,+}^{loc} = L_4^{loc}(\mathbb{R}_+;\mathbf{L}_{4,\mu}) \cap L_2^{loc}(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_\infty^{loc}(\mathbb{R}_+;\mathbf{H}_\mu) \cap \left\{v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+;\mathbf{H}_\mu{}^{-r})\right\}.$$

We denote by $\mathcal{K}_\mu^+$ the set of all weak solutions of the problem (1). Recall that for any $U \in \mathbf{H}$ there exist at least one trajectory $u(\cdot) \in \mathcal{K}_\mu^+$ such that $u(0) = U(x)$. Therefore, the trajectory space $\mathcal{K}_\mu^+$ of the problem (1) is not empty and is sufficiently large.

It is clear that $\mathcal{K}_\mu^+ \subset \mathcal{F}_+^{loc}$ and the trajectory space $\mathcal{K}_\mu^+$ is translation invariant, that is, if $u(s) \in \mathcal{K}_\mu^+$, then $u(h+s) \in \mathcal{K}_\mu^+$ for all $h \geq 0$. Therefore,

$$S(h)\mathcal{K}_\mu^+ \subseteq \mathcal{K}_\mu^+, \quad \forall h \geq 0.$$

We now define metrics $\rho_{t_1,t_2}(\cdot,\cdot)$ on the spaces $\mathcal{F}_{t_1,t_2}$ using the norms of the spaces $L_2(t_1,t_2;\mathbf{H})$:

$$\rho_{0,M}(u,v) = \left(\int_0^M \|u(s) - v(s)\|_{\mathbf{H}}^2 ds\right)^{1/2}, \quad \forall u(\cdot), v(\cdot) \in \mathcal{F}_{0,M}.$$

These metrics generate the topology Θ_+^{loc} in $\mathcal{F}_+^{loc}$ (respectively $\Theta_{\mu,+}^{loc}$ in $\mathcal{F}_{\mu,+}^{loc}$). Recall that a sequence $\{v_k\} \subset \mathcal{F}_+^{loc}$ converges to $v \in \mathcal{F}_+^{loc}$ as $k \to \infty$ in Θ_+^{loc} if $\|v_k(\cdot) - v(\cdot)\|_{L_2(0,M;\mathbf{H})} \to 0$ $(k \to \infty)$ for each $M > 0$. The topology Θ_+^{loc} is metrizable. We consider this topology in the trajectory space $\mathcal{K}_\mu^+$ of (1). The translation semigroup $\{S(t)\}$ acting on $\mathcal{K}_\mu^+$ is continuous in the topology Θ_+^{loc}.

Following the general scheme of [1; Section 2], we define bounded sets in $\mathcal{K}_\mu^+$ using the Banach space $\mathcal{F}_{+,\mu}^b$. We clearly have

$$\mathcal{F}_{+,\mu}^b = L_4^b(\mathbb{R}_+;\mathbf{L}_{4,\mu}) \cap L_2^b(\mathbb{R}_+;\mathbf{V}_\mu) \cap L_\infty(\mathbb{R}_+;\mathbf{H}_\mu) \cap \left\{v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}_+;\mathbf{H}_\mu^{-r})\right\}.$$

In an analogous way, we have

$$\mathcal{F}_+^b = L_4^b(\mathbb{R}_+;\mathbf{L}_4) \cap L_2^b(\mathbb{R}_+;\mathbf{V}) \cap L_\infty(\mathbb{R}_+;\mathbf{H}) \cap \left\{v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R}_+;\mathbf{H}^{-r})\right\};$$

$\mathcal{F}_+^b$ and $\mathcal{F}_{+,\mu}^b$ are subspaces of $\mathcal{F}_+^{loc}$ and $\mathcal{F}_{+,\mu}^{loc}$, respectively.

Consider the translation semigroup $\{S(t)\}$ on $\mathcal{K}_\mu^+$, $S(t) : \mathcal{K}_\mu^+ \to \mathcal{K}_\mu^+$, $t \geq 0$.

Let $\mathcal{K}_\mu$ be the kernel of the problem (1) that consists of all weak complete solutions $u(s), \in \mathbb{R}$, of the system bounded in the space

$$\mathcal{F}_\mu^b = L_4^b(\mathbb{R};\mathbf{L}_{4,\mu}) \cap L_2^b(\mathbb{R};\mathbf{V}_\mu) \cap L_\infty(\mathbb{R};\mathbf{H}_\mu) \cap \left\{v \;\Big|\; \frac{\partial v}{\partial t} \in L_{4/3}^b(\mathbb{R};\mathbf{H}_\mu^{-r})\right\}.$$

In analogous way we define $\mathcal{F}^b$.

The definition of trajectory attractor was given in [1] (see also [9]).

Proposition 1.1. The problem (1) has the trajectory attractors $\mathfrak{A}_\mu$ in the topological space Θ_+^{loc}. The set $\mathfrak{A}_\mu$ is uniformly (w.r.t. $\mu \in (0,1)$) bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc}. Moreover,

$$\mathfrak{A}_\mu = \Pi_+\mathcal{K}_\mu,$$

the kernel $\mathcal{K}_\mu$ is non-empty and uniformly (w.r.t. $\mu \in (0,1)$) bounded in $\mathcal{F}^b$. Recall that the spaces $\mathcal{F}_+^b$ and Θ_+^{loc} depend on μ.

The proof of this proposition almost coincides with the proof given in [9] for a particular case. The existence of an absorbing set that is bounded in $\mathcal{F}_+^b$ and compact in Θ_+^{loc} is proved using Lemma 1.1 similar to [9].

We note that

$$\mathfrak{A}_\mu \subset \mathcal{B}_0(R), \quad \forall \mu \in (0,1),$$

where $\mathcal{B}_0(R)$ is a ball in $\mathcal{F}_+^b$ with a sufficiently large radius R. The Aubin-Lions-Simon Lemma from [1; Section 2] implies that

$$\mathcal{B}_0(R) \Subset L_2^{loc}(\mathbb{R}_+; \mathbf{H}^{1-\delta}), \tag{5}$$
$$\mathcal{B}_0(R) \Subset C^{loc}(\mathbb{R}_+; \mathbf{H}^{-\delta}), \quad 0 < \delta \leq 1. \tag{6}$$

Using compact inclusions (5) and (6), we strengthen the attraction to the constructed trajectory attractor.

Corollary 1.1. For any set $\mathcal{B} \subset \mathcal{K}_\mu^+$ bounded in $\mathcal{F}_+^b$ we have

$$\mathrm{dist}_{L_2(0,M;H^{1-\delta})}\left(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\mu\right) \to 0 \ (t \to \infty),$$
$$\mathrm{dist}_{C([0,M];H^{-\delta})}\left(\Pi_{0,M} S(t)\mathcal{B}, \Pi_{0,M}\mathcal{K}_\mu\right) \to 0 \ (t \to \infty),$$

where M is an arbitrary positive number.

2 Homogenized (limit) problem

Let M_i be 1-periodic solution to a problem

$$\Delta_\xi \left(M_i + \xi_i\right) = 0 \text{ in } \square \setminus \omega, \qquad \frac{\partial M_i}{\partial \nu_\xi} = \nu_i \text{ on } S(x), \tag{7}$$

having zero mean values over the cell of periodicity. Denote by $\langle \cdot \rangle$ the integral over the set $\square \cap \omega$.

The case $\theta > 1$. The homogenized (limit) problem has the form

$$\begin{cases} \dfrac{\partial u_0}{\partial t} - (1+\alpha \mathrm{i}) \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0}{\partial x_j}\right) - \\ \qquad -R(x)u_0 + (1+\beta(x)\mathrm{i})\left|u_0\right|^2 u_0 = \left|\square \cap \omega\right| g(x), & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega, t > 0, \\ u_0 = U(x), & x \in \Omega, t = 0. \end{cases} \tag{8}$$

We consider weak solution to the problem (8), i.e. the function $u_0 = u_0(x,t)$, $x \in \Omega$, $t \geq 0$,

$$u_0 \in L_4^{loc}(\mathbb{R}_+; \mathbf{L}_4) \cap L_2^{loc}(\mathbb{R}_+; \mathbf{V}) \cap L_\infty^{loc}(\mathbb{R}_+; \mathbf{H}) \cap \left\{ v \;\middle|\; \frac{\partial v}{\partial t} \in L_{4/3}^{loc}(\mathbb{R}_+; \mathbf{H}^{-r}) \right\},$$

satisfying the integral identity

$$-\int_{\mathbb{R}_+}\int_\Omega u_0 \frac{\partial v}{\partial t} dtdx + (1+\alpha\mathrm{i})\int_{\mathbb{R}_+}\int_\Omega \sum_{i,j=1}^{d} \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j} \right\rangle \frac{\partial u_0}{\partial x_i}\frac{\partial v}{\partial x_j} dtdx -$$
$$- \int_{\mathbb{R}_+}\int_\Omega \Big(R(x)u_0 - (1+\beta(x)\mathrm{i})\left|u_0\right|^2 u_0 \Big) v dtdx = \int_{\mathbb{R}_+}\int_\Omega \left|\square \cap \omega\right| g(x)\, v \, dtdx$$

for any function $v \in C_0^\infty(\mathbb{R}_+; \mathbf{V} \cap \mathbf{L}_4)$.

Remark 2.1. It should be noted that $M_i(x,\xi)$ are not defined in the whole Ω. Applying the technique of the symmetric extension allows to extend $M_i(x,\xi)$ into the interior of the “holes” retaining the regularity of these functions. We keep the same notation for the extended functions.

3 Auxiliaries

3.1 General reasoning

We investigate the asymptotic behavior of the solution $u_\mu(x)$ as $\mu \to 0$ of the following boundary-value problem in the domain Ω_μ :

$$\begin{cases} -(1+\alpha\mathrm{i})\,\Delta u_\mu = g(x) & \text{in} \quad \Omega_\mu, \\ (1+\alpha\mathrm{i})\,\dfrac{\partial u_\mu}{\partial \nu_\mu} + \mu^\theta q\left(x, \dfrac{x}{\mu}\right) u_\mu = 0 & \text{on} \quad S_\mu, \\ u_\mu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{9}$$

where n_μ is the internal normal to the boundary of "holes" $q(x,\xi)$ is a sufficiently smooth 1-periodic in ξ function.

Definition 3.1. Function $u_\mu \in H^1(\Omega_\mu, \partial\Omega)$ is a solution of problem (9), if the following integral identity

$$(1+\alpha\mathrm{i}) \int_{\Omega^\mu} \nabla u_\mu(x)\,\nabla v(x)\,dx + \mu^\theta \int_{S_\mu} q\left(x, \frac{x}{\mu}\right) u_\mu(x) v(x)\,ds = \int_{\Omega^\mu} g(x)\,v(x)\,dx$$

holds true for any function $v \in H^1(\Omega_\mu, \partial\Omega)$.

Here, we use the standard notation $H^1(\Omega^\mu, \partial\Omega)$ for the closure of the set of $C^\infty(\overline{\Omega}^\mu)$-functions vanishing in a neighborhood of $\partial\Omega$, by the $H^1(\Omega^\mu)$ norm.

In [1] we showed that $\theta = 1$ is a critical value for problem (9); in what follows we prove that the dissipation dominates if $\theta < 1$ and is neglectable if $\theta > 1$.

3.2 Subcritical case $\theta > 1$

This section deals with problem (9) in the case $\theta > 1$. Substituting the expression

$$u_\mu(x) = u_0(x) + \mu^{\theta-1} u_{1,-1}\left(x, \frac{x}{\mu}\right) + \dots + \mu u_{0,1}\left(x, \frac{x}{\mu}\right) + \mu^\theta u_{1,0}\left(x, \frac{x}{\mu}\right) + \\ + \mu^2 u_{0,2}(x, \frac{x}{\mu}) + \mu^{\theta+1} u_{1,1}(x, \frac{x}{\mu}) + \dots + \mu^{k\theta+l} u_{k,l}(x, \frac{x}{\mu}) + \dots \tag{10}$$

in equation (9) and taking into account an evident relation

$$\frac{\partial}{\partial x}\zeta\left(x, \frac{x}{\mu}\right) = \left(\frac{\partial}{\partial x}\zeta(x,\xi) + \frac{1}{\mu}\frac{\partial}{\partial \xi}\zeta(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}},$$

we obtain, after simple transformations, the following formal equality

$$
\begin{aligned}
-\frac{g(x)}{1+\alpha \mathrm{i}} = \Delta_x u_\mu(x) \cong \Delta_x u_0(x) + \mu^{\theta-1}\left(\Delta_x u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{\theta-2}\left(\nabla_x, \nabla_\xi u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+\mu^{\theta-3}\left(\Delta_\xi u_{1,-1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu\left(\Delta_x u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\left(\nabla_x, \nabla_\xi u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+\frac{1}{\mu}\left(\Delta_\xi u_{0,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu^{\theta}\left(\Delta_x u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{\theta-1}\left(\nabla_x, \nabla_\xi u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+\mu^{\theta-2}\left(\Delta_\xi u_{1,0}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu^{2}\left(\Delta_x u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu 2\left(\nabla_x, \nabla_\xi u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+\left(\Delta_\xi u_{0,2}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \mu^{\theta+1}\left(\Delta_x u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{\theta}\left(\nabla_x, \nabla_\xi u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+\mu^{\theta-1}\left(\Delta_\xi u_{1,1}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \dots + \mu^{k\theta+l}\left(\Delta_x u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + 2\mu^{k\theta+l-1}\left(\nabla_x, \nabla_\xi u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \\
+\mu^{k\theta+l-2}\left(\Delta_\xi u_{k,l}(x,\xi)\right)\Big|_{\xi=\frac{x}{\mu}} + \dots \qquad (11)
\end{aligned}
$$

Similarly, on S_μ we get

$$
\begin{aligned}
0 = \frac{\partial u_\mu}{\partial \nu_\mu} + \mu^{\theta}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_\mu \cong \left(\nabla_x u_0, \nu_\mu\right) + \mu^{\theta-1}\left(\nabla_x u_{1,-1}, \nu_\mu\right) + \mu^{\theta}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_0 + \dots + \\
+\mu^{\theta-2}\left(\nabla_\xi u_{1,-1}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{2\theta-1}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_{1,-1} + \mu\left(\nabla_x u_{0,1}, \nu_\mu\right) + \\
+\left(\nabla_\xi u_{0,1}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{\theta+1}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_{0,1} + \mu^{\theta}\left(\nabla_x u_{1,0}, \nu_\mu\right) + \mu^{\theta-1}\left(\nabla_\xi u_{1,0}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \\
+\mu^{2\theta}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_{1,0} + \mu^{2}\left(\nabla_x u_{0,2}, \nu_\mu\right) + \mu\left(\nabla_\xi u_{0,2}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{\theta+2}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_{0,2} + \\
+\mu^{\theta+1}\left(\nabla_x u_{1,1}, \nu_\mu\right) + \mu^{\theta}\left(\nabla_\xi u_{1,1}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{2\theta+1}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_{1,1} + \dots + \\
+\mu^{k\theta+l}\left(\nabla_x u_{k,l}, \nu_\mu\right) + \mu^{k\theta+l-1}\left(\nabla_\xi u_{k,l}\big|_{\xi=\frac{x}{\mu}}, \nu_\mu\right) + \mu^{(k+1)\theta+l}\frac{q\left(x,\frac{x}{\mu}\right)}{1+\alpha \mathrm{i}}u_{k,l} + \dots \qquad (12)
\end{aligned}
$$

Note that the normal vector ν_μ depends on x and $\frac{x}{\mu}$ in Ω_μ. Considering, as usually, x and $\xi = \frac{x}{\mu}$ as independent variables, we represent ν_μ in Ω_μ in the following form:

$$
\nu_\mu(x, \frac{x}{\mu}) = \widetilde{\nu}(x,\xi)\Big|_{\xi=\frac{x}{\mu}} + \mu\nu'_\mu(x,\xi)\Big|_{\xi=\frac{x}{\mu}},
$$

where $\widetilde{\nu}$ is a normal to $S(x) = \{\xi \mid F(x,\xi) = 0\}$,

$$
\nu'_\mu = \nu' + O(\mu).
$$

Collecting all the terms with like powers of μ in (11) and (12), we arrive at the following auxiliary problems:

$$
\begin{cases}
\Delta_\xi u_{1,-1}(x,\xi) = 0 & \text{in} \quad \omega, \\
\dfrac{\partial u_{1,-1}(x,\xi)}{\partial \nu} = 0 & \text{on} \quad S(x),
\end{cases} \qquad (13)
$$

$$\left\{\begin{array}{rcll} \Delta_\xi u_{1,0}\,(x,\xi) &=& -2\,(\nabla_\xi,\nabla_x u_{1,-1}(x,\xi)) & \text{in} \quad \omega, \\ \dfrac{\partial u_{1,0}(x,\xi)}{\partial\nu} &=& -\,(\nabla_x u_{1,-1}(x,\xi),\tilde\nu) & \text{on} \quad S(x), \end{array}\right. \tag{14}$$

and problem

$$\left\{\begin{array}{lll} \Delta_\xi u_{0,1}\,(x,\xi) = 0 & \text{in} & \omega, \\ \dfrac{\partial u_{0,1}(x,\xi)}{\partial\nu} = -\,(\nabla_x(u_0(x)),\tilde n) & \text{on} & S, \end{array}\right. \tag{15}$$

to be solved in the space of 1-periodic in ξ functions; here x is a parameter, $\omega := \Big\{\xi\in\mathbb{T}^d \,|\, F(x,\xi)>0\Big\}$. The problem (15) is the standard "cell" problem appearing in the case of Neumann conditions on the boundary of holes. The solvability condition

$$\int_{S(x)} (\nabla_x u_0(x),\tilde\nu(\xi))\;d\sigma = 0$$

for problem (15) is clearly satisfied, and its solution forms the first "internal" corrector in (10).

It follows from (13) that $u_{1,-1}$ does not depend on ξ. In fact, for our purposes, it suffices to put $u_{1,-1}\equiv 0$. Then $u_{1,0}\equiv 0$ solves (14).

In the next step, we collect all the terms of order μ^0 in (11) and of order μ^1 in (12). This yields

$$\left\{\begin{array}{rcll} \Delta_\xi u_{0,2}\,(x,\xi) &=& -\dfrac{g(x)}{1+\alpha \mathrm{i}} - \Delta_x u_0(x) - 2\,(\nabla_\xi,\nabla_x u_{0,1}(x,\xi)) & \text{in} \quad \omega, \\ \dfrac{\partial u_{0,2}(x,\xi)}{\partial\nu} &=& -\,(\nabla_x u_{0,1}(x,\xi),\tilde\nu) - (\nabla_\xi u_{0,1}(x,\xi),\nu') - (\nabla_x u_0(x),\nu') & \text{on} \quad S(x). \end{array}\right. \tag{16}$$

If we represent $u_{0,1}(x,\xi) = (\nabla_x u_0(x), M(x,\xi))$, where $M(x,\xi) = (M_1(x,\xi),\ldots,\,M_d(x,\xi))$ solves problem (7), then (16) takes the form

$$\left\{\begin{array}{rcll} \Delta_\xi u_{0,2}\,(x,\xi) &=& -\dfrac{g(x)}{1+\alpha \mathrm{i}} - \Delta_x u_0(x) - 2\displaystyle\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\;\frac{\partial M_i(x,\xi)}{\partial\xi_j} - & \\ && -2\displaystyle\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\frac{\partial^2 M_i(x,\xi)}{\partial\xi_j\,\partial x_j} & \text{in} \quad \omega, \\ \dfrac{\partial u_{0,2}(x,\xi)}{\partial\nu} &=& -\displaystyle\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j} M_i(x,\xi)\tilde\nu_j - \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\frac{\partial M_i(x,\xi)}{\partial x_j}\;\tilde\nu_j - & \\ && -\displaystyle\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\frac{\partial M_i(x,\xi)}{\partial\xi_j}\;\nu'_j - \sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\nu'_i & \text{on} \quad S(x). \end{array}\right.$$

Writing down the compatibility condition in the last problem, we get the following equation:

$$\int_{\square\cap\omega}\left(\frac{g(x)}{1+\alpha \mathrm{i}} + \Delta_x u_0(x) + 2\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j}\;\frac{\partial M_i(x,\xi)}{\partial\xi_j} + 2\sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\frac{\partial^2 M_i(x,\xi)}{\partial\xi_j\,\partial x_j}\right)d\xi =$$

$$= \int_Q\left(\sum_{i,j=1}^d \frac{\partial^2 u_0(x)}{\partial x_i\,\partial x_j} M_i(x,\xi)\tilde\nu_j + \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\frac{\partial M_i(x,\xi)}{\partial x_j}\;\tilde\nu_j + \right.$$

$$\left. + \sum_{i,j=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\frac{\partial M_i(x,\xi)}{\partial\xi_j}\;\nu'_j + \sum_{i=1}^d \frac{\partial u_0(x)}{\partial x_i}\;\nu'_i\right)d\sigma.$$

In the same way, as in [1] we find the homogenized problem:

$$\begin{cases} (1+\alpha \mathrm{i}) \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\right) + |\Box \cap \omega| g(x) = 0 \quad \text{in} \quad \Omega, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u_0(x) = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{17}$$

The integral identity for problem (17) reads

$$(1+\alpha \mathrm{i})\int_{\Omega} \sum_{i,j=1}^{d} \left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial \xi_j}\right\rangle \frac{\partial u_0(x)}{\partial x_i}\frac{v(x)}{x_j} = \int_{\Omega} |\Box \cap \omega|\, g(x)\, v(x)\, dx$$

for any function $v \in \overset{\circ}{H^1} (\Omega)$.

Theorem 3.1. Suppose that $g(x) \in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small μ problem (9) has the unique solution and the following convergence

$$\|u_0 - u_\mu\|_{H^1(\Omega_\mu)} \longrightarrow 0$$

takes place, where u_0 is a solution of the problem (17).

3.2.1 Auxiliary propositions

Lemma 3.1. Under the conditions of Theorem 3.1 the inequality

$$\int_{\Omega^\mu} |\nabla v|^2 dx + \mu^\theta \int_{S_\mu} q\left(x, \frac{x}{\mu}\right) v^2 ds \geq C_{13}\|v\|^2_{H^1(\Omega_\mu)}$$

holds for any $v \in H^1(\Omega_\mu, \partial\Omega)$.

Lemma 3.2. For any $v \in H^1(\Omega_\mu)$

$$\left|\int_{S_\mu} q\left(x, \frac{x}{\mu}\right) u_0(x)\, v(x)\ ds\right| \leq C_{14}\, \mu^{-1}\, \|u_0\|_{H^1(\Omega_\mu)}\|v\|_{H^1(\Omega_\mu)}.$$

Proof of the Theorem 3.1. The proof of this assertion is based on this lemma, and it can be found in [20].

We omit their proof.

3.3 Supercritical case $\theta < 1$

This section deals with problem (9) in the case $\theta < 1$. The following assertion is valid.

Theorem 3.2. Suppose that $g(x) \in C^1(\mathbb{R}^d)$ and that $q(x,\xi)$ is smooth enough nonnegative function. Then, for any sufficiently small μ problem (9) has the unique solution and the following convergence

$$\|u_\mu\|_{H^1(\Omega_\mu)} \longrightarrow 0$$

takes place as $\mu \to 0$.

Proof. Keeping in mind Lemma 5 from [21], we get from the integral identity the estimate

$$\|u_\mu\|_{H^1(\Omega_\mu)} \leq C.$$

Acting in the same way as in [21], we deduce

$$\int\limits_{\Omega_\mu} u_\mu^2\,dx \le C\Big(\mu\int\limits_{S_\mu} q\Big(x,\frac{x}{\mu}\Big)u_\mu^2 ds + \mu\|u_\mu\|^2_{H^1(\Omega_\mu)}\Big).$$

On the other hand,

$$\Big|\mu\int\limits_{S_\mu} q\Big(x,\frac{x}{\mu}\Big)u_\mu^2 ds\Big| \le \mu^{1-\theta}\|g(x)\|_{L_2(\Omega)}\|u_\mu\|_{L_2(\Omega_\mu)} + O(\mu^{1-\theta}).$$

Combining these estimates, bearing in mind the uniform boundedness of u_μ in $H^1(\Omega_\mu)$, we complete the proof.

4 The main assertion

4.1 The case $\theta > 1$

Theorem 4.1. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\mu \to \overline{\mathfrak{A}} \ \text{ as } \mu \to 0+\,. \tag{18}$$

Moreover,

$$\mathcal{K}_\mu \to \overline{\mathcal{K}} \ \text{ as } \mu \to 0+ \ \text{ in } \ \Theta^{loc}. \tag{19}$$

Remark 4.1. Recall that the functions from the sets $\mathfrak{A}_\mu$ and $\mathcal{K}_\mu$ are defined in the perforated domains Ω_μ. However, all these functions can be prolonged insides the holes in such a way that their norms in the spaces $\mathbf{H}, \mathbf{V}$, and $\mathbf{L}_p$ (without perforation) remain almost the same (are equivalent with the constants independent of the small parameter) as in the perforated spaces $\mathbf{H}_\mu, \mathbf{V}_\mu$, and $\mathbf{L}_{p,\mu}$ (the prolongation of functions defined in perforated domains, see, for instance, in [5; Ch.VIII]). So, in Theorem 4.1, we measure all the distances in the spaces without perforation.

Proof. It is clear that (19) implies (18). Therefore it is sufficient to prove (19), that is, for every neighbourhood $\mathcal{O}(\overline{\mathcal{K}})$ in Θ^{loc} there exists $\mu_1 = \mu_1(\mathcal{O}) > 0$ such that

$$\mathcal{K}_\mu \subset \mathcal{O}(\overline{\mathcal{K}}) \text{ for } \mu < \mu_1. \tag{20}$$

Suppose that (20) is not true. Then, there exists a neighbourhood $\mathcal{O}'(\overline{\mathcal{K}})$ in Θ^{loc}, a sequence $\mu_k \to 0+ \ (k \to \infty)$, and a sequence $u_{\mu_k}(\cdot) = u_{\mu_k}(s) \in \mathcal{K}_{\mu_k}$ such that

$$u_{\mu_k} \notin \mathcal{O}'(\overline{\mathcal{K}}) \ \text{ for all } k \in \mathbb{N}. \tag{21}$$

The function $u_{\mu_k}(s), s \in \mathbb{R}$ is the solutions to the problem

$$\begin{cases} \dfrac{\partial u_{\mu_k}}{\partial t} = (1+\alpha \mathrm{i})\Delta u_{\mu_k} + R\left(x,\dfrac{x}{\mu_k}\right) u_{\mu_k} - \left(1+\beta\left(x,\dfrac{x}{\mu_k}\right)\mathrm{i}\right)|u_{\mu_k}|^2 u_{\mu_k} + g\,(x)\,, & x\in\Omega_{\mu_k},\\ (1+\alpha \mathrm{i})\dfrac{\partial u_{\mu_k}}{\partial \nu} + \mu_k^\theta q\Big(x,\dfrac{x}{\mu_k}\Big)u_{\mu_k} = 0, & x\in S_{\mu_k}, t>0,\\ u_{\mu_k} = 0, & x\in\partial\Omega,\\ u_{\mu_k} = U(x), & x\in\Omega_{\mu_k}, t=0. \end{cases} \tag{22}$$

on the entire time axis $t \in \mathbb{R}$. To obtain the uniform in μ estimate of the solution, we use the following Lemmata (see [22; Ch. III, §5] and [23] respectively).

We obtain the estimate using the integral identity (3), by means of Lemma 1.1. More precise the sequence $\{u_{\mu_k}(x,s)\}$ is bounded in $\mathcal{F}^b$, that is,

$$\|u_{\mu_k}\|_{\mathcal{F}^b} = \sup_{t\in\mathbb{R}} \|u_{\mu_k}(t)\| + \\ + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\mu_k}(s)\|_1^2 ds\right)^{1/2} + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|u_{\mu_k}(s)\|_{\mathbf{L}_4}^4 ds\right)^{1/4} + \\ + \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \left\|\frac{\partial u_{\mu_k}}{\partial t}(s)\right\|_{\mathbf{H}^{-r}}^{4/3} ds\right)^{3/4} \leq C \text{ for all } k\in\mathbb{N}. \quad (23)$$

The constant C must not depend on μ.

Hence there exists a subsequence $\{u_{\mu_k'}(x,s)\} \subset \{u_{\mu_k}(x,s)\}$ which we label the same such that

$$u_{\mu_k}(x,s) \to u(s) \text{ as } k\to\infty \text{ in } \Theta^{loc},$$

where $u(x,s)\in\mathcal{F}^b$ and $u(s)$ satisfies (23) with the same constant C. Due to (23) we have $u_{\mu_k}(x,s) \rightharpoonup u(x,s)$ $(k\to\infty)$ weakly in $L_2^{loc}(\mathbb{R};\mathbf{V})$, weakly in $L_4^{loc}(\mathbb{R};\mathbf{L}_4)$, $*$-weakly in $L_\infty^{loc}(\mathbb{R}_+;\mathbf{H})$ and $\frac{\partial u_{\mu_k}(x,s)}{\partial t} \rightharpoonup \frac{\partial u(x,s)}{\partial t}$ $(k\to\infty)$ weakly in $L_{4/3,w}^{loc}(\mathbb{R};\mathbf{H}^{-r})$. We claim that $u(x,s)\in\overline{\mathcal{K}}$. We have already proved that $\|u\|_{\mathcal{F}^b}\leq C$. So we have to establish that $u(x,s)$ is a weak solution of (8).

According to the auxiliary problem in the case $\theta>1$, we have

$$(1+\alpha\mathrm{i})\int_{-M}^{M}\int_{\Omega_{\mu_k}} \nabla u_{\mu_k}\nabla\psi dxdt + \mu_k^\theta \int_{-M}^{M}\int_{S_{\mu_k}} q\Big(x,\frac{x}{\mu_k}\Big) u_{\mu_k}\psi d\sigma dt + \int_{-M}^{M}\int_{\Omega_{\mu_k}} g(x)\psi dxdt \to$$

$$(1+\alpha\mathrm{i})\int_{-M}^{M}\int_{\Omega}\sum_{i,j=1}^{d}\left\langle \delta_{ij} + \frac{\partial M_i(x,\xi)}{\partial\xi_j}\right\rangle \frac{\partial u_0(x,t)}{\partial x_i}\frac{\partial\psi}{\partial x_j} dxdt + \int_{-M}^{M}\int_{\Omega} |\square\cap\omega|\, g(x)\psi dxdt$$

as $k\to\infty$.

Let us prove that

$$R\left(x,\frac{x}{\mu_k}\right) u_{\mu_k}(x,s) \rightharpoonup \bar{R}(x)u(x,s) \quad (24)$$

and

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)|u_{\mu_k}(x,s)|^2 u_{\mu_k}(x,s) \rightharpoonup \left(1+\bar{\beta}(x)\mathrm{i}\right)|u(x,s)|^2 u(x,s) \quad (25)$$

as $k\to\infty$ weakly in $L_{4/3,w}^{loc}\left(\mathbb{R};\mathbf{L}_{4/3}\right)$.

We fix an arbitrary number $M>0$. The sequence $\{u_{\mu_k}(x,s)\}$ is bounded in $L_4(-M,M;\mathbf{L}_4)$ (see (23)). Then the sequence $\{|u_{\mu_k}(x,s)|^2 u_{\mu_k}(x,s)\}$ is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$. Since $\{u_{\mu_k}(x,s)\}$ is bounded in $L_2(-M,M;\mathbf{V})$ and $\left\{\dfrac{\partial u_{\mu_k}(x,s)}{\partial t}\right\}$ is bounded in $L_{4/3}(-M,M;\mathbf{H}^{-r})$ we can assume that $u_{\mu_k}(x,s)\to u(x,s)$ as $k\to\infty$ strongly in $L_2(-M,M;\mathbf{L}_2)$ and therefore

$$u_{\mu_k}(x,s)\to u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M).$$

It follows that

$$|u_{\mu_k}(x,s)|^2 u_{\mu_k}(x,s) \to |u(x,s)|^2 u(x,s) \text{ a.e. in } (x,s)\in\Omega\times(-M,M). \quad (26)$$

We have

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-\left(1+\bar\beta(x)\mathrm{i}\right)|u(x,s)|^2u(x,s)=$$
$$=\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)\left(|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-|u(x,s)|^2u(x,s)\right)+$$
$$+\left(\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)-\left(1+\bar\beta(x)\mathrm{i}\right)\right)|u(x,s)|^2u(x,s). \quad (27)$$

Let us show that both summand in the right-hand side of (27) converges to zero as $k\to\infty$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

The sequence

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)\left(|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-|u(x,s)|^2u(x,s)\right)$$

tends to zero as $k\to\infty$ almost everywhere in $(x,s)\in\Omega\times(-M,M)$ (see (26)) and is bounded in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ (see (2)). Therefore Lemma 1.3 from [24] implies that

$$\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)\left(|u_{\mu_k}(x,s)|^2u_{\mu_k}(x,s)-|u(x,s)|^2u(x,s)\right)\rightharpoonup 0 \text{ as } k\to\infty$$

weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

The sequence

$$\left(\left(1+\beta\left(x,\frac{x}{\mu_k}\right)\mathrm{i}\right)-\left(1+\bar\beta(x)\mathrm{i}\right)\right)|u(x,s)|^2u(x,s)$$

also approaches zero as $k\to\infty$ weakly in $L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$ because, by the assumption $\beta\left(x,\frac{x}{\mu}\right)\rightharpoonup\bar\beta(x)$ as $k\to\infty$ *-weakly in $L_{\infty,w}\left(-M,M;\mathbf{L}_2\right)$ and $|u(x,s)|^2u(x,s)\in L_{4/3}\left(-M,M;\mathbf{L}_{4/3}\right)$.

We have proved (25). The convergence (24) is proved similarly. Using (24) and (25), we pass to the limit in the equation (22) as $k\to\infty$ in the space $D'\left(\mathbb{R}_+;\mathbf{H}^{-r}\right)$ and obtain that the function $u(x,s)$ satisfies the equation (8).

Consequently, $u\in\overline{\mathcal{K}}$. We have proved above that $u_{\mu_k}\longrightarrow u$ as $k\to\infty$ в Θ^{loc}. Assumption $u_{\mu_k}\notin\mathcal{O}'(\overline{\mathcal{K}})$ (see (21)) implies $u\notin\mathcal{O}'(\overline{\mathcal{K}})$, and, hence, $u\notin\overline{\mathcal{K}}$. We arrive at the contradiction that completes the proof of the theorem.

4.2 The case $\theta<1$

Considering the convergence in Theorem 3.2, we get the following assertion.

Theorem 4.2. The following limit holds in the topological space Θ_+^{loc}

$$\mathfrak{A}_\mu\to 0 \text{ as } \mu\to 0+.$$

Moreover,

$$\mathcal{K}_\mu\to 0 \text{ as } \mu\to 0+ \text{ in } \Theta^{loc}.$$

Author Contributions

All authors contributed equally to this work.

Conflict of Interest

The authors declare no conflict of interest.

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Локальды периодты кеуектері бар орталарда Гинсбург-Ландау теңдеулерінің аттракторларының орташалау: суб- және суперкритикалық жағдайлары

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Теңдеуде және шекаралық шарттарында тез тербелмелі мүшелері бар Гинсбург-Ландау теңдеуі тесік облыста қарастырылған. Бұл теңдеудің траекториялық аттракторлары әлсіз мағынада «оғаш мүшесі» (әлеуеті) бар орташаланған Гинсбург-Ландау теңдеуінің траекториялық аттракторларына жуықтайтыны дәлелденді. Ол үшін В.В. Чепыжовтың және М.И. Вишиктің эволюциялық теңдеулердің траекториялық аттракторлары туралы мақалалары мен монографияларының әдістемесі қолданылған. Сондай–ақ, XX ғасырдың соңында пайда болған орташалау әдістері пайдаланылған. Алдымен асимптотикалық әдістерді асимптотиканы формальды құру үшін қолданамыз, содан кейін асимптотикалық қатарлардың негізгі мүшелерін функционалды талдау және интегралды бағалау әдістерін қолдана отырып таңдаймыз. Сәйкесінше, көмекші әлсіз топологиялы функционалды кеңістікті анықтай отырып, шекті (орташаланған) теңдеуін аламыз және осы теңдеу үшін траекториялық аттракторы бар екенін дәлелдейміз. Содан кейін негізгі теоремаларды тұжырымдап, оны көмекші леммалардың көмегімен дәлелдейміз. Бұл теңдеудің траекториялық аттракторлары субкритикалық жағдайда орташаланған Гинсбург-Ландау теңдеуінің траекториялық аттракторына әлсіз түрде жинақталатынын және суперкритикалық жағдайда жоғалып кететінін дәлелдейміз.

Кілт сөздер: аттракторлар, орташалау, Гинсбург-Ландау теңдеулері, сызықтық емес теңдеулер, әлсіз жинақтылық, тесік облыс, кеуекті орта.

Усреднение аттракторов уравнений Гинзбурга-Ландау в средах с локально периодическими препятствиями: суб- и суперкритические случаи

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Рассмотрено уравнение Гинзбурга-Ландау с быстро осциллирующими членами в уравнении и граничных условиях в перфорированной области. Приведено доказательство того, что траекторные аттракторы этого уравнения в слабом смысле сходятся к траекторным аттракторам усредненного уравнения Гинзбурга-Ландау. Для этого мы используем подход из статей и монографий В.В. Чепыжова и М.И. Вишика о траекторных аттракторах эволюционных уравнений, а также применяем методы усреднения, появившиеся в конце XX века. Сначала используем асимптотические методы для формального построения асимптотик, далее обосновываем вид главных членов асимптотических рядов с помощью методов функционального анализа и интегральных оценок. Определяя соответствующие вспомогательные функциональные пространства со слабой топологией, мы выводим предельное (усредненное) уравнение и доказываем существование траекторного аттрактора для этого уравнения. Затем формулируем основные теоремы и доказываем их с помощью вспомогательных лемм. Кроме того, доказываем, что траекторные аттракторы этого уравнения сходятся в слабом смысле к траекторным аттракторам усреднённого уравнения Гинзбурга-Ландау в субкритическом случае и исчезают — в суперкритическом.

Ключевые слова: аттракторы, усреднение, уравнения Гинзбурга-Ландау, нелинейные уравнения, слабая сходимость, перфорированная область, пористая среда.

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