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DESCRIPTION OF THE CLOSURE OF THE SET OF INFINITELY DIFFERENTIABLE COMPACTLY SUPPORTED FUNCTIONS IN A WEIGHTED SOBOLEV SPACE

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Abstract
The paper describes the closure of infinitely differentiable compactly supported functions in certain second-order weighted Sobolev space depending on the degree of singularity of weight functions at zero and at infinity.

Keywords Weight function · Weighted Sobolev space · Set of compactly supported functions · Closure · Singularity of weight function

Mathematics Subject Classification (2010) 46E35

Introduction

Let $I = (0, \infty)$ and $1 < p, q < \infty$. Let u, υ, and r be a.e. positive functions such that u and υ are locally integrable functions and r is a continuously differentiable function on the interval I. Moreover, $r^{-1} \in L_1^{loc}(I)$ and $\upsilon^{-p'} \in L_1^{loc}(I)$, where $p' = \frac{p}{p-1}$.

We assume that $D_r^2 f(t) = \frac{d}{dt} r(t) \frac{df(t)}{dt}$ and $D_r^1 f(t) = r(t) \frac{df(t)}{dt}$. Let $C_0^\infty(I)$ be the set of infinitely differentiable compactly supported functions.

Let $W^2_{p,\upsilon}(r, I)$ be a set of functions $f : I \to \mathbb{R}$, having generalized derivatives together with functions $D_r^1 f(t)$ on the interval I with the finite norm

$$\| f \|_{W^2_{p,\upsilon}(r,I)} = \|\upsilon D_r^2 f\|_p + |D_r^1 f(1)| + |f(1)|, \tag{1.1}$$

where $\| \cdot \|_p$ is the standard norm of the space $L_p(I)$.

By the conditions on the functions υ and r, we have that $C_0^\infty(I) \subset W^2_{p,\upsilon}(r, I)$. Denote by $\mathring{W}^2_{p,\upsilon}(r, I)$ the closure of the set $C_0^\infty(I)$ with respect to the norm (1.1).

Depending on the degree of singularity of the functions υ^{-1} and r^{-1} at zero and at infinity, the functions $f \in W^2_{p,\upsilon}(r, I)$ have finite limits $\lim\limits_{t\to 0^+} f(t) = f(0)$, $\lim\limits_{t\to 0^+} D_r^1 f(t) = D_r^1 f(0)$, $\lim\limits_{t\to\infty} f(t) = f(\infty)$ and $\lim\limits_{t\to\infty} D_r^1 f(t) = D_r^1 f(\infty)$ or do not have them. The

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elements of the set $\mathring{W}^2_{p,\upsilon}(r,I)$ can differ from the elements of the set $W^2_{p,\upsilon}(r,I)$ only by their behavior in the neighborhoods of zero and infinity. In other words, depending on the behavior of the weight functions υ^{-1} and r^{-1} at zero and at infinity, the set $\mathring{W}^2_{p,\upsilon}(r,I)$ can or can not coincide with the space $W^2_{p,\upsilon}(r,I)$. If not, then it is necessary to describe its direct complement to the entire space. The description of the closure of the set of infinitely differentiable compactly supported functions in various weighted Sobolev spaces is the subject of numerous papers (see, e.g., [1–7]). In this paper, in terms of boundary values, the elements of the set $\mathring{W}^2_{p,\upsilon}(r,I)$ are described depending on the degree of singularity of the functions υ^{-1} and r^{-1} at zero and at infinity, which allows posing boundary value problems for an equation with degenerative or singular coefficients at a finite point and at infinity.

Description of the set $\mathring{W}^2_{p,\upsilon}(r,I)$ in terms of elements of the set $W^2_{p,\upsilon}(r,I)$

Let $I_0=(0,1]$ and $I_\infty=[1,\infty)$. Let the sets $W^2_{p,\upsilon}(r,I_0)$ $(\mathring{W}^2_{p,\upsilon}(r,I_0))$ and $W^2_{p,\upsilon}(r,I_\infty)$ $(\mathring{W}^2_{p,\upsilon}(r,I_\infty))$ be the set of restrictions of functions $f\in W^2_{p,\upsilon}(r,I)$ $(f\in \mathring{W}^2_{p,\upsilon}(r,I))$ to the intervals I_0 and I_∞, respectively. In what follows, the functions $f\in W^2_{p,\upsilon}(r,I)$ and their restrictions to the intervals I_0 and I_∞ will be denoted by the same letter.
Let $\mathring{C}^\infty(I_0)$ and $\mathring{C}^\infty(I_\infty)$ be the set of restrictions of functions from $C_0^\infty(I)$ to the intervals I_0 and I_∞, respectively. Then, $\mathring{W}^2_{p,\upsilon}(r,I_i)$, as the set of restrictions of functions from $\mathring{W}^2_{p,\upsilon}(r,I_i)$ to the interval I_i, is the closure of the set $\mathring{C}^\infty(I_i)$ with respect to the norm

$$\|f\|_{W^2_{p,\upsilon}(r,I_i)}=\|\upsilon D^2_r f\|_{p,I_i}+|D^1_r f(1)|+|f(1)|$$

of the space $W^2_{p,\upsilon}(r,I_i)$, where $i=0,\infty$.
Assume that

$$\Phi_0(I_\infty)=\{c\chi_{I_\infty}(t):\ c\in\mathbb{R}\},$$

$$\Phi_1(I_\infty)=\{c\chi_{I_\infty}(t)\int_1^t r^{-1}(x)dx:\ c\in\mathbb{R}\},$$

$$\Phi_2(I_\infty)=\{c_1\chi_{I_\infty}(t)+c_2\chi_{I_\infty}(t)\int_1^t r^{-1}(x)dx:\ c_1\in\mathbb{R},\ c_2\in\mathbb{R}\},$$

where χ_{I_∞} is the characteristic function of the interval I_∞ and

$$R_0W^2_{p,\upsilon}(r,I_\infty)=\{f\in W^2_{p,\upsilon}(r,I_\infty)\ :f(\infty)=0\},$$

$$R_1W^2_{p,\upsilon}(r,I_\infty)=\{f\in W^2_{p,\upsilon}(r,I_\infty)\ :D^1_r f(\infty)=0\},$$

$$R_2W^2_{p,\upsilon}(r,I_\infty)=\{f\in W^2_{p,\upsilon}(r,I_\infty)\ :f(\infty)=D^1_r f(\infty)=0\}.$$

Theorem 2.1 *Let* $1<p<\infty$.

(i) *If* $\upsilon^{-1}\notin L_{p'}(I_\infty)$, $r^{-1}\notin L_1(I_\infty)$ *or* $r^{-1}\in L_1(I_\infty)$, $\int_1^\infty \upsilon^{-p'}(t)\left(\int_t^\infty r^{-1}(x)dx\right)^{p'}dt=\infty$, *then*
$$\mathring{W}^2_{p,\upsilon}(r,I_\infty)=W^2_{p,\upsilon}(r,I_\infty).$$

(ii) *If* $\upsilon^{-1} \notin L_{p'}(I_\infty)$, $r^{-1} \in L_1(I_\infty)$ *and* $\int_1^\infty \upsilon^{-p'}(t)\left(\int_t^\infty r^{-1}(x)dx\right)^{p'} dt < \infty$, *then*

$$\mathring{W}^2_{p,\upsilon}(r, I_\infty) = R_0 W^2_{p,\nu}(r, I_\infty) \quad \text{and} \quad W^2_{p,\nu}(r, I_\infty) = \mathring{W}^2_{p,\upsilon}(r, I_\infty) \dotplus \Phi_0(I_\infty).$$

(iii) *If* $\upsilon^{-1} \in L_{p'}(I_\infty)$, $r^{-1} \notin L_1(I_\infty)$ *and* $\int_1^\infty \upsilon^{-p'}(t)\left(\int_1^t r^{-1}(x)dx\right)^{p'} dt = \infty$, *then*

$$\mathring{W}^2_{p,\upsilon}(r, I_\infty) = R_1 W^2_{p,\nu}(r, I_\infty) \quad \text{and} \quad W^2_{p,\nu}(r, I_\infty) = \mathring{W}^2_{p,\upsilon}(r, I_\infty) \dotplus \Phi_1(I_\infty).$$

(iv) *If* $\upsilon^{-1} \in L_{p'}(I_\infty)$ *and* $r^{-1} \in L_1(I_\infty)$, *then*

$$\mathring{W}^2_{p,\upsilon}(r, I_\infty) = R_2 W^2_{p,\nu}(r, I_\infty) \quad \text{and} \quad W^2_{p,\nu}(r, I_\infty) = \mathring{W}^2_{p,\upsilon}(r, I_\infty) \dotplus \Phi_2(I_\infty).$$

Proof Let $f \in W^2_{p,\upsilon}(r, I_\infty)$ and $\upsilon^{-1} \in L_{p'}(I_\infty)$. Then

$$\int_1^\infty |D_r^2 f(t)|dt \le \|\upsilon^{-1}\|_{p',I_\infty} \|\nu D_r^2 f\|_p < \infty.$$

Therefore, there exists $D_r^1 f(\infty)$ and

$$|D_r^1 f(\infty)| \le \|f\|_{W^2_{p,\upsilon}(r,I_\infty)}. \tag{2.1}$$

If $f \in \mathring{W}^2_{p,\upsilon}(r, I_\infty)$, then there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathring{C}^\infty(I_\infty)$ such that

$$\lim_{n\to\infty} \|f - f_n\|_{W^2_{p,\upsilon}(r,I_\infty)} = 0. \tag{2.2}$$

From (2.1) and (2.2), it follows that $D_r^1 f(\infty) = 0$ for $f \in \mathring{W}^2_{p,\upsilon}(r, I_\infty)$.

Let $r^{-1} \in L_1(I_\infty)$ and

$$\int_1^\infty \upsilon^{-p'}(t)\left(\int_t^\infty r^{-1}(x)dx\right)^{p'} dt < \infty \tag{2.3}$$

hold. For $f \in W^2_{p,\upsilon}(r, I_\infty)$, we have $D_r^1 f(t) = \int_1^t D_r^2 f(s)ds + D_r^1 f(1)$. Due to (2.3), we get

$$\int_1^\infty \left|\frac{df(t)}{dt}\right|dt = \int_1^\infty \left|r^{-1}(t) \int_1^t D_r^2 f(s)ds + r^{-1}(t) D_r^1 f(1)\right|dt$$

$$\le \int_1^\infty |D_r^2 f(s)| \int_s^\infty r^{-1}(t)dt\, ds + |D_r^1 f(1)| \int_1^\infty r^{-1}(t)dt \le C\|f\|_{W^2_{p,\upsilon}(r,I_\infty)},$$

i.e., there exists the finite $f(\infty)$ and we obtain that $|f(\infty)| \le C\|f\|_{W^2_{p,\upsilon}(r,I_\infty)}$. Hence, due to (2.2), we have that $f(\infty) = 0$ for $f \in \mathring{W}^2_{p,\upsilon}(r, I_\infty)$. Then, from the conditions (ii), (iii), and (iv), we respectively deduce that $R_i W^2_{p,\nu}(r, I_\infty) \supset \mathring{W}^2_{p,\upsilon}(r, I_\infty)$, $i = 0, 1, 2$. In addition, the relation $\mathring{W}^2_{p,\upsilon}(r, I_\infty) \subset W^2_{p,\upsilon}(r, I_\infty)$ is obvious.

Now, for the proof of Theorem 2.1, it is enough to show that

$$R_i W^2_{p,\nu}(r, I_\infty) \subset \mathring{W}^2_{p,\upsilon}(r, I_\infty),\ i = 0, 1, 2, \quad \text{and} \quad W^2_{p,\nu}(r, I_\infty) \subset \mathring{W}^2_{p,\upsilon}(r, I_\infty) \tag{2.4}$$

hold under the conditions (ii), (iii), (iv), and (i), respectively.

Let $f \in \mathring{W}^2_{p,\upsilon}(r, I_\infty)$ and $\{g_n\}_{n=1}^\infty \subset \mathring{C}^\infty(I_\infty)$ such that

$$\lim_{n\to\infty} \|f - g_n\|_{W^2_{p,v}(r,I_\infty)} = 0. \tag{2.5}$$

Let $\varphi : I_\infty \to \mathbb{R}$ be an infinitely differentiable function such that $0 \le \varphi \le 1$, $\varphi(t) = 1$ for $1 \le t \le \frac{3}{2}$ and $\varphi(t) = 0$ for $t \ge 2$. The functions f and g_n, $n \ge 1$, can be represented as $f(t) = \tilde{f}(t) + \Psi(t,f)$ and $g_n(t) = \tilde{g}_n + \Psi(t, g_n)$, where

$$\Psi(t,f) = \varphi(t)\left(D^1_r f(1) \int_1^t r^{-1}(x)dx + f(1) \right). \tag{2.6}$$

From (2.6) and from

$$D^1_r \Psi(t,f) = r(t)\varphi'(t)\left(D^1_r f(1) \int_1^t r^{-1}(x)dx + f(1) \right) + \varphi(t) D^1_r f(1)$$

we have that $\tilde{f}(1) = D^1_r \tilde{f}(1) = 0$ and $\tilde{g}_n(1) = D^1_r \tilde{g}_n(1) = 0$. Moreover, $\Psi(t,f) = 0$ and $\Psi(t, g_n) = 0$ for $t \ge 2$ and

$$\|vD^2_r \Psi(\cdot, f - g_n)\|_{p,I_\infty} \le \|vD^2_r \varphi\|_{p,(1,2)} \left(|D^1_r f(1) - D^1_r g_n(1)| \int_1^2 r^{-1}(x)dx + |f(1) - g_n(1)| \right)$$

$$+ \|v\varphi'\|_{p,(1,2)} |D^1_r f(1) - D^1_r g_n(1)|.$$

The latter, together with (2.5), gives that $\lim_{n\to\infty} \|vD^2_r \Psi(\cdot, f - g_n)\|_{p,I_\infty} = 0$. Therefore, $\lim_{n\to\infty} \|\tilde{f} - \tilde{g}_n\|_{W^2_{p,v}(r,I_\infty)} = \lim_{n\to\infty} \|f - g_n\|_{W^2_{p,v}(r,I_\infty)} = 0$ and the behavior of the functions f and g_n at infinity determines the behavior of the functions $\tilde{f}$ and $\tilde{g}_n$, respectively. Thus, without loss of generality, we can assume that $f(1) = D^1_r f(1) = 0$ for $f \in W^2_{p,v}(r, I_\infty)$ and for $f \in \mathring{C}^\infty(I_\infty)$.

Based on this convention, the space norm $W^2_{p,v}(r, I_\infty)$ has the form $\|f\|_{W^2_{p,v}(r,I_\infty)} = \|vD^2_r f\|_p$. Then, assuming that $D^2_r f = g$ for $f \in W^2_{p,v}(r, I_\infty)$, we have $g \in L_{p,v}(I_\infty)$, and inversely, assuming that $f(t) = \int_1^t \int_s^t r^{-1}(x)dx g(s)ds$ for $g \in L_{p,v}(I_\infty)$, we get $f \in W^2_{p,v}(r, I_\infty)$. Moreover, this one-to-one correspondence is isometric. Consequently, for all $F \in \left(W^2_{p,v}(r, I_\infty)\right)^*$, there exists $h = h(F) \in L_{p',v^{-1}}(I_\infty)$ such that

$$F(f) = \int_1^\infty h(t) D^2_r f(t) dt, \quad \forall f \in W^2_{p,v}(r, I_\infty).$$

Let

$$\mathcal{B} = \left\{ F \in \left(W^2_{p,v}(r, I_\infty)\right)^* : F(\varphi) = 0, \ \forall \varphi \in \mathring{C}^\infty(I_\infty) \right\}.$$

The set $\mathcal{B}$ is the annihilator of $\mathring{C}^\infty(I_\infty)$, i.e., $\mathcal{B} = \left[\mathring{C}^\infty(I_\infty)\right]^\perp$. Due to the reflexivity of the space $W^2_{p,v}(r, I_\infty)$ and the density of $\mathring{C}^\infty(I_\infty)$ in $\mathring{W}^2_{p,v}(r, I_\infty)$, we get

$$\mathcal{B}^\perp = \left(\left[\mathring{C}^\infty(I_\infty)\right]^\perp \right)^\perp = \overline{\mathring{C}^\infty(I_\infty)} = \mathring{W}^2_{p,v}(r, I_\infty),$$

i.e.,

$$\mathring{W}^2_{p,v}(r, I_\infty) = \left\{ f \in W^2_{p,v}(r, I_\infty) : F(f) = 0, \ \forall F \in \mathcal{B} \right\}.$$

Whence, it follows that the inclusions (2.4) hold if and only if

$$F(f)=0,\ \ \forall f\in R_iW^2_{p,v}(r,I_\infty),\ \ i=0,1,2,$$

and

$$F(f)=0,\ \ \forall f\in W^2_{p,v}(r,I_\infty)$$

hold for all $F\in\mathcal{B}$. Since $F(\varphi)=0$, $\forall\varphi\in\mathring{C}^\infty(I_\infty)\supset C_0^\infty(I_\infty)$ for all $F\in\mathcal{B}$, then

$$F(\varphi)=\int_1^\infty h(t)D_r^2\varphi(t)dt=0,\ \ \forall\varphi\in C_0^\infty(I_\infty).$$

Therefore, the function h is a generalized solution of the equation

$$(D_r^2)^*h(t)=D_r^2h(t)=0. \tag{2.7}$$

The general solution to the equation (2.7) has the form

$$h(t)=\beta_0+\beta_1H(t),\ \ \beta_0\in\mathbb{R},\ \ \beta_1\in\mathbb{R}, \tag{2.8}$$

where $H(t)=\int_1^t r^{-1}(x)dx$ or $H(t)=\int_t^\infty r^{-1}(x)dx$ if $r^{-1}\in L_1(I_\infty)$.

By the condition, we have that $h\in L_{p',v^{-1}}(I_\infty)$. From the conditions of (i) of Theorem 2.1, it is easy to see that $h\in L_{p',v^{-1}}(I_\infty)$ if and only if $\beta_0=\beta_1=0$, i.e., $h(t)\equiv 0$. Then, $F(f)=0$, $\forall f\in W^2_{p,v}(r,I_\infty)$ for all $F\in\mathcal{B}$. Hence,

$$\mathring{W}^2_{p,v}(r,I_\infty)\supset W^2_{p,v}(r,I_\infty).$$

Therefore, under the conditions of (i) of Theorem 2.1, we have that

$$\mathring{W}^2_{p,v}(r,I_\infty)\equiv W^2_{p,v}(r,I_\infty) \tag{2.9}$$

holds. Let us show that (ii) of Theorem 2.1 also holds. By the conditions of (ii) of Theorem 2.1, we have that $v^{-1}\notin L_{p'}(I_\infty)$, and from (2.8), it follows that $\beta_0=0$, while from $r^{-1}\in L_1(I_\infty)$ and (2.3), we obtain that $h(t)=\beta_1\int_t^\infty r^{-1}(x)dx\in L_{p',v^{-1}}(I_\infty)$. Then, for all $f\in R_0W^2_{p,v}(r,I_\infty)$ and for all $F\in(W^2_{p,v}(r,I_\infty))^*$, we get

$$F(f)=\beta_1\int_1^\infty\left(\int_t^\infty r^{-1}(x)dx\right)D_r^2f(t)dt=\beta_1D_r^1f(t)\int_t^\infty r^{-1}(x)dx\Big|_1^\infty-\beta_1\int_1^\infty f'(t)dt$$

$$=\beta_1\lim_{t\to\infty}D_r^1f(t)\int_t^\infty r^{-1}(x)dx. \tag{2.10}$$

If the last limit is equal to zero, then for all $f\in R_0W^2_{p,v}(r,I_\infty)$ and for all $F\in\mathcal{B}$, we obtain that $F(f)=0$, i.e., $\mathring{W}^2_{p,v}(r,I_\infty)\supset R_0W^2_{p,v}(r,I_\infty)$. Therefore,

$$\mathring{W}^2_{p,v}(r,I_\infty)\equiv R_0W^2_{p,v}(r,I_\infty)\quad\text{and}\quad W^2_{p,v}(r,I_\infty)=\mathring{W}^2_{p,v}(r,I_\infty)\dotplus\Phi_0(I_\infty). \tag{2.11}$$

Let us show that the limit (2.10) is equal to zero. Since, due to (2.3), for $f\in W^2_{p,v}(r,I_\infty)$, we have that

$$\int_1^\infty r^{-1}(x)\int_1^t|D_r^2f(t)|dtdx=\int_1^\infty\left(\int_t^\infty r^{-1}(x)dx\right)|D_r^2f(t)|dt<\infty,$$

then $\lim_{z\to\infty}\int_z^\infty r^{-1}(x)\int_1^t |D_r^2 f(t)|dtdx = 0$. Hence, the equality to zero of the limit (2.10) follows from the relation

$$|D_r^1 f(t)|\int_t^\infty r^{-1}(x)dx \le \int_1^t |D_r^2 f(s)|ds\int_t^\infty r^{-1}(x)dx \le \int_t^\infty r^{-1}(x)\int_1^x |D_r^2 f(t)|dtdx.$$

Therefore, the proof of the statement (ii) of Theorem 2.1 is complete.

Let us turn to the statement (iii) of Theorem 2.1. From $v^{-1}\in L_{p'}(I_\infty)$ and

$$\int_1^\infty v^{-p'}(t)\left(\int_1^t r^{-1}(x)dx\right)^{p'}dt = \infty,$$

it follows that $h(t)=\beta_0$. Therefore, for all $f\in R_1W^2_{p,v}(r,I_\infty)$ and for all $F\in\mathcal{B}$, we have

$$F(f) = \beta_0\int_1^\infty D_r^2 f(s)ds = \beta_0(D_r^1 f(\infty) - D_r^1 f(1)) = 0,$$

which shows that $\mathring{W}^2_{p,v}(r,I_\infty)\supset R_1W^2_{p,v}(r,I_\infty)$, i.e.,

$$\mathring{W}^2_{p,v}(r,I_\infty)\equiv R_1W^2_{p,v}(r,I_\infty) \quad\text{and}\quad W^2_{p,v}(r,I_\infty) = \mathring{W}^2_{p,v}(r,I_\infty)\dotplus\Phi_1(I_\infty). \tag{2.12}$$

Thus, the proof of the statement (iii) of Theorem 2.1 is complete.

Now, consider the last statement (iv) of Theorem 2.1. Let the conditions of the statement (iv) hold. Then, $h(t)=\beta_0+\beta_1\int_1^t r^{-1}(x)dx\in L_{p',v^{-1}}(I_\infty)$, and for all $F\in\mathcal{B}$ and for all $f\in R_2W^2_{p,v}(r,I_\infty)$, we have

$$F(f) = \int_1^\infty h(t)D_r^2 f(t)dt = \int_1^\infty\left(\beta_0+\beta_1\int_1^t r^{-1}(x)dx\right)D_r^2 f(t)dt$$

$$= \beta_1\int_1^\infty\left(\int_1^t r^{-1}(x)dx\right)D_r^2 f(t)dt = \beta_1\left(D_r^1 f(t)\int_1^t r^{-1}(x)dx\Big|_1^\infty - \int_1^\infty f'(t)dt\right)$$

$$= \beta_1\lim_{t\to\infty} D_r^1 f(t)\int_1^t r^{-1}(x)dx = 0,$$

which gives that $\mathring{W}^2_{p,v}(r,I_\infty)\supset R_2W^2_{p,v}(r,I_\infty)$. Therefore,

$$\mathring{W}^2_{p,v}(r,I_\infty)\equiv R_2W^2_{p,v}(r,I_\infty) \quad\text{and}\quad W^2_{p,v}(r,I_\infty) = \mathring{W}^2_{p,v}(r,I_\infty)\dotplus\Phi_2(I_\infty). \tag{2.13}$$

The proof of the statement (iv) of Theorem 2.1 is complete. Combining (2.9), (2.11), (2.12), and (2.13), we get that all statements of Theorem 2.1 hold. The proof of Theorem 2.1 is complete. □

Now, consider the space $W^2_{p,v}(r,I_0)$. Let

$$\mathcal{L}_0W^2_{p,v}(r,I_0) = \left\{f\in W^2_{p,v}(r,I_0) : f(0) = D_r^1(0) = 0\right\},$$

$$\mathcal{L}_1W^2_{p,v}(r,I_0) = \left\{f\in W^2_{p,v}(r,I_0) : D_r^1(0) = 0\right\},$$

$$\mathcal{L}_0 W^2_{p,v}(r,I_0) = \{f \in W^2_{p,v}(r,I_0) : f(0)=0\}.$$

Assume that

$$\Phi_0(I_0) = \{c\chi_{I_0}(t) : c \in \mathbb{R}\},$$

$$\Phi_1(I_0) = \{c\chi_{I_0}(t) \int_t^1 r^{-1}(x)dx : c \in \mathbb{R}\},$$

$$\Phi_2(I_0) = \{c_1\chi_{I_0}(t) + c_2\chi_{I_0}(t) \int_t^1 r^{-1}(x)dx : c_1 \in \mathbb{R}, c_2 \in \mathbb{R}\}.$$

In the same way as Theorem 2.1, one can prove

Theorem 2.2 *Let* $1 < p < \infty$.

(i) *If* $\upsilon^{-1} \notin L_{p'}(I_0)$, $r^{-1} \notin L_1(I_0)$ *or* $r^{-1} \in L_1(I_0)$, $\int_0^t \upsilon^{-p'}(t)\left(\int_0^t r^{-1}(x)dx\right)^{p'} dx = \infty$, *then*

$$\mathring{W}^2_{p,\upsilon}(r,I_0) = W^2_{p,\upsilon}(r,I_0).$$

(ii) *If* $\upsilon^{-1} \notin L_{p'}(I_0)$, $r^{-1} \in L_1(I_0)$ *and* $\int_0^1 \upsilon^{-p'}(t)\left(\int_0^t r^{-1}(x)dx\right)^{p'} dt < \infty$, *then*

$$\mathring{W}^2_{p,\upsilon}(r,I_0) = \mathcal{L}_0 W^2_{p,\upsilon}(r,I_0) \quad \text{and} \quad W^2_{p,\upsilon}(r,I_0) = \mathring{W}^2_{p,\upsilon}(r,I_0) \dotplus \Phi_0(I_0).$$

(iii) *If* $\upsilon^{-1} \in L_{p'}(I_0)$, $r^{-1} \notin L_1(I_0)$ *and* $\int_0^1 \upsilon^{-p'}(t)\left(\int_t^1 r^{-1}(x)dx\right)^{p'} dt = \infty$, *then*

$$\mathring{W}^2_{p,\upsilon}(r,I_0) = \mathcal{L}_1 W^2_{p,\upsilon}(r,I_0) \quad \text{and} \quad W^2_{p,\upsilon}(r,I_0) = \mathring{W}^2_{p,\upsilon}(r,I_0) \dotplus \Phi_1(I_0).$$

(iv) *If* $\upsilon^{-1} \in L_1(I_0)$ *and* $r^{-1} \in L_1(I_0)$, *then*

$$\mathring{W}^2_{p,\upsilon}(r,I_0) = \mathcal{L}_2 W^2_{p,\upsilon}(r,I_0) \quad \text{and} \quad W^2_{p,\upsilon}(r,I_0) = \mathring{W}^2_{p,\upsilon}(r,I_0) \dotplus \Phi_2(I_0).$$

General case

Now, combining the statements of Theorems 2.1 and 2.2, we obtain a description of the set $\mathring{W}^2_{p,\upsilon}(r,I)$ in terms of elements of the set $W^2_{p,\upsilon}(r,I)$.

Let $\psi_0(t)$ and $\psi_\infty(t)$ be infinitely differentiable functions on I such that $\psi_0(t)=1$, $\psi_\infty(t)=0$ for $t \in [0,\frac{1}{2}]$, $\psi_0(t)=0$, $\psi_\infty(t)=1$ for $t \in [2,\infty)$, $\psi_0(t) > 0$, $\psi_\infty(t) > 0$ for $t \in (\frac{1}{2},2)$, and $\psi_(t) + \psi_\infty(t) = 1$ for all $t \in I$.

Theorem 3.1 *Let* $1 < p < \infty$ *and the conditions of the statement* (*iv*) *of Theorem* 2.2 *hold*.

(i) If the conditions of the statement (i) of Theorem 2.1 hold, then

$$\mathring{W}^2_{p,v}(r,I) = \left\{ f \in W^2_{p,v}(r,I) : \ f(0) = D^1_r f(0) = 0 \right\}$$

and

$$W^2_{p,v}(r,I) = \mathring{W}^2_{p,v}(r,I) \dotplus \psi_0 \Phi_2(I_0).$$

(ii) If the conditions of the statement (ii) of Theorem 2.1 hold, then

$$\mathring{W}^2_{p,v}(r,I) = \left\{ f \in W^2_{p,v}(r,I) : \ f(0) = D^1_r f(0) = f(\infty) = 0 \right\}$$

and

$$W^2_{p,v}(r,I) = \mathring{W}^2_{p,v}(r,I) \dotplus \psi_0 \Phi_2(I_0) \dotplus \psi_\infty \Phi_0(I_\infty).$$

(iii) If the conditions of the statement (iii) of Theorem 2.1 hold, then

$$\mathring{W}^2_{p,v}(r,I) = \left\{ f \in W^2_{p,v}(r,I) : \ f(0) = D^1_r f(0) = D^1_r f(\infty) = 0 \right\}$$

and

$$W^2_{p,v}(r,I) = \mathring{W}^2_{p,v}(r,I) \dotplus \psi_0 \Phi_2(I_0) \dotplus \psi_\infty \Phi_1(I_\infty).$$

(iv) If the conditions of the statement (iv) of Theorem 2.1 hold, then

$$\mathring{W}^2_{p,v}(r,I) = \left\{ f \in W^2_{p,v}(r,I) : \ f(0) = D^1_r f(0) = f(\infty) = D^1_r f(\infty) = 0 \right\}$$

and

$$W^2_{p,v}(r,I) = \mathring{W}^2_{p,v}(r,I) \dotplus \psi_0 \Phi_2(I_0) \dotplus \psi_\infty \Phi_2(I_\infty). \tag{3.1}$$

Proof The statement (**i**) is obvious since it follows from the statement (*iv*) of Theorem 2.2. Let us prove the statement (**iv**). In order to prove this, it is enough to show that (3.1) holds. Let $f \in W^2_{p,v}(r,I)$ and the statement (*iv*) of Theorem 2.2 and the statement (*iv*) of Theorem 2.1 hold. Then, these statements imply that $\varphi_0 \in \Phi_2(I_0)$ and $\varphi_\infty \in \Phi_2(I_\infty)$ exist and

$$\lim_{t\to 0^+}(f(t) - \varphi_0(t)) = \lim_{t\to 0^+} D^1_r(f(t) - \varphi_0(t)) = 0,$$

$$\lim_{t\to \infty}(f(t) - \varphi_\infty(t)) = \lim_{t\to \infty} D^1_r(f(t) - \varphi_\infty(t)) = 0$$

hold. Since $f(t) = \psi_0(t)f(t) + \psi_\infty(t)f(t)$ and $D^1_r f(t) = \psi_0(t)D^1_r f(t) + \psi_\infty(t)D^1_r f(t)$ for all $t \in I$, then

$$\lim_{t\to 0^+}(f(t) - \psi_0(t)\varphi_0(t)) = \lim_{t\to 0^+} \psi_0(t)(f(t) - \varphi_0(t)) = 0,$$

$$\lim_{t\to 0^+} D^1_r(f(t) - \psi_0(t)\varphi_0(t)) = \lim_{t\to 0^+} \psi_0(t)D^1_r(f(t) - \varphi_0(t)) = 0.$$

Similarly, $\lim_{t\to\infty}(f(t) - \psi_\infty(t)\varphi_\infty(t)) = \lim_{t\to\infty} D^1_r(f(t) - \psi_\infty(t)\varphi_\infty(t)) = 0.$

Therefore, for the function $g_f(t) = f(t) - \psi_0(t)\varphi_0(t) - \psi_\infty(t)\varphi_\infty(t)$, we get that $g_f(0) = D_r^1 g_f(0) = g_f(\infty) = D_r^1 g_f(\infty) = 0$, i.e., $g_f \in \mathring{W}^2_{p,v}(r, I)$ and $f(t) = g_f(t) + \psi_0(t)\varphi_0(t) + \psi_\infty(t)\varphi_\infty(t)$. Since the sets $\mathring{W}^2_{p,v}(r, I)$, $\psi_0\Phi_2(I_0)$, and $\psi_\infty\Phi_2(I_\infty)$ are pairwise disjoint, the right-hand side of (3.1) contains the left-hand side.

Inversely, let $g \in \mathring{W}^2_{p,v}(r, I)$ and $\bar{\varphi}_0 \in \Phi_2(I_0)$, $\bar{\varphi}_\infty \in \Phi_2(I_\infty)$. We assume that $\bar{f} = g + \bar{\psi}$, where $\bar{\psi} = \psi_0\bar{\varphi}_0 + \psi_\infty\bar{\varphi}_\infty$. Since ψ_0 and ψ_∞ are infinitely differentiable functions, we have

$$\int_1^\infty |vD_r^2\bar{\psi}(t)|^p dt = \int_{1/2}^2 |vD_r^2\bar{\psi}(t)|^p dt \le \sup_{1/2\le t\le 2} |D_r^2\bar{\psi}(t)|^p \int_{1/2}^2 v^p(t)dt < \infty.$$

Hence, $\bar{f} \in W^2_{p,v}(r, I)$ and the left-hand side of (3.1) contains the right-hand side. The proof of the equality (3.1) is complete. The other statements of Theorem 3.1 can be proved in the same way. The proof of Theorem 3.1 is complete. □

Similarly, we have

Theorem 3.2 *Let* $1 < p < \infty$ *and the conditions of the statement* (*iv*) *of Theorem* 2.1 *hold.*

(**i**) If the conditions of the statement (i) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,v}(r, I) = \left\{f \in W^2_{p,v}(r, I) :\ f(\infty) = D_r^1 f(\infty) = 0\right\}$$

and

$$W^2_{p,v}(r, I) = \mathring{W}^2_{p,v}(r, I) \dotplus \psi_\infty\Phi_2(I_\infty).$$

(**ii**) If the conditions of the statement (ii) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,v}(r, I) = \left\{f \in W^2_{p,v}(r, I) :\ f(0) = D_r^1 f(\infty) = f(\infty) = 0\right\}$$

and

$$W^2_{p,v}(r, I) = \mathring{W}^2_{p,v}(r, I) \dotplus \psi_0\Phi_0(I_0) \dotplus \psi_\infty\Phi_2(I_\infty).$$

(**iii**) If the conditions of the statement (iii) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,v}(r, I) = \left\{f \in W^2_{p,v}(r, I) :\ D_r^1 f(0) = f(\infty) = D_r^1 f(\infty) = 0\right\}$$

and

$$W^2_{p,v}(r, I) = \mathring{W}^2_{p,v}(r, I) \dotplus \psi_0\Phi_1(I_0) \dotplus \psi_\infty\Phi_2(I_\infty).$$

The following theorem collects cases where only one of $f(t)$, $D_r^1 f(t)$ has a finite value at zero or at infinity.

Theorem 3.3 *Let* $1 < p < \infty$.

(**i**) If the conditions of the statement (ii) of Theorems 2.1 and 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{f \in W^2_{p,\upsilon}(r,I) : \ f(0) = f(\infty) = 0\right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0\Phi_0(I_0) \dotplus \psi_\infty\Phi_0(I_\infty).$$

(ii) If the conditions of the statements (iii) of Theorems 2.1 and 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{f \in W^2_{p,\upsilon}(r,I) : \ D^1_r f(0) = D^1_r f(\infty) = 0\right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0\Phi_1(I_0) \dotplus \psi_\infty\Phi_1(I_\infty).$$

(iii) If the conditions of the statement (ii) of Theorem 2.1 and the conditions of the statement (iii) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{f \in W^2_{p,\upsilon}(r,I) : \ f(0) = D^1_r f(\infty) = 0\right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0\Phi_0(I_0) \dotplus \psi_\infty\Phi_1(I_\infty).$$

(iv) If the conditions of the statement (iii) of Theorem 2.1 and the conditions of the statement (ii) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{f \in W^2_{p,\upsilon}(r,I) : \ D^1_r f(0) = f(\infty) = 0\right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0\Phi_1(I_0) \dotplus \psi_\infty\Phi_0(I_\infty).$$

The last case is as follows.

Theorem 3.4 *Let* $1 < p < \infty$. *Then*,

(i) If the conditions of the statement (i) of Theorem 2.1 and the conditions of the statement (ii) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{f \in W^2_{p,\upsilon}(r,I) : \ f(0) = 0\right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0\Phi_0(I_0).$$

(**ii**) If the conditions of the statement (i) of Theorem 2.1 and the conditions of the statement (iii) of Theorem 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{ f \in W^2_{p,\upsilon}(r,I) : \; D^1_r f(0) = 0 \right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0 \Phi_1(I_0).$$

(**iii**) If the conditions of the statement (i) of Theorem 2.2 and the conditions of the statement (ii) of Theorem 2.1 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{ f \in W^2_{p,\upsilon}(r,I) : \; f(\infty) = 0 \right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_\infty \Phi_0(I_\infty).$$

(**iv**) If the conditions of the statement (i) of Theorem 2.2 and the conditions of the statement (iii) of Theorem 2.1 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = \left\{ f \in W^2_{p,\upsilon}(r,I) : \; D^1_r f(\infty) = 0 \right\}$$

and

$$W^2_{p,\upsilon}(r,I) = \mathring{W}^2_{p,\upsilon}(r,I) \dotplus \psi_0 \Phi_1(I_0) \dotplus \psi_\infty \Phi_1(I_\infty)$$

(**v**) If the conditions of the statements (i) of Theorems 2.1 and 2.2 hold, then

$$\mathring{W}^2_{p,\upsilon}(r,I) = W^2_{p,\upsilon}(r,I).$$

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