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ON THE BOUNDEDNESS OF A GENERALIZED FRACTIONAL-MAXIMAL OPERATOR IN LORENTZ SPACES

In this paper considers a generalized fractional-maximal operator, a special case of which is the classical fractional-maximal function. Conditions for the function Φ, which defines a generalized fractional-maximal function, and for the weight functions w and v, which determine the weighted Lorentz spaces $\Lambda_p(v)$ and $\Lambda_q(w)$ $(1 < p \leq q < \infty)$ under which the generalized maximal-fractional operator is bounded from one Lorentz space $\Lambda_p(v)$ to another Lorentz space $\Lambda_q(w)$ are obtained. For the classical fractional maximal operator and the classical maximal Hardy-Littlewood function such results were previously known. When proving the main result, we make essential use of an estimate for a nonincreasing rearrangement of a generalized fractional-maximal operator. In addition, we introduce a supremal operator for which conditions of boundedness in weighted Lebesgue spaces are obtained. This result is also essentially used in the proof of the main theorem.
Key words: fractional-maximal function, non-increasing rearrangement, generalized fractional-maximal operator, weighted Lorentz spaces, supremal operator.

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Жалпыланған бөлшекті-максималды оператордың Лоренц кеңістіктеріндегі шенелгендігі

Жұмыста жалпыланған бөлшекті-максималды оператор қарастырылады, оның дербес жағдайы классикалық бөлшекті-максималды функция болып табылады. Жалпыланған бөлшекті-максималды функцияны анықтайтын Φ функциясы үшін және $\Lambda_p(v)$ және $\Lambda_q(w)$ $(1 < p \leq q < \infty)$ салмақты Лоренц кеңістіктерін анықтайтын w және v салмақты функциялары үшін жалпыланған бөлшекті-максималды оператор бір Лоренц $\Lambda_p(v)$ кеңістігінен басқа Лоренц $\Lambda_q(w)$ кеңістігіне шенелген болуының шарттары алынған. Классикалық бөлшекті-максималды оператор және классикалық Харди-Литлвуд максималды функциясы үшін мұндай нәтижелер бұрын белгілі болды. Негізгі нәтижені дәлелдеу кезінде жалпыланған бөлшекті-максималды оператордың өспейтін алмастыруының бағалауы маңызды түрде пайдаланылады. Сонымен қатар, біз өлшемді Лебег кеңістіктерінде шенелгендік шарттары алынған супремалды операторды енгіземіз. Бұл нәтиже негізгі теореманы дәлелдеуде де қолданылады.
Түйін сөздер: бөлшекті-максималды функция, өспейтін алмастыру, жалпыланған бөлшекті-максималды оператор, салмақты Лоренц кеістіктері, максималды оператор.

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Об ограниченности обобщенного дробно-максимального оператора в пространствах Лоренца

В работе рассматривается обобщенный дробно-максимальный оператор, частным случаем которого является классическая дробно-максимальная функция. Получены условия на функцию Φ, определяющую обобщенную дробно-максимальную функцию, и на весовые функции w и v определяющие весовые пространства Лоренца $\Lambda_p(v)$ и $\Lambda_q(w)$ $(1 < p \leq q < \infty)$, при которых обобщенный дробно-максимальный оператор является ограниченным из одного пространства Лоренца $\Lambda_p(v)$ в другое пространство Лоренца $\Lambda_q(w)$. Для классического дробно-максимального оператора и классической максимальной функции Харди-Литтлвуда такие результаты ранее были известны. При доказательстве основного результата существенно используется оценка невозрастающей перестановки обобщенного дробно-максимального оператора. Кроме того, в рассмотрение вводится супремальный оператор, для которого получены условия ограниченности в весовых пространствах Лебега. Этот результат так же существенно используется при доказательстве основной теоремы.
Ключевые слова: дробно-максимальная функция, неовзрастающая перестановка, обобщенный дробно-максимальный оператор, весовые пространства Лоренца, супремальный оператор.

1 Introduction

The classical Hardy-Littlewood maximal operator $M := M_0$ for $f \in L^1_{Loc}(\mathbb{R}^n)$ is defined by

$$(Mf)(x) = \sup_{r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is open ball from $\mathbb{R}^n$ with the center on the point $x \in \mathbb{R}^n$ and radius $r > 0$.

The classical fractional maximal operator for $\alpha \in [0,n)$ is defined at $f \in L^1_{Loc}(\mathbb{R}^n)$ by

$$(M_\alpha f)(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int\limits_{B(x,r)} |f(y)|dy, \quad (0 \leq \alpha < n),$$

when $\alpha = 0$ we get $M_0 = M$.

Definition 1. Let $\Phi : (0,\infty) \to (0,\infty)$. The generalized fractional maximal function $M_\Phi f$ is defined for the function $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$(M_\Phi f)(x) = \sup_{r>0} \Phi(r) \int\limits_{B(x,r)} |f(y)|dy.$$

The generalized fractional-maximal function in this form was defined in [1-2]. In the case $\Phi(r) = r^{\alpha-n}$, $\alpha \in (0;n)$ we obtain the classical fractional maximal function $M_\alpha f$.

The boundedness of the classical maximal operator and the classical fractional maximal operator in the Lebesgue spaces $L_p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ are well known [3-6].

The boundedness of the Hardy-Littlewood maximal operator M in the classical Lorentz Space $\Lambda_p(v)$ was considered in [7], the boundedness of the classical fractional-maximal operator in Lorentz spaces was considered in [8].

Note that various variants of the generalized fractional-maximal function were previously considered in [9-15]. For such variants of a generalized fractional-maximal function, the questions of boundedness in Lorentz spaces were considered in [8], [11]. In this article, for a generalized fractional-maximal function $M_\Phi f$, we obtain boundedness conditions from the Lorentz space $\Lambda_p(v)$ to another Lorentz space $\Lambda_q(w)$.

2 Preliminaries

Let $L_0 = L_0(\mathbb{R}^n)$ be the set of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$; $\dot{L}_0 = \dot{L}_0(\mathbb{R}^n)$ be the set of functions $f \in L_0$, for which the non-increasing rearrangement of the f^* is not identical to infinity. Non-increasing rearrangement f^* for the function f defined by the equality:

$$f^*(t) = inf\{y \in [0;\infty) : \ \ \lambda_f(y) \le t\}, \ \ t \in \mathbb{R}_+ = (0;\infty),$$

where

$$\lambda_f(y) = \mu_n \left\{x \in \mathbb{R}^n : \ \ |f(x)| > y\right\}, \quad y \in [0,\infty)$$

is the Lebesgue distribution function.

We denote by $L_0^+(0,\infty)$ the set of all nonnegative measurable functions on $(0,\infty)$, and by $L_0^+(0,\infty;\downarrow)$ the set of all nonincreasing functions from $L_0^+(0,\infty)$. The symbol $\chi_{(a,b)}$ stands for the characteristic function of an interval $(a,b) \subset (0,\infty)$. We use the letter C for a positive constant, independent of appropriate parameters and not necessarily the same at each occurrence.

The function $f^{**}(t)$ is defined by the following equality:

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(\tau)d\tau, \ t \in \mathbb{R}_+.$$

It is known that $0 \le f^* \downarrow;\ f^*(t+0) = f^*(t),\ t \in \mathbb{R}_+;\ f^*$ is equally measurable with $|f|$, i.e.

$$\mu_1 \left\{t \in \mathbb{R}_+ : \ f^*(t) > y\right\} = \mu_n \left\{x \in \mathbb{R}^n : \ |f(x)| > y\right\},$$

here μ - is the Lebesgue measure (in $\mathbb{R}^n$ or on $\mathbb{R}_+$, respectively, see [5]).

A function $\Phi : (0;\infty) \to (0;\infty)$ is said to be quasi-increasing (quasi-decreasing), if for some $C > 0$

$$\Phi(t_1) \le C\Phi(t_2) \quad \big(\Phi(t_2) \le C\Phi(t_1)\big)$$

holds whenever $0 < t_1 \le t_2 < \infty$.

Definition 2 ([1]). Let $n \in \mathbb{N}$ and $R \in (0;\infty]$. A function $\Phi : (0;R) \to \mathbb{R}_+$ belongs to the class $B_n(R)$ if the following conditions hold:

(1) Φ is non-increasing and continuous on $(0;R)$;

(2) there exists $C > 0$ such that

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \le C\Phi(r)r^n, \ \ r \in (0,R). \tag{1}$$

For example,

$$\Phi(\rho) = \rho^{\alpha-n} \in B_n(\infty) \ (0 < \alpha < n);$$

$$\Phi(\rho) = \rho^{\alpha-n}\Big(ln\frac{eR}{\rho}\Big)^{\beta} \in B_n(R), \text{if } R \in \mathbb{R}_+ \text{ and } \begin{cases} \alpha = 0, \ \beta < 0, \\ 0 < \alpha < n, \ 0 < \beta < R, \\ \alpha = n, \ \beta > 0. \end{cases}$$

For $\Phi \in B_n(R)$ the following estimate also holds

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \geq n^{-1}\Phi(r)r^n,\ r \in (0,R).$$

Therefore

$$\int_0^r \Phi(\rho)\rho^{n-1}d\rho \cong \Phi(r)r^n,\ r \in (0,R),$$

$$\Phi \in B_n(R) \Rightarrow \{0 \leq \Phi \downarrow;\ \Phi(r)r^n. \uparrow,\ r \in (0,R)\}. \tag{2}$$

It follows from (2) that for any $\alpha \in [1;\infty)$ there exists $\beta = \beta(\alpha, c, n) \in [1;\infty)$ (where c is the constant from (1)) such that

$$\Big\{\rho, r \in (0;R); \alpha^{-1} \leq \frac{\rho}{r} \leq \alpha\Big\} \Rightarrow \beta^{-1} \leq \frac{\Phi(\rho)}{\Phi(r)} \leq \beta.$$

Note the well-known equivalence result of N.K. Bari and S.B. Stechkin [16]:

$$(1) \Leftrightarrow \exists\gamma \in (0;n) \text{ such that } \Phi(r)r^{\gamma} quasi - increasing\ on\ (0;R).$$

3 Main results

For the classical Hardy-Littlewood maximal operator $M := M_0$, the rearrangement inequality

$$C_1 f^{**}(t) \leq (Mf)^*(t) \leq C_2 f^{**}(t), \quad t \in (0,\infty),$$

holds ([5], Chapter 3, Th 3.8).

Let given $p \in (1,\infty)$ and non-negative measurable function v on $(0,\infty)$, the classical weighted Lorentz space $\Lambda^p(v)$ is the set of all measurable functions f on R^n such that the quantity

$$\|f\|_{\Lambda^p(v)} = \Big(\int_0^\infty (f^*(t))^p v(t)dt\Big)^{\frac{1}{p}}$$

is finite [5].

Let a function u be non-negative and measurable on $\mathbb{R}$ and let weighted Lebesque spaces $L_{p,u}(\mathbb{R})$ be the space of all functions f measurable on $\mathbb{R}$ for which

$$\|f\|_{L_{p,u}(\mathbb{R})} = \Big(\int_0^\infty |f(t)|^p u(t)dt\Big)^{\frac{1}{p}} < \infty.$$

The following theorem is the main result of this paper.

Theorem 1. Let $\Phi \in B_n(\infty)$, $1 < p \leq q < \infty$, and let ω, v be non-negative and measurable functions on $(0,\infty)$. Then the generalized fractional maximal operator M_Φ is bounded from $\Lambda^p(v)$ into $\Lambda^q(\omega)$ if and only if there exists a positive constant C such that

$$r(\Phi(r)^{\frac{1}{n}})\Big(\int_0^r w(t)dt\Big)^{\frac{1}{q}} \leq C\Big(\int_0^r v(t)dt\Big)^{\frac{1}{p}} \tag{3}$$

and

$$\Big(\int\limits_r^\infty t^{q(\gamma/n-1)}w(t)dt\Big)^{\frac{1}{q}}\Big(\int\limits_0^r \big(t^{-1}\int\limits_0^t v(y)dy\big)^{-p'}\Big)^{\frac{1}{p'}} \le C$$

hold for all $r\in(0,\infty)$.

Theorem 1 for the classical fractional maximal operator was proved in [8].

The following results are used in the proof of Theorem 1.

The Theorem 2 gives an estimate for a non-increasing rearrangement of a generalized fractional-maximal function.

Theorem 2.[1] Let $\Phi\in B_n(\infty)$. Then there exist a positive constant C, depending on n such that

$$(M_\Phi f)^*(t)\le C\sup_{t<s<\infty} s\Phi(s^{1/n})f^{**}(s),\quad t\in(0,\infty), \tag{4}$$

for every $f\in L^1_{loc}(\mathbb{R}^n)$.

Inequality (4) is sharp in the sense that for every $\varphi\in L_0^+(0,\infty;\downarrow)$ there exists a function $f\in L^+\mathbb{R}^n$ such that $f^*=\varphi$ a.e. on $(0,\infty)$ and

$$(M_\Phi f)^*(t)\ge C\sup_{t<s<\infty} s\Phi(s^{1/n})f^{**}(s),\quad t\in(0,\infty)$$

where C is a positive constant which depends only on n.

For the case of the class of fractional maximal functions M_γ a similar theorem was proved in [8].

Theorem 3. [7] Suppose that $w(x)$ and $u(x)$ are nonnegative measurable functions on $(0,\infty)$. If $1<p\le q<\infty$, then the Hardy-Littlewood maximal operator M is bounded from $\Lambda^p(v)$A to $\Lambda^q(w)$, for all non-negative and non-increasing functions f, if and only if both of the following conditions hold:

$$\Big(\int\limits_0^r w(x)dx\Big)^{\frac{1}{q}}\le A\Big(\int\limits_0^r v(x)dx\Big)^{\frac{1}{p}}, \text{for all } r>0;$$

$$\Big(\int\limits_r^\infty x^{-q}w(x)dx\Big)^{\frac{1}{q}}\Big(\int\limits_0^r \big(x^{-1}\int\limits_0^x v(y)dy\big)^{-p'}v(x)dx\Big)^{\frac{1}{p'}}\le B, \text{for all } r>0.$$

We begin by proving a weighted norm inequality for the operator R_Φ defined at $\varphi\in L_0^+(0,\infty;\downarrow)$ by

$$(R_\Phi\varphi)(t)=\sup_{t<\tau<\infty}\tau\Phi(\tau^{\frac{1}{n}})\varphi(\tau).$$

Lemma 1. Let $n\in\mathbb{N}$, $1<p\le q<\infty$, and let ω, v be non-negative and measurable functions on $(0,\infty)$ with v satisfying $\int\limits_0^x v(t)dt<\infty$ for every $x\in(0,\infty)$. Then there is a positive constant C such that the inequality

$$\|R_\Phi\varphi\|_{L_{q,\omega(\mathbb{R}^+)}}\le C\|\varphi\|_{L_{p,v(\mathbb{R}^+)}} \tag{5}$$

holds for every $\varphi\in L_0^+(0,\infty;\downarrow)$ if and only if (3) holds for all $r\in(0,\infty)$.

Proof of Lemma 4.

Necessity. It is clear that $R_\Phi \chi_{(0,r)}(\tau) = \Phi(r) r^{1/n} \chi_{(0,r)}(t)$ for any $r \in (0,\infty)$, $t \in (0,\infty)$. The necessity of (3) follows by testing (5) on $\varphi = \chi_{(0,r)}$.

By using monotonicity of $r\Phi(r^{\frac{1}{n}}) \uparrow$ for the left-hand side of inequality (5) we have:

$$\left(\int_0^\infty \left[(R_\Phi \varphi)(t)\right]^q w(t)dt\right)^{\frac{1}{q}} = \left(\int_0^\infty \left[\sup_{t<\tau<\infty} \tau\Phi(\tau^{\frac{1}{n}})\chi_{(0,r)}(t)\right]^q w(t)dt\right)^{\frac{1}{q}} =$$

$$= \left(\int_0^r \left[\sup_{t<\tau<r} \tau\Phi(\tau^{\frac{1}{n}})\right]^q w(t)dt\right)^{\frac{1}{q}} = r\Phi(r^{\frac{1}{n}})\left(\int_0^r w(t)dt\right)^{\frac{1}{q}}.$$

For the right-hand side of (5), we get:

$$\|\varphi\|_{L_{p,v}(\mathbb{R}^+)} = \left(\int_0^\infty \chi_{(0,r)}(t)v(t)dt\right)^{\frac{1}{p}} = \left(\int_0^r v(t)dt\right)^{\frac{1}{p}},$$

that is (3) is satisfied.

Sufficiency. Let (3) be satisfied and let $\omega \neq 0$ on a set of positive measure. Then (3) entails $\int_0^\infty v(t)dt = \infty$. Consequently, there is an increasing sequence $\{r_k\}_{k\in Z}$ in $(0,\infty)$ such that

$$\int_0^{r_k} v(t)dt = 2^k, \quad k \in Z. \tag{6}$$

It clearly suffices to verify (5) for continuous φ having compact support in $[0,\infty)$ and $\varphi \not\equiv 0$. For such φ, the set $A \subset Z$ given by

$$A = \{k \in Z : (R_\Phi\varphi)(r_{k-1}) > (R_\Phi\varphi)(r_k)\}$$

is not empty. Take $k \in A$ and define

$$z_k = \begin{cases} 0 \;\; if \;\; (R_\Phi\varphi)(t) = (R_\Phi\varphi)(r_{k-1}), \;\; t \in (0, r_{k-1}) \\ \min\{\mathrm{r}_j : (R_\Phi\varphi)(r_j) = (R_\Phi\varphi)(r_{k-1})\}, \quad \text{otherwise.} \end{cases}$$

Then we obtain

$$(R_\Phi\varphi)(t) = (R_\Phi\varphi)(r_{k-1}), \;\; k \in A, \;\; t \in [z_k, r_{k-1}).$$

Moreover, by the definition of A, the supremum appearing in the definition of $(R_\Phi\varphi)(r_{k-1})$ is attained in $[r_{k-1}, r_k)$. Therefore for every $k \in A$ and $t \in [z_k, r_k)$, we have

$$(R_\Phi\varphi)(t) \le (R_\Phi\varphi)(r_{k-1}) = \sup_{r_{k-1}<\tau<r_k} \tau\Phi(\tau^{\frac{1}{n}})\varphi(\tau) \le r_k(\Phi(r_k)^{\frac{1}{n}})\varphi(r_{k-1}). \tag{7}$$

Thus by (7) we get

$$\Big(\int\limits_0^\infty \big[(R_\Phi\varphi)(t)\big]^q w(t)dt\Big)^{\frac{1}{q}} = \Big(\sum_{k\in A}\int\limits_{z_k}^{r_k} \big[(R_\Phi\varphi)(t)\big]^q w(t)dt\Big)^{\frac{1}{q}} \leq$$

$$\leq \Big(\sum_{k\in A} r_k(\Phi(r_k^{\frac{1}{n}}))\varphi^q(r_{k-1})\int\limits_0^{r_k} w(t)dt\Big)^{\frac{1}{q}} \leq$$

$$\leq C\Big(\sum_{k\in A}\varphi^q(r_{k-1})\Big(\int\limits_0^{r_k} v(t)dt\Big)^{q/p}\Big)^{\frac{1}{q}}.$$

Consequently by (3), (6) and by monotonicity of φ and the inequality $q \geq p$ we get

$$\Big(\int\limits_0^\infty \big[(R_\Phi\varphi)(t)\big]^q w(t)dt\Big)^{\frac{1}{q}} \leq 4^{1/p}C\Big(\sum_{k\in A}\varphi^q(r_{k-1})\Big(\int\limits_{r_{k-2}}^{r_{k-1}} v(t)dt\Big)^{q/p}\Big)^{\frac{1}{q}} \leq$$

$$\leq 4^{1/p}C\Big(\sum_{k\in A}\Big(\int\limits_{r_{k-2}}^{r_{k-1}} \varphi^p(t)v(t)dt\Big)^{q/p}\Big)^{\frac{1}{q}} \leq$$

$$\leq 4^{1/p}C\Big(\int\limits_0^\infty \varphi^p(t)v(t)dt\Big)^{\frac{1}{p}}.$$

Lemma 1 is proved.

Proof of Theorem 1. We need to prove that

$$\|(M_\Phi f)\|_{\Lambda^q(\omega)} \leq C\|f\|_{\Lambda^p(v)}$$

By Theorem 2 and Lemma 1 we give

$$\|(M_\Phi f)\|_{\Lambda^q(w)} = \Big(\int\limits_0^\infty \big[(M_\Phi f)^*(t)\big]^q w(t)dt\Big)^{\frac{1}{q}} \leq$$

$$\leq C\Big(\int\limits_0^\infty \big[\sup_{t<\tau<\infty}\tau\Phi(\tau^{1/n})f^{**}(\tau)\big]^q w(t)dt\Big)^{\frac{1}{q}} =$$

$$= C\Big(\int\limits_0^\infty \big[(R_\Phi f^{**})(t)\big]^q w(t)dt\Big)^{\frac{1}{q}} \leq C\Big(\int\limits_0^\infty \big[(f^{**})(t)\big]^p v(t)dt\Big)^{\frac{1}{p}} =$$

$$= C\Big(\int\limits_0^\infty \big[\frac{1}{t}\int\limits_0^t f^*(t)\big]^p v(t)dt\Big)^{\frac{1}{p}}.$$

Here based on Theorem 3 we have:

$$\|(M_\Phi f)\|_{\Lambda^q(w)} \leq C\Big(\int\limits_0^\infty \big(f^*(t)\big)^p v(t)dt\Big)^{\frac{1}{p}} = \|f\|_{L^p(v)}$$

Theorem 1 is proved.

4 Conclusion

In this paper, we considered the generalized fractional-maximal operator. For such operator, necessary and sufficient conditions of boundedness from one weighted Lorentz space to another weighted Lorentz space are obtained. The found conditions are imposed on the weight functions that define the Lorentz spaces and on the function that defines the generalized fractional-maximal operator.

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