

A note on the generalized Weierstrass representation

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Abstract

The study of the relation between the Weierstrass inducing formulae for constant mean curvature surfaces and the completely integrable euclidean nonlinear σ -model suggests a connection among integrable σ -models in a background and other type of surfaces. We show how a generalization of the Weierstrass representation can be achieved and we establish a connection with the Weingarten surfaces. We suggest also a possible generalization for two-dimensional surfaces immersed in a flat space R^8 with Euclidean metric.

1 Introduction

It is well known that conformal immersions of the minimal surfaces into a 3-dimensional euclidean space are described by a classical Weierstrass representation (WR)

$$x^+ = 2i \int_{z_0}^z (\bar{\psi}_1^2 dz' - \bar{\psi}_2^2 d\bar{z}'), \quad x^- = 2i \int_{z_0}^z (\psi_2^2 dz' - \psi_1^2 d\bar{z}'), \quad x_3 = -2 \int_{z_0}^z (\psi_2 \bar{\psi}_1 dz' + \psi_1 \bar{\psi}_2 d\bar{z}') \quad (1)$$

where $x^\pm = x_1 \pm ix_2$ and $\bar{\psi}_1, \psi_2$ are arbitrary analytic functions. Generalization of WR was proposed by Konopelchenko in [1] (see, also [2-4]) in order to study constant mean curvature surfaces. In this case ψ_1, ψ_2 satisfy the following infinite-dimensional Hamiltonian system

$$\psi_{1z} = 2H(|\psi_1|^2 + |\psi_2|^2)\psi_2, \quad \psi_{2\bar{z}} = -2H(|\psi_1|^2 + |\psi_2|^2)\psi_1. \quad (2)$$

One can assume, without loss of generality $H = \frac{1}{2}$. Then, system (1) takes the 2-dimensional Dirac-like equation

$$\psi_{1z} = u\psi_2, \quad \psi_{2\bar{z}} = -u\psi_1 \quad (3)$$

where

$$u = |\psi_1|^2 + |\psi_2|^2. \quad (4)$$

Using the standard formulas, we find that the first fundamental form on the surface is given by

$$I = u^2 dz d\bar{z} \quad (5)$$

at this the Gaussian curvature is

$$K = -\frac{4}{u^2} [\log u]_{z\bar{z}}. \quad (6)$$

In this paper we would like to consider a generalization of the equation (3), study its connection with certain nonlinear σ - models and the corresponding surfaces in a 3-dimensional Euclidean space, obtained via a suitable Weierstrass-like formula.

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2 Generalized Weierstrass representation

First, differently from the system (3), we postulate the equations

$$\psi_{1z} = u\psi_2 + h_1\psi_1, \quad \psi_{2\bar{z}} = -u\psi_1 + h_2\psi_2, \quad (7a)$$

$$\bar{\psi}_{1\bar{z}} = u\bar{\psi}_2 + \bar{h}_1\bar{\psi}_1, \quad \bar{\psi}_{2z} = -u\bar{\psi}_1 + \bar{h}_2\bar{\psi}_2, \quad (7b)$$

where h_j are complex functions to be characterized by further requirements. We call the equations (7) the generalized Konopelchenko system (GKS). Second, we postulate that the generalized Weierstrass representation (GWR) has the form

$$x^+ = 2 \int_{\gamma} F^+(z, \bar{z})dz + G^+(z, \bar{z})d\bar{z} \quad (8a)$$

$$x^- = 2 \int_{\gamma} F^-(z, \bar{z})dz + G^-(z, \bar{z})d\bar{z} \quad (8b)$$

$$x_3 = -2 \int_{\gamma} F_3(z, \bar{z})dz + G_3(z, \bar{z})d\bar{z} \quad (8c)$$

where

$$F^+ = \rho(\bar{\psi}_1^2 - g\bar{\psi}_2^2), \quad F^- = \rho(\psi_2^2 - g\psi_1^2), \quad F_3 = \rho(\bar{\psi}_1\psi_2 + g\psi_1\bar{\psi}_2) \quad (9a)$$

$$G^+ = \bar{\rho}(\bar{\psi}_2^2 - \bar{g}\bar{\psi}_1^2), \quad G^- = \bar{\rho}(\psi_1^2 - \bar{g}\psi_2^2), \quad G_3 = \bar{\rho}(\psi_1\bar{\psi}_2 + \bar{g}\bar{\psi}_1\psi_2) \quad (9b)$$

with ρ, g are complex functions to be defined and γ is any curve from a fixed point to z . Since the coordinates x_i do not depend on the choice of the curve γ in the complex plane C (but only its end points), the functions F_i, G_i must satisfy the following conditions

$$F_{\bar{z}}^+ = G_z^+, \quad F_{\bar{z}}^- = G_z^-, \quad F_{3\bar{z}} = G_{3z}. \quad (10)$$

These show that the integrands are total differentials. Substituting (9) into (10), we get the following restrictions for the functions h_i, ρ and g

$$[\rho_{\bar{z}} + 2\rho\bar{h}_1 + (\bar{\rho}\bar{g})_z]\bar{\psi}_1^2 - [\bar{\rho}_z + 2\bar{\rho}\bar{h}_2 + (\rho g)_{\bar{z}}]\bar{\psi}_2^2 + 2u(\rho + \bar{\rho})\bar{\psi}_1\bar{\psi}_2 + 2(\bar{\rho}\bar{g}\bar{\psi}_1\bar{\psi}_{1z} - \rho g\bar{\psi}_2\bar{\psi}_{2\bar{z}}) = 0, \quad (11a)$$

$$[\rho_{\bar{z}} + 2\rho h_2 + (\bar{\rho}\bar{g})_z]\psi_2^2 - [\bar{\rho}_z + 2\bar{\rho}h_1 + (\rho g)_{\bar{z}}]\psi_1^2 - 2u(\rho + \bar{\rho})\psi_1\psi_2 - 2(\rho g\psi_1\psi_{1\bar{z}} - \bar{\rho}\bar{g}\psi_2\psi_{2z}) = 0, \quad (11b)$$

$$[\rho_{\bar{z}} + \rho(\bar{h}_1 + h_2)]\bar{\psi}_1\psi_2 - [\bar{\rho}_z + \bar{\rho}(h_1 + \bar{h}_2)]\psi_1\bar{\psi}_2 + u(\rho - \bar{\rho})(|\psi_2|^2 - |\psi_1|^2) + (\rho g\psi_1\bar{\psi}_2)_{\bar{z}} - (\bar{\rho}\bar{g}\bar{\psi}_1\psi_2)_z = 0. \quad (11c)$$

3 The associated generalized σ -model of the GKS

In the paper [6] it was shown that the Konopelchenko system (3) is equivalent to a scalar second order equation, which is the stereographic projection of the euclidean nonlinear σ - model. Inspired by the transformation used there, also in this we define a function ω

$$\omega = \frac{\psi_1}{\psi_2}. \quad (12)$$

Now, computing the derivatives of ω with the use of system (7) we find

$$\omega_z = \psi_2^2 \left(1 + |\omega|^2 \right) + h\omega, \quad (13)$$

where we have introduced the notation $h = h_1 - \bar{h}_2$. Solving the previous relation with respect to ψ_2^2 , we obtain an "inverse" formulas for ψ_1, ψ_2 , i.e.

$$\psi_1 = \epsilon \omega \frac{\sqrt{\bar{\omega}_z - \bar{h}\omega}}{1 + |\omega|^2}, \quad \psi_2 = \epsilon \frac{\sqrt{\omega_z - h\omega}}{1 + |\omega|^2}, \quad \epsilon^2 = 1. \quad (14)$$

The formulae (14) generalize those used in [6] and they coincide with for $h = 0$. Then, system (7) allows to find an equation for ω , precisely

$$\omega_{z\bar{z}} = 2 \frac{\omega_z \omega_{\bar{z}}}{1 + |\omega|^2} \bar{\omega} + (h\omega)_{\bar{z}} + 2(\omega_z - h\omega) h_2 - 2 \frac{|\omega|^2}{1 + |\omega|^2} [\omega_{\bar{z}} h - \omega_z \bar{h} + \omega |h|^2]. \quad (15)$$

We can show that in the case $h = 0$ the equation (15) fits both the usual euclidean nonlinear σ -model and the Ernst-like system studied by Schief in [11].

4 The $O(3)$ σ -model

Now, let us consider the simplest reductions of the above equations.

i) Let us consider first the case $h_2 = \bar{h}_1 = g = 0$, $\rho = i$. Then, the conditions (11) are obviously satisfied and the GKS (7) reduces to the usual KS (3). Moreover, the system (15) reduces to the usual euclidean $O(3)$ σ -model

$$\omega_{z\bar{z}} = 2 \frac{\omega_z \omega_{\bar{z}}}{1 + |\omega|^2} \bar{\omega} \quad (16)$$

the integrability properties of which are well known [12] and its connection connected with the constant mean curvature surfaces is described in [6], [7] and [9].

ii) Now let $h_2 = \bar{h}_1 = 0$, $\rho = i$. The restrictions (11) now dictate strong conditions on the function g , i.e.

$$(\bar{g}\bar{\psi}_1^2)_z + (g\bar{\psi}_2^2)_{\bar{z}} = 0, \quad (17a)$$

$$(\bar{g}\psi_2^2)_z + (g\psi_1^2)_{\bar{z}} = 0, \quad (17b)$$

$$(g\psi_1\bar{\psi}_2)_{\bar{z}} + (\bar{g}\bar{\psi}_1\psi_2)_z = 0. \quad (17c)$$

This equations were found in [9] and they have as a solution $g = J(z)/u^2$, where $J(z)$ is an arbitrary analytic function. The corresponding Weierstrass formula provides a non isothermic set of coordinates for the mean constant curvature surfaces, since the corresponding scalar equation is still (15).

5 The σ -model in a curved space

Now we would like to consider the restriction

$$h_2 = \bar{h}_1 = -\frac{\rho_{\bar{z}}\omega_z + \rho_z\omega_{\bar{z}}}{4\text{Re}}(\rho)\omega_z, \quad (18)$$

where the real function ρ satisfies the equation

$$\rho_{z\bar{z}} = 0. \quad (19)$$

In this case we have $h = 0$ and the equation (15) becomes

$$\omega_{z\bar{z}} = 2 \frac{\omega_z \omega_{\bar{z}}}{1 + |\omega|^2} \bar{\omega} + 2h_2\omega_z. \quad (20)$$

Furthermore, the conditions (11) provide a constraint on g . Equation (20) represents a nonlinear σ -model in a curved space, and it can be interpreted also as a 2-dimensional isotropic magnet,

whose squared saturation moment is a harmonic function [5]. Moreover, it is known that this system is integrable [5]. In fact, the equation (20) is the compatibility condition for the linear 2×2 matrix system

$$\Psi_z = U\Psi, \quad \Psi_{\bar{z}} = V\Psi \quad (21)$$

where the potential matrices are given by

$$U = \frac{\rho}{\varrho + \rho} S_z S, \quad V = -\frac{\rho}{\varrho - \rho} S_z S \quad (22)$$

with

$$S = \frac{i}{1 + |\omega|^2} \begin{pmatrix} 1 - |\omega|^2 & -2\bar{\omega} \\ -2\omega & |\omega|^2 - 1 \end{pmatrix}, \quad \varrho = i\beta - u + \sqrt{(u - \gamma)(u + \gamma)}, \quad \gamma = \rho + i\beta. \quad (23)$$

The function γ is analytic in a suitable region and $u \in C$ is a "hidden" spectral parameter, which does not depend on the coordinates.

6 The Ernst-type equation

Weakening the reality condition on ρ used in the previous section, we consider the case

$$h_2 = \bar{h}_1 = -\frac{p_{\bar{z}}\omega_z + p_z\omega_{\bar{z}}}{4\text{Re}(p)\omega_z} = \frac{f}{2\omega_z}, \quad p_{z\bar{z}} = 0, \quad (24)$$

where we introduced $p = \rho + i\sigma$, both these functions are harmonic, and

$$f = -\frac{1}{2} \frac{p_{\bar{z}}\omega_z + p_z\omega_{\bar{z}}}{\text{Re}(p)}. \quad (25)$$

Now the equation (15) takes the form

$$\omega_{z\bar{z}} = 2\frac{\omega_z\omega_{\bar{z}}}{1 + |\omega|^2}\bar{\omega} + f, \quad p_{z\bar{z}} = 0, \quad (26)$$

and it was considered by Schief [11] as a particular reduction of the static Loewner system in connection with the study of the Weingarten surfaces of Class 2. The system is integrable and this property was studied in [13]. With these assumptions the GKS (7) takes the form

$$\psi_{1z} = u\psi_2 + \frac{\bar{f}}{2\bar{\omega}_{\bar{z}}}\psi_1, \quad \psi_{2\bar{z}} = -u\psi_1 + \frac{f}{2\omega_z}\psi_2 \quad (27a)$$

$$\bar{\psi}_{1\bar{z}} = u\bar{\psi}_2 + \frac{f}{2\omega_z}\bar{\psi}_1, \quad \bar{\psi}_{2z} = -u\bar{\psi}_1 + \frac{\bar{f}}{2\bar{\omega}_{\bar{z}}}\bar{\psi}_2 \quad (27b)$$

where

$$u = |\psi_1|^2 + |\psi_2|^2 = \frac{(\omega_z\bar{\omega}_{\bar{z}})^{1/2}}{1 + |\omega|^2}. \quad (28)$$

We notice that both the systems (26) and (27) are conformally invariant under the transformations

$$\begin{aligned} z' &= \xi(z), & \bar{z}' &= \bar{\xi}(\bar{z}), \\ \omega' &= \omega, & p' &= p, & \psi'_1 &= (\bar{\xi}_{\bar{z}})^{-\frac{1}{2}}\psi_1, & \psi'_2 &= (\xi_z)^{-\frac{1}{2}}\psi_2, \end{aligned}$$

where $\xi(z)$ is an arbitrary analytic function on the complex plane. This property is important from the geometrical point of view, since it is the realization of the invariance under parametrization of the 2-dimensional surfaces, which are in the motivations of the present work. Furthermore, this fact indicates that the system admits conservation laws, which would be useful in the further analysis.

Now, we consider certain reductions of the system (27). Besides the trivial example $p = \text{const}$, which leads to the usual system (3), we can consider the following special cases.

i) Case $p = p(\bar{z})$ leads to the quasi-linear system with peculiar non-constant coefficients

$$\psi_{1z} = u\psi_2 - \frac{\bar{p}_z}{4\text{Re}(p)}\psi_1, \quad \psi_{2\bar{z}} = -u\psi_1 - \frac{p_{\bar{z}}}{4\text{Re}(p)}\psi_2. \quad (29)$$

ii) Case $p = p(z)$. In this case from (7) we obtain a system which is highly nonlinear in the ψ' s, and which can be written in form

$$\psi_{1z} = u\psi_2 - \epsilon\omega \frac{\bar{p}_{\bar{z}}(\bar{\omega}_{\bar{z}})^{-\frac{1}{2}}\bar{\omega}_z}{4\text{Re}(p)(1+|\omega|^2)}, \quad \psi_{2\bar{z}} = -u\psi_1 - \epsilon \frac{p_z(\omega_z)^{-\frac{1}{2}}\omega_{\bar{z}}}{4\text{Re}(p)(1+|\omega|^2)}. \quad (30)$$

Our aim is now to deduce directly from the above expressions the integrals (8), solving implicitly the system of constraints (11). This result can be achieved looking at a conservation law form of the system (26). In doing so, we introduce the "spin" S matrix in (23). Then, equation (26) is equivalent, in the sense of the stereographic projection, to the matrix equation

$$(\rho[S, S_{\bar{z}}] + 2\sigma_{\bar{z}}S)_z + (\rho[S, S_z] + 2\sigma_zS)_{\bar{z}} = 0, \quad (31)$$

which has the conservation law form. For brevity we introduce the matrices

$$K(\rho, \sigma, \bar{z}) = \rho[S, S_{\bar{z}}] + 2\sigma_{\bar{z}}S, \quad M(\rho, \sigma, z) = \rho[S, S_z] + 2\sigma_zS, \quad (32)$$

which allows us to rewrite (31) in the form

$$K_z + M_{\bar{z}} = 0. \quad (33)$$

The explicit expressions of matrices K and M in terms of ω are

$$K = m \begin{pmatrix} \bar{\omega}\omega_{\bar{z}} - \bar{\omega}_{\bar{z}}\omega & -(\bar{\omega}_{\bar{z}} + \bar{\omega}^2\omega_{\bar{z}}) \\ \omega_{\bar{z}} + \omega^2\bar{\omega}_{\bar{z}} & -(\bar{\omega}\omega_{\bar{z}} - \bar{\omega}_{\bar{z}}\omega) \end{pmatrix} + n_1 \begin{pmatrix} 1 - |\omega|^2 & -2\bar{\omega} \\ -2\omega & |\omega|^2 - 1 \end{pmatrix} \quad (34a)$$

and

$$M = m \begin{pmatrix} \bar{\omega}\omega_z - \bar{\omega}_z\omega & -(\bar{\omega}_z + \bar{\omega}^2\omega_z) \\ \omega_z + \omega^2\bar{\omega}_z & -(\bar{\omega}\omega_z - \bar{\omega}_z\omega) \end{pmatrix} + n_2 \begin{pmatrix} 1 - |\omega|^2 & -2\bar{\omega} \\ -2\omega & |\omega|^2 - 1 \end{pmatrix}, \quad (34b)$$

where

$$m = -\frac{4\rho}{(1+|\omega|^2)^2}, \quad n_1 = \frac{2i\sigma_{\bar{z}}}{1+|\omega|^2}, \quad n_2 = \frac{2i\sigma_z}{1+|\omega|^2}. \quad (35)$$

From the above expression it is easy to prove the properties

$$K(\bar{\rho}, \bar{\sigma}, \bar{z}) = -M^\dagger(\rho, \sigma, z), \quad K(\rho, \sigma, \bar{z}) = -M^\dagger(\bar{\rho}, \bar{\sigma}, z) = M(\rho, \sigma, \bar{z}). \quad (36)$$

We need to use also the following relations

$$\omega_z = \frac{u^2}{\psi_2^2}, \quad \bar{\omega}_z = -\frac{T}{\psi_2^2}, \quad T = \bar{\psi}_1\psi_{2z} - \bar{\psi}_1z\psi_2 = 2\frac{\omega_z\bar{\omega}_z}{(1+|\omega|^2)^2}. \quad (37)$$

We note that T satisfies the relation

$$T_{\bar{z}} = \frac{f}{\omega_z} T - \frac{u^2 \bar{f}}{\bar{\omega}_{\bar{z}}}, \quad (38)$$

which breaks the analyticity property possessed by the analogous quantity in the theory of the constant mean curvature surfaces reviewed above [6, 9]. Using the formula (14) with $h = 0$, we now find the expressions of matrices K and M in terms of ψ 's

$$K = -4\rho \begin{pmatrix} -\psi_1 \bar{\psi}_2 & -\bar{\psi}_2^2 \\ \psi_1^2 & \psi_1 \bar{\psi}_2 \end{pmatrix} + \frac{2i\sigma_{\bar{z}}}{u} \begin{pmatrix} |\psi_2|^2 - |\psi_1|^2 & -2\bar{\psi}_1 \bar{\psi}_2 \\ -2\psi_1 \psi_2 & |\psi_1|^2 - |\psi_2|^2 \end{pmatrix} + \frac{4\rho \bar{T}}{u^2} \begin{pmatrix} \bar{\psi}_1 \psi_2 & -\bar{\psi}_1^2 \\ \psi_2^2 & -\bar{\psi}_1 \psi_2 \end{pmatrix} \quad (39)$$

and

$$M = -4\rho \begin{pmatrix} \bar{\psi}_1 \psi_2 & -\bar{\psi}_1^2 \\ \psi_2^2 & -\bar{\psi}_1 \psi_2 \end{pmatrix} + \frac{2i\sigma_z}{u} \begin{pmatrix} |\psi_2|^2 - |\psi_1|^2 & -2\bar{\psi}_1 \bar{\psi}_2 \\ -2\psi_1 \psi_2 & |\psi_1|^2 - |\psi_2|^2 \end{pmatrix} + \frac{4\rho T}{u^2} \begin{pmatrix} -\psi_1 \bar{\psi}_2 & -\bar{\psi}_2^2 \\ \psi_1^2 & \bar{\psi}_2 \psi_1 \end{pmatrix}. \quad (40)$$

In this notation the three conservation laws read

$$- [4\rho(\psi_1 \bar{\psi}_2 + \frac{\bar{T}}{u^2} \bar{\psi}_1 \psi_2) + \frac{2i\sigma_{\bar{z}}}{u} (|\psi_2|^2 - |\psi_1|^2)]_z + [4\rho(\bar{\psi}_1 \psi_2 + \frac{4\bar{T}}{u^2} \psi_1 \bar{\psi}_2) - \frac{2i\sigma_z}{u} (|\psi_2|^2 - |\psi_1|^2)]_{\bar{z}} = 0, \quad (41a)$$

$$[4\rho(-\psi_1^2 + \frac{\bar{T}}{u^2} \psi_2^2) - \frac{4i\sigma_{\bar{z}}}{u} \psi_1 \psi_2]_z + [4\rho(-\psi_2^2 + \frac{4T}{u^2} \psi_1^2) - \frac{4i\sigma_z}{u} \psi_1 \psi_2]_{\bar{z}} = 0, \quad (41b)$$

$$[4\rho \bar{\psi}_2^2 - \frac{\bar{T}}{u^2} \bar{\psi}_1^2) - \frac{4i\sigma_{\bar{z}}}{u} \bar{\psi}_1 \bar{\psi}_2]_z + [4\rho(\bar{\psi}_1^2 - \frac{4T}{u^2} \bar{\psi}_2^2 - \frac{4i\sigma_z}{u} \bar{\psi}_1 \bar{\psi}_2)]_{\bar{z}} = 0. \quad (41c)$$

As a result of these equations, we introduce the six real-valued functions

$$F_1 = 4\rho(\bar{\psi}_1^2 - \frac{4T}{u^2} \bar{\psi}_2^2) - \frac{4i\sigma_z}{u} \bar{\psi}_1 \bar{\psi}_2 + 4\bar{\rho}(\psi_2^2 - \frac{T}{u^2} \psi_1^2) - \frac{4i\bar{\sigma}_z}{u} \psi_1 \psi_2, \quad (42a)$$

$$F_2 = 4\rho(\bar{\psi}_1^2 - \frac{4T}{u^2} \bar{\psi}_2^2) - \frac{4i\sigma_z}{u} \bar{\psi}_1 \bar{\psi}_2 - 4\bar{\rho}(\psi_2^2 + \frac{T}{u^2} \psi_1^2) + \frac{4i\bar{\sigma}_z}{u} \psi_1 \psi_2, \quad (42b)$$

$$F_3 = -2[4\rho(\bar{\psi}_1 \psi_2 + \frac{4\bar{T}}{u^2} \psi_1 \bar{\psi}_2) - \frac{2i\sigma_z}{u} (|\psi_2|^2 - |\psi_1|^2)], \quad (42c)$$

and

$$G_1 = -[4\bar{\rho}(\psi_1^2 - \frac{4\bar{T}}{u^2} \psi_2^2 - \frac{4i\bar{\sigma}_z}{u} \psi_1 \psi_2 + 4\rho \bar{\psi}_2^2 - \frac{\bar{T}}{u^2} \bar{\psi}_1^2) + \frac{4i\sigma_{\bar{z}}}{u} \bar{\psi}_1 \bar{\psi}_2], \quad (43a)$$

$$G_2 = 4\bar{\rho}(\psi_1^2 - \frac{4\bar{T}}{u^2} \psi_2^2 - \frac{4i\bar{\sigma}_z}{u} \psi_1 \psi_2 - 4\rho \bar{\psi}_2^2 + \frac{\bar{T}}{u^2} \bar{\psi}_1^2) - \frac{4i\sigma_{\bar{z}}}{u} \bar{\psi}_1 \bar{\psi}_2, \quad (43b)$$

$$G_3 = 2[4\rho(\psi_1 \bar{\psi}_2 + \frac{\bar{T}}{u^2} \bar{\psi}_1 \psi_2) + \frac{2i\sigma_{\bar{z}}}{u} (|\psi_2|^2 - |\psi_1|^2)], \quad (43c)$$

which enter into the following exact forms

$$x_1(z, \bar{z}) = i \int_{\gamma} F_1(z', \bar{z}') dz' + G_1(z', \bar{z}') d\bar{z}', \quad (44a)$$

$$x_2(z, \bar{z}) = \int_{\gamma} F_2(z', \bar{z}') dz' + G_2(z', \bar{z}') d\bar{z}', \quad (44b)$$

$$x_3(z, \bar{z}) = \int_{\gamma} F_3(z', \bar{z}') dz' + G_3(z', \bar{z}') d\bar{z}', \quad (44c)$$

integrated on an arbitrary curve γ in the complex plane, connecting an initial fixed point to a final one z . The conservation laws (41) guarantee that the quantities x_i do not depend on the choice of the curve γ in the complex plane C . Since the functions $x_i(z, \bar{z})$ are real, they can be considered as components of the radial vector

$$\mathbf{r}(z, \bar{z}) = (x_1, x_1, x_1) \quad (45)$$

of an orientable, connected surface immersed in R^3 and locally parametrized by z . The properties of such surfaces have to be determined, but they are closely related to the Weingarten surfaces of Class 2. The reason for this is based on the observation that the tangent vectors can be represented by the formulae (32), after the identification $S \rightarrow \mathbf{S}$, where \mathbf{S} is a unimodular real vector of \mathbf{R}^3 . Then interpreting such a vector as the normal vector to the surface, expressions (32) are similar to the equations (4.8) in [11], which are the so - called Lelievre formulas for the Weingarten surfaces of Class 2. The only difference resides in the last terms of both expressions, which in our case are proportional to the normal vector and not to its derivatives. However the coefficients also change, while in the Schief case one has the same coefficient .

7 The Weierstrass representation for surfaces immersed into R^8

Quite recently in [9], a generalization of the WR of equations corresponding to CP^2 harmonic maps was presented. This generalization allows to study 2-dimensional surfaces immersed in a flat space R^8 with Euclidean metric. Analogously, the generalized spin model (31) also gives a new class of surfaces immersed into R^8 . To do so, first we rewrite the equation (31) in terms of the projector as

$$(\rho P P_y)_y + (\rho P P_x)_x + \sigma_x P_x + \sigma_y P_y = 0 \quad (46a)$$

$$\rho_{xx} + \rho_{yy} = 0, \quad \sigma_{xx} + \sigma_{yy} = 0 \quad (46b)$$

where

$$P = \frac{1}{1 + |\omega|^2} \begin{pmatrix} 1 & \bar{\omega} \\ \omega & |\omega|^2 \end{pmatrix}. \quad (47)$$

Now let us consider the equation (46a) when the projector P is a 3×3 matrix with the structure

$$P = \frac{1}{1 + |\omega_1|^2 + |\omega_2|^2} \begin{pmatrix} 1 & \bar{\omega}_1 & \bar{\omega}_2 \\ \omega_1 & |\omega_1|^2 & \omega_1 \bar{\omega}_2 \\ \omega_2 & \bar{\omega}_1 \omega_2 & |\omega_2|^2 \end{pmatrix}. \quad (48)$$

As in the previously, in this case we have the conservation laws

$$K_z + M_{\bar{z}} = 0, \quad (49)$$

with the matrices

$$K = [P_{\bar{z}}, P] + \sigma_{\bar{z}} P, \quad M = [P_z, P] + \sigma_z P. \quad (50)$$

Then, as in section 3, we can introduce two pairs of complex functions ψ_i, ϕ_i as

$$\omega_i = \frac{\psi_i}{\phi_i} \quad (51)$$

and express K, M in terms of ψ_i, ϕ_i . In this way, the equation (46) allows us to define eight new conservation laws. Since the results of the calculations are the same as in Ref.[9], we will not present them here. Only we remark that we can construct eight real quantities $X_{\{i\}}(z, z)$ which can be used to study various geometrical aspects of the surface.

8 Summary and concluding remarks

The main aim of this paper is to propose a generalization of the Konopelchenko system (3), involved in the study of constant mean curvature surfaces. Supplying the general structure (7) with constraints on the functions h_1 , h_2 , ρ and g we can recover known examples and, in particular we obtained the Weierstrass - like representation (41) - (42) - (43) for surfaces of Weingarten type. However, the integrability properties of the system (27), its explicit solutions and the concrete use of formulae (44) is far to be achieved. Finally, we have shown that this approach can be pursued also in the case of certain surfaces immersed in R^8 .

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