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The Hardy–Littlewood Theorem for Fourier–Haar Series

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Abstract—An interpolation theorem for a class of net spaces is proved. In terms of Fourier–Haar coefficients, we obtain a test for a function to belong to the net space $N_p^q(M)$, where $1 < p < \infty$ and M is the set of all closed intervals in $[0,1]$. As a corollary, we derive an analog of the Hardy–Littlewood theorem for Fourier–Haar series.

Key words: *trigonometric series, interpolation, Lebesgue measure, Fourier–Haar series, net space, Paley's inequalities, Peetre functional.*

The following result for trigonometric series is well known.

The Hardy–Littlewood Theorem [1]. *Suppose that $1 < p < \infty$ and $f \sim \sum_{k=0}^{\infty} a_k \cos kx$. If $\{a_k\}_{k=0}^{\infty}$ is a monotone nonincreasing sequence, then for f to belong to $L_p[0,\pi]$ it is necessary and sufficient that the following series be convergent*:

$$\sum_{k=0}^{\infty} k^{p-2} a_k^p. \tag{1}$$

There is an analog of this theorem for monotone functions [2]:

If f is monotone and $f \sim \sum_{k=0}^{\infty} a_k \cos kx$, then for f to belong to $L_p[0,\pi]$ it is necessary and sufficient that the series (1) *be convergent.*

As is seen from these results, the conditions for monotone functions and functions with monotone coefficients to belong to the space L_p are the same, namely, the convergence of the series (1).

The situation is quite different for series with respect to the Haar system. In [3], P. L. Ul′yanov proved that if the Fourier–Haar coefficients $\{c_k\}$ are monotone, then for a function f to belong to the space $L_p[0,1]$, $1 < p < \infty$, it is necessary and sufficient that $\{c_k\}_{k=1}^{\infty}$ belong to l_2, i.e., that the series $\sum_{k=1}^{\infty} |c_k|^2$ be convergent.

In particular, the following assertion was proved in [3]:

Suppose that $\{\chi_k\}_{k=1}^{\infty}$ is the Haar system, $1 < p < \infty$, and $f \sim \sum_{k=1}^{\infty} c_k \chi_k$. If $f(x)$ is a monotone function, then for f to belong to L_{pq} it is necessary and sufficient that the following series be convergent:

$$\left(\sum_{k=0}^{\infty} 2^{k(1/2-1/p)q} \left(\sup_{2^k \le \nu < 2^{k+1}} |c_\nu| \right)^q \right)^{1/q}. \tag{2}$$

Inequalities that are analogs of Paley's inequalities [1] were also obtained in [3].

Our methods of study are based on interpolation theorems (proved in Sec. 1) for a particular class of net spaces.

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1. INTERPOLATION PROPERTIES OF NET SPACES

Suppose that μ is the Lebesgue measure on $\mathbb{R}^n$ and M is a fixed family of sets of finite measure from $\mathbb{R}^n$. In what follows, M is called a "*net*." For a function $f(x)$ defined and integrable on each set e from M, we define the function

$$\bar{f}(t, M) = \sup_{\substack{e\in M\\ |e|>t}} \frac{1}{|e|}\left|\int_e f(x)\,d\mu\right|,$$

where the supremum is taken over all sets $e \in M$ whose measure is $|e| \stackrel{\text{def}}{=} \mu e > t$, $t \in (0, \infty)$. In the case $\sup\{|e| : e \in M\} = \alpha < \infty$ and $t > \alpha$, set $\bar{f}(t, M) = 0$. The function $\bar{f}(t, M)$ is called the *average of the function f over the net M*.

By $N_{pq}(M)$, $0 < p, q \le \infty$, we denote the set of functions f such that for $q < \infty$

$$\|f\|_{N_{pq}(M)} = \left(\int_0^\infty (t^{1/p}\bar{f}(t, M))^q \frac{dt}{t}\right)^{1/q} < \infty$$

and for $q = \infty$

$$\|f\|_{N_{p\infty}(M)} = \sup_{t>0} t^{1/p}\bar{f}(t, M) < \infty.$$

Note several properties of the spaces N_{pq}:

(a) If $M_1 \subset M_2$, then $N_{pq}(M_2) \hookrightarrow N_{pq}(M_1)$.
(b) For $0 < q \le q_1 \le \infty$, we have $N_{pq}(M) \hookrightarrow N_{pq_1}(M)$.
(c) If the net M satisfies $\sup_{e\in M} |e| = \alpha < \infty$, then for $0 < p < p_1 \le \infty$, $0 < q, q_1 \le \infty$ we have $N_{p_1q_1}(M) \hookrightarrow N_{pq}(M)$.

Suppose that (A_0, A_1) is a consistent pair of Banach spaces [4] and

$$K(t, a; A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \qquad a \in A_0 + A_1,$$

is the Peetre functional.

For $0 < q < \infty$, $0 < \theta < 1$, we have

$$(A_0, A_1)_{\theta,q} = \left\{a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta}K(t, a))^q \frac{dt}{t}\right)^{1/q} < \infty\right\},$$

and for $q = \infty$,

$$(A_0, A_1)_{\theta,\infty} = \left\{a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta}K(t, a) < \infty\right\}.$$

In [5], the following theorem was proved.

Theorem 1. *Suppose that $0 < p_0 < p_1 < \infty$, $0 < q \le \infty$, and $0 < \theta < 1$.*
If M is an arbitrary net in $\mathbb{R}^n$, then

$$(N_{p_0q_0}(M), N_{p_1q_1}(M))_{\theta q} \hookrightarrow N_{pq}(M),$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$.

The goal in this section is to prove the following assertion.

Theorem 2. *Suppose that M is the set of all closed intervals from $\mathbb{R}$,*

$$0<\theta<1, \quad 1\le p_0<p_1<\infty, \quad 1\le q_0, q_1, q\le\infty, \quad 1/p=(1-\theta)/p_0+\theta/p_1.$$

Then

$$(N_{p_0q_0}(M), N_{p_1q_1}(M))_{\theta q}=N_{pq}(M).$$

Lemma 1. *If the family M satisfies the following inequality for any $\tau>0$ and some $r>0$:*

$$\inf_{|\phi|\le\bar f(\tau)}\Bigl(\sup_{s\ge\tau} s^{1/r}\overline{(f-\phi)}(s\,;M)\Bigr)\le c\sup_{\tau\ge s>0} s^{1/r}\bar f(s\,;M), \tag{3}$$

where c is a constant independent of f and τ, then for $r<p_0<p_1<\infty$, $0<\theta<1$ we have

$$(N_{p_0q_0}(M), N_{p_1q_1}(M))_{\theta q}=N_{pq}(M),$$

where $1/p=(1-\theta)/p_0+\theta/p_1$, $1\le q, q_0, q_1\le\infty$.

Proof. Let us show that if the assumption of the lemma is satisfied, then

$$(N_{r\infty}(M), N_{\infty\infty}(M))_{\theta q}=N_{pq}(M), \qquad 1\le q\le\infty;$$

in that case, by the reiteration theorem for the real-variable method [4], we obtain the assertion of the lemma.

From the condition on the net M, we obtain

$$K(t,f)\le K(t,f\,;N_{r\infty},N_{\infty\infty})\le c\inf_{|\phi|\le\bar f(\tau)}\sup_{\tau\ge s>0} s^{1/r}\overline{(f-\phi)}(s)+t\bar f(\tau).$$

Taking into account the fact that $\bar f(s)$ is monotone, for $\tau=t^r$ we have

$$K(t,f)\le r\sup_{\tau\ge s}\int_0^s \bar f(y)y^{1/r-1}\,dy+2r\int_0^{t^r}\bar f(y)y^{1/r-1}\,dy=3r\int_0^{t^r}\bar f(y)y^{1/r-1}\,dy$$

or

$$\|f\|_{(N_{r\infty},N_{\infty\infty})_{\theta q}}\le 3r\biggl(\int_0^\infty\biggl(t^{-\theta}\int_0^{t^r}\bar f(y)y^{1/r-1}\,dy\biggr)^q\frac{dt}{t}\biggr)^{1/q}.$$

Further, substituting $y\to yt^r$ in the inner integral and applying the generalized Minkowski inequality, we obtain

$$\|f\|_{(N_{r\infty}(M),N_{\infty\infty}(M))_{\theta q}}\le c\int_0^1\biggl(\int_0^\infty(t^{-\theta}\bar f(yt^r))^q\frac{dt}{t}\biggr)^{1/q}y^{1/r}\,\frac{dy}{y}.$$

Finally, substituting $yt^r\to t$ in the inner integral, we can write

$$\|f\|_{(N_{r\infty}(M),N_{\infty\infty}(M))_{\theta q}}\le c\biggl(\int_0^1 y^{\theta/r-1}\,dy\biggr)\|f\|_{N_{pq}(M)}=c_1\|f\|_{N_{pq}(M)}$$

or

$$N_{pq}(M)\hookrightarrow(N_{r\infty}(M),N_{\infty\infty}(M))_{\theta q},$$

where $1/p=(1-\theta)/r$.

On the other hand, Theorem 1 yields the inverse embedding. $\square$

Proof of Theorem 2. Suppose that M is the set of all closed intervals from $\mathbb{R}$. Let us show that for $r=1$ this net satisfies condition (3) from Lemma 1.

For a function f and fixed $\tau>0$, we define the function $\phi_0(x)$ as follows. Let us divide $\mathbb{R}$ into disjoint half-intervals $\{I_k\}_{k\in\mathbb{Z}}$ of measure $|I_k|=\tau$. Let

$$\phi_0(x)=\frac{1}{|I_k|}\int_{I_k} f(y)\,dy \qquad \text{for } x\in I_k .$$

Note that $|\phi_0(x)|\le \bar f(\tau,M)$ and $\int_{I_k}(f(x)-\phi_0(x))\,dx=0$ for all $k\in\mathbb{Z}$. Then, for any closed interval $I\colon |I|\ge\tau$, we have

$$\begin{aligned}\left|\int_I (f-\phi_0)(x)\,dx\right| &= \left|\sum_{k=m}^{m+N}\int_{I_k}(f(x)-\phi_0(x))\,dx+\int_{I\cap I_{m-1}}(f-\phi_0)(x)\,dx\right.\\ &\qquad\left.+\int_{I\cap I_{m+N+1}}(f(x)-\phi_0(x))\,dx\right|\\ &\le \left|\int_{I\cap I_{m-1}} f(x)\,dx\right|+\left|\int_{I\cap I_{m+N+1}} f(x)\,dx\right|\\ &\qquad+(|I\cap I_{m-1}|+|I\cap I_{m+N+1}|)\bar f(\tau).\end{aligned}$$

Or, for $s_1=|I\cap I_{m-1}|\le\tau$, $s_2=|I\cap I_{m+N+1}|\le\tau$, we can write

$$\left|\int_I (f-\phi_0)(x)\,dx\right|\le s_1\bar f(s_1,M)+s_2\bar f(s_2,M)+2\tau\bar f(\tau,M);$$

therefore,

$$\left|\int_I (f-\phi_0)(x)\,dx\right|\le 4\sup_{\tau\ge s>0} s\bar f(s,M).$$

Thus,

$$\begin{aligned}\inf_{|\phi|\le\bar f(\tau)}\sup_{s\ge\tau} s\overline{(f-\phi_0)} &\le \sup_{s\ge\tau} s\overline{(f-\phi_0)}(s)=\sup_{s\ge\tau} s\sup_{|I|\ge s}\frac{1}{|I|}\left|\int_I (f-\phi_0)(x)\,dx\right|\\ &\le 4\sup_{\tau\ge t>0} t\bar f(t)\sup_{s\ge\tau}\sup_{|I|\ge s}\frac{s}{|I|}=4\sup_{\tau\ge t>0} t\bar f(t).\end{aligned}$$

Therefore, the net M satisfies the assumption of Lemma 1, and hence Theorem 2 is proved. □

2. THE HARDY–LITTLEWOOD THEOREM AND PALEY-TYPE INEQUALITIES

Theorem 3. *Suppose that $1<p<\infty$ and M is the set of all closed intervals in $\mathbb{R}$. Then for f to belong to $N_{pq}(M)$ it is necessary and sufficient that the following condition hold for the sequence of its Fourier–Haar coefficients $\{c_n\}_{n=1}^{\infty}$:*

$$\left(\sum_{k=0}^{\infty} 2^{k(1/2-1/p)q}|b_k|^q\right)^{1/q}<\infty,$$

where $b_k=\sup_{2^k\le\nu<2^{k+1}}|c_\nu|$, $k=0,1,2,\dots$, and

$$\|f\|_{N_{pq(M)}}\sim\left(\sum_{k=0}^{\infty} 2^{k(1/2-1/p)q}|b_k|^q\right)^{1/q}.$$

Proof. Suppose that $1 < p < \infty$ and $f \sim \sum_{k=1}^{\infty} c_k \chi_k$. Let us prove the inequality

$$\|b\|_{l^\sigma_\infty} \le c\|f\|_{N_{p\infty}(M_0)}, \tag{4}$$

where $\sigma = 1/2 - 1/p$. By definition,

$$\begin{aligned}\|b\|_{l^\sigma_\infty} &= \sup_{k\ge 0} 2^{k(1/2-1/p)} \max_{2^k\le\nu<2^{k+1}} |c_\nu| \le 2\sup_{k\ge 0} 2^{k(1/2-1/p)} \max_{1\le\nu\le 2^k} 2^{k/2}\left|\int_{(\nu-1)/2^k}^{\nu/2^k} f(x)\,dx\right| \\ &\le 2\sup_{Q\in M} \frac{1}{|Q|^{1/p'}}\left|\int_Q f(x)\,dx\right| = 2\|f\|_{N_{p\infty}(M)}.\end{aligned}$$

Thus, inequality (4) is proved for an arbitrary p, $1 < p < \infty$. The space with norm $\|c\| = \|b\|_{l^\sigma_q}$ is a retract of the space $l^\sigma_q(l_\infty)$ [6]. Therefore, from the interpolation properties of the spaces $l^\sigma_q(l_\infty)$ and Theorem 2, we obtain the "strong" inequality

$$\|b\|_{l_q^{1/2-1/p}} \le c\|f\|_{N_{pq}(M)}$$

for arbitrary $1 < p < \infty$ and $1 \le q \le \infty$.

Let us verify the reverse inequality acting by the same scheme. Suppose that Q is an arbitrary closed interval from the net M. Then

$$\frac{1}{|Q|^{1/p'}}\left|\int_Q f(x)\,dx\right| \le \sum_{k=1}^{\infty} \frac{1}{|Q|^{1/p'}}\left|\int_Q \sum_{\nu=2^{k-1}}^{2^k-1} c_\nu \chi_\nu(x)\,dx\right| = \sum_{k=1}^{\infty} \frac{1}{|Q|^{1/p}}\left|\sum_{2^{k-1}\le\nu<2^k} c_\nu \int_Q \chi_\nu(x)\,dx\right|.$$

It follows from the definition of the functions $\chi_\nu(x)$ that at most the two summands in the sum

$$\sum_{2^{k-1}\le\nu<2^k} c_\nu \int_Q \chi_\nu(x)\,dx$$

are nonzero, namely, those summands in which the supports of the functions χ_ν contain the endpoints of the closed interval Q. Therefore, we have

$$\begin{aligned}\frac{1}{|Q|^{1/p'}}\left|\int_Q f(x)\,dx\right| &\le 2\sum_{k=1}^{\infty} \frac{1}{|Q|^{1/p'}} \max_{2^{k-1}\le\nu<2^k} |c_\nu| \cdot 2^{k/2} \min\left(|Q|, \frac{1}{2^k}\right) \\ &\le 2\sum_{k=1}^{\infty} 2^{1/2-1/p'} \max_{2^{k-1}\le\nu<2^k} |c_\nu|.\end{aligned}$$

Since the choice of Q is arbitrary, we obtain

$$\|f\|_{N_{p\infty}(M)} = \sup_{|Q|\in M} \frac{1}{|Q|^{1/p'}}\left|\int_Q f(x)\,dx\right| \le c\|a_k\|_{l_1^{1/2-1/p'}}.$$

From the interpolation properties of the spaces $l^\sigma_r(l_\infty)$ and Theorem 2, we obtain

$$\|f\|_{N_{pq}(M)} \le c\|b\|_{l_q{}^{1/2-1/p}} = c\left(\sum_{k=0}^{\infty} 2^{q(1/2-1/p)k} \sup_{2^k\le\nu<2^{k+1}} |c_\nu|^q\right)^{1/q}.$$

Theorem 3 is proved. □

A function f is said to be *monotone* (*nonincreasing*) *in the extended sense* if for any $x \in (0,1]$ the following inequality holds:

$$|f(x)| \le \frac{B}{|x|}\left|\int_0^x f(y)\,dy\right|.$$

Theorem 4. *Suppose that* $1 < p < \infty$, f *is monotone in the extended sense. Then for* f *to belong to* $L_{pq}[0,1]$ *it is necessary and sufficient that the sequence of its Fourier–Haar coefficients* $\{c_n\}_{n=1}^{\infty}$ *satisfy the inequality*

$$\left(\sum_{k=0}^{\infty} 2^{k(1/2-1/p)q} |b_k|^q\right)^{1/q} < \infty.$$

The assertion of this theorem follows from Theorem 3, since for monotone (in the extended sense) functions when M is the set of all closed intervals from $[0,1]$, the norms of the spaces $N_{pq}(M)$ and $L_{pq}[0,1]$ are equivalent (we have two-sided inequalities).

A. V. Maslov [7] obtained Paley-type inequalities for Fourier–Haar series: suppose that there exists a monotone sequence of positive numbers $\{v_m\}_{m=0}^{\infty}$ such that

$$\sum_{m=0}^{\infty} (v_m(m+1))^{-1} < \infty.$$

Then:

(1) For any $p \in (1,2)$ and all functions $f \in L_p$, the following inequalities are valid:

$$C_1 \sum_{m=0}^{\infty} |c_m(f)|^p (v_m(m+1))^{p/2-1} \le \|f\|_p^p \le C_2 \sum_{m=0}^{\infty} |c_m(f)|^p (m+1)^{p/2-1}.$$

(2) For any $p > 2$ and all functions $f \in L_p$, the following inequalities are valid:

$$C_3 \sum_{m=0}^{\infty} |c_m(f)|^p (m+1)^{p/2-1} \le \|f\|_p^p \le C_4 \sum_{m=0}^{\infty} |c_m(f)|^p (v_m(m+1))^{p/2-1}.$$

The next theorem complements these results in a certain sense.

Theorem 5. *Suppose that* $\{\chi_k(x)\}_{k=1}^{\infty}$ *is the Haar system,* $f \sim \sum_{k=1}^{\infty} c_k \chi_k(x)$, *and*

$$\vartheta = \{\vartheta_k\} = \left\{\frac{1}{2^{k/2}} \sum_{\nu=2^{k-1}}^{2^k-1} c_\nu\right\}_{k=1}^{\infty}, \qquad u = \{u_k\} = \left\{2^{k/2} \max_{2^{k-1} \le \nu < 2^k} |c_\nu|\right\}_{k=1}^{\infty}.$$

(a) *If* $1 < p < 2$, $p' = p/(p-1)$, *and* $f \in L_p[0,1]$, *then the following inequality is valid*:

$$\|\vartheta\|_{l_{p'q}} \le c\|f\|_{L_{pq}[0,1]}.$$

(b) *If* $p > 2$ *and* $u \in l_{p'q}$, *then* $f \in L_{pq}[0,1]$ *and the following inequality is valid*:

$$\|f\|_{L_{pq}[0,1]} \le c\|u\|_{l_{p'q}}.$$

Proof. Consider the system of functions

$$\varphi_k(x) = \frac{1}{2^{k/2}} \sum_{\nu=2^{k-1}}^{2^k-1} \chi_\nu(x), \qquad k = 1, 2, \dots .$$

This system of functions is orthogonal, normalized, and bounded. The Fourier coefficients of the function $f(x)$ with respect to this system are of the form

$$\int_0^1 f(x)\varphi_k(x)\,dx = \int_0^1 f(x)\frac{1}{2^{k/2}}\sum_{\nu=2^{k-1}}^{2^k-1}\chi_\nu(x)\,dx = \frac{1}{2^{k/2}}\sum_{\nu=2^{k-1}}^{2^k-1} c_\nu = \vartheta_k.$$

For $1 < p < 2$, from Paley's inequality we obtain

$$\|\vartheta\|_{l_{p'q}} \le c\|f\|_{L_{pq[0,1]}}.$$

Let us prove assertion (b) of Theorem 5. For $p = \infty$, the following inequality holds:

$$\|f\|_{L_\infty} = \sup|f(x)| \le \sup_x \sum_{k=1}^{\infty}\biggl|\sum_{\nu=2^{k-1}}^{2^k-1} c_\nu\chi_\nu(x)\biggr| \le \sum_{k=1}^{\infty} 2^{k/2}\max_{2^{k-1}\le\nu<2^k}|c_\nu| = \|u\|_{l_1}.$$

For $p = 2$, from Parseval's relation we obtain

$$\|f\|_{L_2} = \biggl(\sum_{k=1}^{\infty} c_k^2\biggr)^{1/2} = \biggl(\sum_{k=1}^{\infty}\sum_{\nu=2^{k-1}}^{2^k-1} c_\nu^2\biggr)^{1/2} \le \biggl(\sum_{k=1}^{\infty}\Bigl(2^{k/2}\max_{2^{k-1}\le\nu<2^k}|c_\nu|\Bigr)^2\biggr)^{1/2} = \|u\|_{l_2}.$$

Using the interpolation properties of the corresponding spaces, for $2 < p < \infty$ we can write

$$\|f\|_{L_{pq}} \le c\|u\|_{l_{p'q}}. \quad \square$$

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