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SOFT SINGULARITY AND THE FUNDAMENTAL LENGTH

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It is shown that some regular solutions in 5D Kaluza-Klein gravity may have interesting properties if one from the parameters is in the Planck region. In this case the Kretschman metric invariant runs up to a maximal reachable value in nature, i.e. practically the metric becomes singular. This observation allows us to suppose that in this situation the problems with such soft singularity will be much easier resolved in the future quantum gravity then by the situation with the ordinary hard singularity (Reissner-Nordström singularity, for example). It is supposed that the analogous consideration can be applied for the avoiding the hard singularities connected with the gauge charges.

Key words: Planck length, singularity, Kaluza-Klein gravity.

1. INTRODUCTION

Any solution in any theory has some parameters: mass, charge, angular momentum, characteristic length and so on. The properties of the solution depends, of course, on the value of the parameters: the presence

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of event horizon, singularity and so on. In this paper we investigate the case when these properties crucially depend on the parameters values. The reason for this is that the nature may have a natural length: Planck (fundamental) length. In this case one can expect that some properties of the solution will be changed on the level of the fundamental length.

Here we consider the solution of the 5D Kaluza-Klein gravity [1] - [4] which has two parameters: electric charge q and some characteristic length r_0. From the mathematical point of view this solution is regular everywhere. We will show that if $r_0 \approx \ell$ ($\ell \approx 10^{-33} cm$ is the Planck length) then from the physical point of view the situation completely changes: in a part of this spacetime the Kretschman invariant becomes $\approx 1/\ell^4$ and it means that we have a singularity since this value is maximal if the fundamental length does exist.

2. FUNDAMENTAL LENGTH

In the fifties Wheeler has introduced the notion of a Planck length as a minimal achievable length in nature. The physical consequences of the existence of this length are very important. For example, the fluctuations of the metric on this level probably lead to the appearance of a spacetime foam where the fluctuations of a spacetime topology takes place. Unfortunately the mathematical rigorous introduction of this length is possible in a quantum gravity theory only. Nevertheless using the theory of deformations of a Lie algebra connected with given physical theory one can show [6]- [9] that a fundamental length is an essential ingredient of an extension of a physical theory.

The essence of the deformation theory is that one can extend the commutator $[\ ,\ ]_0$ of a Lie algebra $\mathcal{L}_0$ by introducing a new parameter t:

$$[A, B]_t = [A, B]_0 + \sum_{i=1}^{\infty} M_i\left(A, B\right) t^i, \tag{1}$$

where $A, B \in V$ and $M_i\left(A, B\right) \in V$; $[\ ,\ ]_t$ is a new commutator corresponding to the new parameter t (it can be, for example, either the speed of light c or the Planck constant $\hbar$ or another completely new fundamental constant) in a new Lie algebra $\mathcal{L}_t$.

A deformation of $\mathcal{L}_0$ is trivial if the algebra $\mathcal{L}_t$ is isomorphic to $\mathcal{L}_0$, i.e. there is an invertible transformation $T_t : V \rightarrow V$ such that

$$T_t\left([A, B]_t\right) = [T_t A, T_t B]_0 . \tag{2}$$

The most interesting case is with the non-isomorphic deformation from $\mathcal{L}_0$ to $\mathcal{L}_t$. In this case we can obtain a new physical theory with a new fundamental constant t.

Moving by such a way one can show that: (a) the stabilization from the classical mechanics to the relativistic one is possible by introducing a new fundamental constant, the speed of light c; (b) the stabilization from the classical mechanics to the quantum one is possible with the introduction of a new fundamental constant - the Planck constant $\hbar$; (c) stabilization from classical relativistic mechanics to an algebra with a fundamental length is possible [6-8].

In Refs. [10], a deformation of the canonical algebra for kinematical observables of the quantum field theory in Minkowski space-time has been considered under the condition of Lorentz invariance. A relativistic invariant algebra obtained depends on additional fundamental constants M, L and H with the dimensions of mass, length and action, respectively. Possible applications of obtained algebras for descriptions of states of matter under extreme conditions are briefly discussed.

In Refs. [11,12] (resting on the principles of the deformation idea) it is shown that the Lie algebra for the interface of the gravitational and quantum realms is the stabilized form of the Poincaré - Heisenberg algebra which carries three additional parameters and one from them is the fundamental length. It is shown that the stable Snyder-Yang-Mendes Lie algebra is a serious candidate for the symmetries underlying freely falling frames at the interface of quantum and gravitational ideas.

3. THE METRIC

We start with the 5D metric

$$ds^2 = \frac{1}{\Delta(r)}dt^2 - r_0^2\Delta(r)\left[d\chi + \omega(r)dt\right]^2 - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2). \quad (3)$$

The 5D Einstein's equations are

$$R_{AB} - \frac{1}{2}G_{AB}R = 0, \quad (4)$$

where $A, B = 0, 1, 2, 3, 5$. The solution, which was found in Refs. [1–4], we present in the form [5]

$$a = r_0^2 + r^2, \quad (5)$$

$$\Delta = \frac{q}{2r_0}\frac{r^2 - r_0^2}{r^2 + r_0^2}, \quad (6)$$

$$\omega = \frac{4r_0}{q}\frac{r}{r_0^2 - r^2}. \quad (7)$$

The curvature scalar R and the invariant $R_{AB}R^{AB}$ are zero in the consequence of the 5D Einstein's equations (4). The corresponding Kretschman invariant for the metric (3) and the solution (5)-(7) is

$$K = R^{ABCD}R_{ABCD} = \frac{24}{r_0^4}\frac{1 - 2\left(r/r_0\right)^2}{\left[1 + \left(r/r_0\right)^2\right]^4}. \quad (8)$$

We see that the spacetime with the metric (5)-(7) is regular everywhere. But there is one subtlety. This statement is correct only *if nature has not any fundamental length*. If such length exists then we should be more careful. The parameter r_0 describes the region of the spacetime (3)-(7) between two surfaces where $ds^2\left(\pm r_0\right) = 0$. In contrast with the Schwarzschild metric

$$ds^2 = \left(1 - r_g/r\right) dt^2 - \frac{dr^2}{1 - r_g/r} - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \qquad (9)$$

we do not need to introduce a new coordinate system to describe the region with $|r| < r_0$. It is visible that the metric (3), (5)-(7) can be written in the form

$$\begin{aligned} ds^2 = & \frac{2r_0}{q}\frac{r^2 - r_0^2}{r^2 + r_0^2}\left(dt^2 - \frac{q^2}{4}d\chi^2\right) + \frac{r_0^2 r}{r^2 + r_0^2}dtd\chi \\ & - dr^2 - \left(r^2 + r_0^2\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \end{aligned} \qquad (10)$$

In Ref. [5] a connection between the metric (10) and Wheeler's old proposal of electric charge as a wormhole filled with electric flux that flows from one mouth to the other – "the charge without charge" model of electric charge is made. Near to the surface $r = \pm r_0$,

$$ds^2 = (r_0/2)dtd\chi - dr^2 - 2r_0^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) + \mathcal{O}\left(r - r_0\right). \qquad (11)$$

Here immediately we see that we have not any singularity at the points $r = \pm r_0$.

4. SOFT SINGULARITY

The essential thing here is that, if $r_0 = \ell$, then the Kretschman invariant (8) blow up in the region $|r| \leq r_0$:

$$K \approx 1/\ell^4. \qquad (12)$$

As the Planck length ℓ is the minimal length then the maximal possible value of K can be $1/\ell^4$ only. This statement leads to the very strong conclusion: *an external observer sees the region $|r| \leq r_0$ as a soft singularity.* The word "soft" means that it is not ordinary (hard) singularity where $K = \infty$. But here, in the presence of the minimal length, we can say that $\frac{1}{\ell^4} \approx \infty$.

Now we will try to understand how one can apply this result to physics.

4.1 Inner Structure of a Singularity

At first we would like to describe what we have for this situation with $r_0 \approx \ell$. An external observer sees that the volume $r \leq r_0$ has a singularity although from the mathematical point of view the metric is regular one. The resolution of this problem is that we consider the *classical* solutions but one can hope that inside of the region $|r| \leq r_0$ a quantum gravity can gives us some nonsingular answer. The main hope here is that even on the classical level we have some inner structure of the soft singularity consequently the quantum gravity must gives us some nonsingular answer on the problem of an inner structure of the singularity.

In fact the above-mentioned consideration gives us a hint that any singularity where there is a gauge charge (electric or color) actually has an inner nonsingular structure. The regularization process of such singularity can be connected with the fact that near to the (even strong) singularity the gravitational field *in the presence of a gauge field* becomes so strong that it excites the dynamics on the extra dimensions and this dynamics can be described on the language of quantum gravity only.

4.2 Very Naive Model of the Electric Charge

The standard interpretation of the 5D metric (3) gives us the 4D metric

$$ds_4^2 = dt^2/\Delta(r) - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{13}$$

with the 4D electromagnetic potential

$$A_\mu = G_{5\mu}/G_{55} = \{\omega, 0, 0, 0\}. \tag{14}$$

Ordinarily, a 4D metric (13) with $A_t = \omega(r)$ cannot be considered as the 4D spacetime filled with the correct 4D electromagnetic field because of the existence an ubservabled scalar field $G_{55} = \Delta(r)$. But here there is one attenuating circumstance: the parameter $r_0 \approx \ell$. Below it will be shown that in this case $\Delta \approx const$ by $|r| \gg \ell$. It allows us to consider with the big accuracy the 5D spacetime (3) as a 4D spacetime filled with the Coulomb electric field. The measure of inaccuracy of such 5D $\rightarrow$ 4D reduction is $\approx \ell^2/r^2$. Also that is especially interesting: at the center we have a soft singularity region.

For more detailed consideration of this model, let us consider the 5D (χt)–Einstein's Eq. (4) (which is 4D Maxwell equation)

$$(\omega' \Delta 4\pi a)' = 0, \tag{15}$$

where $(\cdots)' = d(\cdots)/dr$. It has the solution [5]

$$\omega' = q/a\Delta. \tag{16}$$

In this case the metric (3), (5)-(7) can be considered as a 5D model of the electric charge since far away from the origin ($r = 0$) the electric field becomes like Coulomb field

$$E = \omega' \approx 1/r^2, \quad |r| \gg r_0. \tag{17}$$

The problem here is that there is a modification of the Coulomb law which can be measured experimentally. One way for the reduction of this correction to the minimum is $r_0 \to 0$. On this way we have only one obstacle: the fundamental length, i.e. $(r_0)_{min} \approx \ell$. In this case we will have the soft singularity and all that we spoke earlier in this occasion. Let us to consider the case with $r_0 = \ell$. At the first we would like to analyze carefully the notion of the electric field in this case.

The 5D Kaluza-Klein gravity after the dimensional reduction indicates that the Maxwell tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{18}$$

This allows us to write the electric field as $E_r = \omega'$. Equation (15), with the electric field defined by (18), can be compared with Maxwell's equations in a continuous medium,

$$\mathrm{div}\mathbf{D} = 0, \tag{19}$$

where $\mathbf{D} = \epsilon \mathbf{E}$ is an electric displacement and ϵ is a dielectric permeability. Comparing Eq. (19) with Eq. (15) we see that the magnitude $q/a = \omega' \Delta$ is like to the electric displacement and the dielectric permeability is $\epsilon = \Delta$. It means that q can be taken as the Kaluza-Klein electric charge because the flux of the electric displacement is $\Phi = 4\pi a \mathrm{D} = 4\pi q$.

For the 4D observer at infinity ($r \to \infty$), we have the following asymptotical behavior of the electric displacement D_r, electric field E_r, and the scalar field $G_{55} = \Delta$:

$$\mathrm{D}_r = \frac{q}{r^2 + r_0^2} \approx \frac{q}{r}\left(1 - \ell^2/r^2\right), \tag{20}$$

$$E_r = \omega' = \frac{2\ell}{r^2 - \ell^2} \approx \frac{2\ell}{r^2}\left(1 + \ell^2/r^2\right), \tag{21}$$

$$\Delta = \frac{q}{2r_0}\frac{r^2 - r_0^2}{r^2 + r_0^2} \approx \frac{q}{2\ell}\left(1 - 2\ell^2/r^2\right), \tag{22}$$

as $r_0 \approx \ell$. The deviation (20) from the Coulomb law can not be measured with the modern technical abilities. For the interpretation of the 5D Kaluza-Klein gravity as 4D electrogravity it is very important to have the dilaton field $G_{55} = const$. From the equation (22) we see that it is true with the accuracy $\approx \ell^2/r^2$. Probably the most important here is that the 4D observer will see the (soft) singularity at the center.

It is interesting to compare this situation with the 4D Reissner-Nordstöm solution. In the last case the metric crucially depends on the relation between the mass m and the electric charge q. If $m^2 > q^2$ $(c = G = 1)$ then there is an event horizon but in the opposite case – a naked singularity. In our 5D case the 4D metric is (13) and we see that G_{tt} metric component (3) asymptotically is

$$g_{tt} \approx \frac{2\ell}{q}\left(1 + \frac{2\ell^2}{r^2}\right). \tag{23}$$

Let us note that one can redefine the time t to remove the factor $2\ell/q$. From this equation we see that we have not the term m/r that means that the mass is zero $(m = 0)$ for this 4D interpretation of the 5D metric (3).

Near to the region $|r| \leq r_0$ we have the soft singularity which hopefully will be regularized in a future quantum gravity theory. After that we will have regular everywhere spacetime with the finite electric field, finite energy of electric field, arbitrary value of electric charge q in contrast with the Reissner-Nordström solution where the topological structure of spacetime depends on the relation between mass m and charge q.

Also it is interesting to note the in Refs. [13,14] the 4D analog (Levi-Civita-Robinson-Bertotti metric [15]) of the 5D gravitational flux tube solution (3), (5)-(7) is considered as a model of the electric charge.

5. CONCLUSIONS

In this notice we have shown that the well known spherically symmetric solution in the 5D Kaluza-Klein gravity has unexpected properties if one of the parameters of this solution is in the Planck region. This conclusion essentially follows from the fact that a minimal length exists in nature. We have shown that in this case an external observer will see the soft singularity. The appearance of such kind singularity is connected with the fact that at the center of this spacetime the Kretschman metric invariant blow up from zero (at $r = r_0/\sqrt{2}$) up to infinity (at $r = 0$). Certainly such situation demand the careful consideration on the basis of quantum gravity. The good news here is that such (soft) singularity has an inner structure in contrast with the hard singularity (Reissner-Nordström solution, for example). It gives a hope that the resolution of the singularity problem in general relativity can be found in quantum gravity.

Mathematically the soft singularity is regular one since near to $r = r_0$ the gravity is so strong that the dynamic on the 5-th dimension is excited. This observation permits us to suppose that *any* singularity having gauge charge (electric or color) can be regularized by the similar way: near to the singularity the superstrong gravitational field excites the metric dynamic on the extra dimensions and the quantum gravity effects smoothes away the singular metric up to a regular quantum one.

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