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MATHEMATICAL PHYSICS

Control Problem with a Point Source of Heat Used as a Controller

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An attempt to control the thermal processing of thin plates by point sources of heat leads to the heat equation

$$u_t - \Delta u = m(t)\delta(\boldsymbol{\omega}(t) - x) - s(t)\delta(\mathbf{r}(t) - x), \quad t \in (0, 1), \quad x \in \Omega, \tag{1}$$

which is considered in the domain $[0, 1] \times \Omega$, where Ω is a bounded domain in $\mathbf{R}^2$. In (1), $m(t)$ and $s(t)$ are scalar nonnegative functions related to the heating and cooling powers, respectively; $\delta(\cdot)$ is an approximation of the Dirac delta function; and $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t))$ and $\mathbf{r}(t) = (r_1(t), r_2(t))$ are vector functions specifying the heated and cooled areas of Ω at time t, respectively.

Equation (1) is supplemented with the initial and boundary conditions

$$u|_{t=0} = g(x), \quad u|_{\partial\Omega} = 0, \tag{2}$$

where $\partial\Omega$ is the boundary of Ω and $g(x)$ defines the initial temperature in Ω.

The problem is formulated as follows.

Problem 1. Let $g(x)$ be a twice continuously differentiable function in Ω that satisfies the condition $g|_{\partial\Omega} = 0$, and let $v(t, x)$ be a twice continuously differentiable function on $[0, 1] \times \Omega$ that satisfies condition (2). Given $\varepsilon > 0$, the goal is to find functions $m(t)$ and $s(t)$, each taking two values $m(t) = 0$ or $m(t) = m_0 > 0$ and $s(t) = 0$ or $s(t) = s_0 > 0$, and continuous vector functions $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t))$ and $\mathbf{r}(t) = (r_1(t), r_2(t))$ such that

$$\sup_{0 \le t \le 1} \|v(t, \cdot) - u(t, \cdot)\|_{L_2(\Omega)} \le \varepsilon,$$

where $u(t, x)$ is the solution of problem (1) satisfying condition (2), $L_2(\Omega)$ is the Hilbert space of square integrable functions in Ω, and $\|\cdot\|_{L_2(\Omega)}$ is the norm in $L_2(\Omega)$.

Problems of the type of problem 1 arise, for example, in the laser processing of metals [1], where $v(t, x)$ corresponds to the thermal regime ensuring the required hardening. It is easy to see that problem 1 with $\varepsilon = 0$ may not have a solution. For applications, however, it is sufficient that problem 1 has a solution for small $\varepsilon > 0$.

To ensure the existence of classical solutions, we assume in what follows that all the data in the problems under consideration are sufficiently smooth. However, if necessary, weak solutions can be considered and the corresponding conditions on the smoothness of the data can be specified.

Assume that $m(t)$ and $s(t)$ satisfy the conditions

$$m(t) = m_0 \quad \text{or} \quad m(t) = 0, \qquad s(t) = s_0 \quad \text{or} \quad s(t) = 0. \tag{3}$$

This corresponds to applications, namely, the sources of heat (for example, a laser beam) and the sources of cold (for example, a liquid nitrogen flow) can be switched on and off.

The function $\delta(x)$ is defined as

$$\delta(x) = \delta_1(x_1)\delta_1(x_2), \quad \delta_1(y)\begin{cases} = \gamma^{-1} & \text{if } |y| \le \dfrac{\gamma}{2} \\ \ge 0 & \text{if } |y| > \dfrac{\gamma}{2}, \end{cases} \tag{3'}$$

where $\gamma > 0$ is a large constant.

Theorem 1. *Suppose that* $0 < \theta < 1$, *and let* $g(x)$ *be a twice continuously differentiable function on* Ω, *while* $v(t, x)$ *be a twice continuously differentiable function on* $[0, 1] \times \Omega$ *such that*

$$g|_{\partial\Omega} = 0, \quad v(t, x)|_{\partial\Omega} = 0, \quad v(t, x)|_{t=0} = g(x), \quad -s_0(1-\theta) \le \frac{\partial}{\partial t}v(t, x) - \Delta_x v(t, x) < m_0(1-\theta). \tag{4}$$

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Then, for any $\varepsilon > \gamma^{-1}$, *there exist functions* $m(t)$ *and* $s(t)$ *satisfying* (3) *and continuous vector functions* $\boldsymbol{\omega}(t)$ *and* $\mathbf{r}(t)$ *such that the solution* $u(t, x)$ *of the problem*

$$u_t - \Delta u = m(t)\delta(\boldsymbol{\omega}(t) - x) - s(t)\delta(\mathbf{r}(t) - x), \quad (5)$$
$$u|_{t=0} = g(x), \quad u(t, x)|_{\partial\Omega} = 0, \quad (t, x) \in (0, 1] \times \Omega$$

satisfies the inequality

$$\sup_{0 \le t \le 1} \|v(t, \cdot) - u(t, \cdot)\|_{L_2(\Omega)} \le \varepsilon C. \quad (6)$$

If the opportunity of cooling is absent, then problem 1 turns into a special case with $s(t) = 0$. Then the following result holds.

Corollary 1. *Let all the conditions of Theorem* 1 *be satisfied and* $s(t) \equiv 0$. *Then, for any* $\varepsilon > \gamma^{-1}$, *there exists a function* $m(t)$ *obeying* (3) *and a continuous vector function* $\boldsymbol{\omega}(t)$ *such that the solution of problem* (5) (*with* $s(t) \equiv 0$) *satisfies estimate* (6).

In addition problem 1, we consider the following one.

Problem 2. Find functions $m(t)$ and $s(t)$ satisfying (3) and continuous vector functions $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t))$ and $\mathbf{r}(t) = (r_1(t), r_2(t))$ such that the solution $u(t, x)$ of Eq. (1) satisfies the conditions

$$u|_{t=0} = g(x), \quad u|_{t=1} = \gamma(x),$$
$$g(x)|_{\partial\Omega} = \gamma(x)|_{\partial\Omega} = 0, \quad (7)$$

where $g(x)$ and $\gamma(x)$ are continuous functions.

Problem 2 is considered in both two- and three-dimensional cases. Before solving problem 2, it is convenient to consider an abstract problem in a Hilbert space.

Let H be a Hilbert space and A be a self-adjoint positive operator in H. Consider the Cauchy problem

$$u' + Au = f(t), \quad t \in (0, 1),$$
$$u(t)|_{t=0} = g_1, \quad (8)$$

where $u(\cdot)$ and $f(\cdot)$ are H-valued functions and $g_1 \in H$.

Consider the following problem.

Problem 3. Find $f(t)$ such that the solution $u(t)$ of Cauchy problem (8) satisfies the conditions

$$u(t)|_{t=1} = g_2, \quad \int_0^1 |f(t)|_H^2 dt < \infty, \quad (8')$$

where g_2 is a given element of H.

Mathematical models of material processing by a laser heat source were described in [1–4], and the following result was proved and used in [2–4].

Theorem 2. *Let*

$$\int_0^1 |Ag(t)|^2 dt < \infty, \quad |Ag_1| < \infty.$$

If

$$f(t) = A(E - e^{-A})^{-1} \times \left[g_2 - e^{-A}g_1 - \int_0^1 e^{-A(t-\tau)}g(\tau)d\tau\right] + g(t), \quad (9)$$

then the solution $u(t)$ *of Cauchy problem* (8) *satisfies condition* (8'). *Conversely, if* $f(t)$ *is such that the solution* $u(t)$ *of problem* (8) *satisfies* (8'), *then* $f(t)$ *can be represented in the form of* (9).

In this theorem, the functions of the operator A are understood in the sense of spectral decompositions. Since problem 3 has a infinite set of solutions, the question arises of optimally choosing $f(t)$.

Definition 1. Let $R(\cdot)$ be a positive continuous function defined on $(-\infty, \infty)$. We say that $f_1(t)$ is an R-optimal solution of problem (8), (8') if

$$\int_0^1 |R(A)f_1(\eta)|_H^2 d\eta = \inf \int_0^1 |R(A)f(\eta)|_H^2 d\eta,$$

where the infimum is taken over all $f(\eta)$ such that the solution of Cauchy problem (8) satisfies the condition $u(1) = g_2$.

Theorem 3. *Let* $A \ge E$, *where* E *is the identity operator. If* $g_1, g_2 \in H$ *and* Ag_2, $R(A)g_1$, $R(A)Ag_2 \in H$, *then an* R-*optimal solution of problem* (8), (8') *exists and is*

$$f(t) = a + y(t) - A(1 - e^{-A})^{-1} \int_0^1 e^{-A(1-\tau)}y(\tau)d\tau,$$

where $y(t) = a + e^{-At}\theta + e^{At}\beta$, $a = A(E - e^{-A})^{-1}(g_2 - e^{-A}g_1)$, *and*

$$\beta = [(e^A - 1)^{-1} + (1 - 3e^{-A})(2 - 2A - 3e^{-A} + e^A)^{-1}]a,$$
$$\theta = (3 - e^A)(2 - 2A - 3e^{-A} + e^A)^{-1}a.$$

Note that the formula for the R-optimal solution is independent of $R(\cdot)$. Therefore, "R-optimal" is hereafter replaced by "optimal."

Theorem 3 is derived from Theorem 2 and the definition of optimality.

From Theorems 2 and 3, replacing H by $L_2(\Omega)$ and A by the Laplacian with the domain $\{u(x)\colon u(x) \in W_2^2(\Omega), u|_{\partial\Omega} = 0\}$, we obtain the following result.

Theorem 4. *If* $g(x)$ *and* $\gamma(x)$ *in problem* 2 *satisfy the conditions*

$$g(x) \in L_2(\Omega), \quad g(x)|_{\partial\Omega} = 0 \ \text{and}$$
$$\gamma(x) \in W_2^2(\Omega), \quad \gamma(x)|_{\partial\Omega} = 0,$$

then, for any $\varepsilon > \gamma^{-1}$, *there exist* $m(t)$, $s(t)$, $\boldsymbol{\omega}(t) = (\omega_1(t), \omega_2(t))$, *and* $\mathbf{r}(t) = (r_1(t), r_2(t))$ *obeying the conditions of problem* 2 *such that the solution* $u(t, x)$ *of problem* (5) *satisfies inequality* (6), *where* $v(t, x)$ *solves the problem*

$$v' - \Delta v = f(t,x), \quad 0 \le t \le 1, \quad x \in \Omega,$$
$$v|_{t=0} = g(x), \quad v|_{t=1} = l(x),$$
$$v|_{\partial\Omega} = 0,$$

with an optimally chosen $f(t, x)$.

A three-dimensional body is supplied with heat through its boundary. Therefore, in the case when Ω is a domain in $\mathbf{R}^3$, we consider the following problem.

Problem 4. Find a function $f(t, x)$ defined on $[0, 1] \times \Omega$ such that the solution of the Cauchy problem

$$\frac{\partial u}{\partial t} - \Delta u + u = 0,$$
$$u|_{t=0} = g(x), \quad \left(\frac{\partial}{\partial n}u - f(t,x)\right)\bigg|_{\partial\Omega} = 0 \qquad (10)$$

satisfies the condition $u(1, x) = l(x)$, where $g(x) \in L_2(\Omega)$ and $\gamma(x) \in W_2^{(2)}(\Omega)$.

Let P denote the operator extending a function defined on $[0, 1] \times \partial\Omega$ to a function defined on $[0, 1] \times \Omega$.

By introducing $v = u(t, x) - (Pf)(t, x)$, problem (10) can be written as

$$\frac{\partial v}{\partial t} - \Delta v + v = -\left(\frac{\partial}{\partial t} - \Delta + E\right)(Pf)(t,x),$$
$$v|_{t=0} = g(x) - (Pf)(0,x), \quad \frac{\partial v}{\partial n}\bigg|_{\partial\Omega} = 0.$$

The condition $u(1, x) = l(x)$ is rewritten as $v|_{t=1} = l(x) - (Pf)(t, x)|_{t=1}$.

Applying Theorem 2, we can prove the following result.

Theorem 5. *Problem* (10) *has a solution satisfying* $u(1, x) = l(x)$ *if and only if*

$$\int_0^1 e^{-A(1-\eta)}\Big[-\frac{\partial}{\partial n}f(\eta,x) + (\Delta - E)(Pf)(\eta,x)$$
$$-A(E - e^{-A})^{-1}(g(x) - (Pf)(0,x) - e^{-A}(l(x)$$
$$-(Pf)(1,x)))\Big]d\eta = 0.$$

Here, A is the operator $-\Delta + E$ with the domain $\left\{u\colon u \in W_2^2(\Omega),\ \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0\right\}$ and E is the unit operator. The functions of operators are understood in the sense of spectral decompositions.

Using this theorem, we can prove a result similar to Theorem 4.

Note that an important point is to describe an algorithm for constructing $m(t)$, $s(t)$, $\boldsymbol{\omega}(t)$, and $\mathbf{r}(t)$, whose existence is stated in Theorems 4, 1, and 2. The detailed proofs of these theorems are constructive, so the corresponding algorithm is easy to design.

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