

Quantization of classical singular solutions in Yang-Mills theory^(*)V. DZHUNUSHALIEV⁽¹⁾ (**) and D. SINGLETON⁽²⁾ (***)⁽¹⁾ *Institute für Mathematik, Universität Potsdam - 14469, Potsdam, Germany
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Summary. — In this paper we apply a variant of Heisenberg's quantization method for strongly interacting, nonlinear fields, to solutions of the classical Yang-Mills field equations which have a bad asymptotic behavior. After quantization we find that the bad features (*i.e.* divergent fields and energy densities) of these solutions are moderated. From these results we argue that in general the n -point Green's functions for Yang-Mills theories can have nonperturbative pieces which cannot be represented as the sum of Feynman diagrams. A formalism for dealing with these non-Feynman pieces via nonassociative field operators is suggested. These methods may also find some application in dealing with high- T_c superconductors.

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1. – Introduction

The quantization of strongly coupled, nonlinear fields is a difficult and still open problem. The success of QED and the Standard Model of electroweak interactions is largely due to the small size of the coupling constants, which allows for the perturbative expansion of physical quantities in terms of a power series in the coupling constants. This Feynman diagrammatic technique is the standard method for dealing with weakly coupled theories. For strongly interacting field theories such as quantum chromodynamics (QCD) this procedure cannot be applied in energy regimes where the coupling constant is large. The theoretical demonstration of confinement and the determination of the mass

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spectrum of light mesons and baryons are two examples where the perturbative techniques of quantum field theory are not good tools. These low-energy QCD phenomenon must be dealt with as completely nonperturbative, quantum effects.

A mathematical way of stating this basic difficulty in QCD is that one needs some non-perturbative method for calculating the n -point Green's functions of the theory. Usually these problems in strongly coupled field theories are dealt with using numerical simulations, but the computing power of modern computers does not yet permit, in general, the same kind of precise comparison between theory and experiment as is found in QED or the electroweak theory. Even if the state of the art in lattice gauge theory calculations increases to the same level of precision, it would still be desirable to have approximate, analytical methods for investigating these nonperturbative aspects of QCD or any strongly coupled field theory. Here we apply a nonperturbative method for calculating the n -point Green's functions of a theory which was originally proposed in the 50's by Heisenberg (see refs. [1-3]) to deal with the strongly coupled, nonlinear Dirac equation. In this method one writes down an infinite system of equations which connects together all the n -point Green's functions of the theory. Then by introducing some physically reasonable approximations one reduces this infinite system of equations into a finite system. Here we apply this quantization procedure to several classical solutions in $SU(2)$ and $SU(3)$ gauge theories. Although these classical solutions possess some interesting properties they also have bad features which are eliminated or reduced after the Heisenberg quantization is applied.

2. – Classical singular solutions in $SU(2)$ and $SU(3)$ gauge theories

In this section we examine several classical solutions for $SU(2)$ and $SU(3)$ Yang-Mills gauge theories. These solutions all have a similar, qualitative asymptotic behavior: one part of the classical gauge potential diverges as a power law and another part oscillates strongly. In this section we follow the notation and conventions of ref. [4].

2.1. The $SU(2)$ gauge "string". – Consider the following cylindrical ansatz for $SU(2)$ Yang-Mills theory:

$$\begin{aligned} (1a) \quad & A_t^1 = f(\rho), \\ (1b) \quad & A_z^2 = v(\rho), \\ (1c) \quad & A_\phi^3 = \rho w(\rho); \end{aligned}$$

z, ρ, ϕ are the standard cylindrical coordinates; $a = 1, 2, 3$ are $SU(2)$ color indices. Inserting this ansatz into the Yang-Mills field equations

$$(2) \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{a\mu}{}_\nu) + f^{abc} F^{b\mu}{}_\nu A_\mu^c = 0$$

yields

$$\begin{aligned} (3a) \quad & f'' + \frac{f'}{\rho} = f(v^2 + w^2), \\ (3b) \quad & v'' + \frac{v'}{\rho} = v(-f^2 + w^2), \end{aligned}$$

$$(3c) \quad w'' + \frac{w'}{\rho} - \frac{w}{\rho^2} = w(-f^2 + v^2).$$

In the case where $w = 0$ eqs. (3) reduce to

$$(4a) \quad f'' + \frac{f'}{\rho} = fv^2,$$

$$(4b) \quad v'' + \frac{v'}{\rho} = -vf^2.$$

The asymptotic behavior of the ansatz functions f, v and the energy density \mathcal{E} are

$$(5a) \quad f \approx 2 \left[x + \frac{\cos(2x^2 + 2\phi_1)}{16x^3} \right],$$

$$(5b) \quad v \approx \sqrt{2} \frac{\sin(x^2 + \phi_1)}{x},$$

$$(5c) \quad \mathcal{E} \propto f'^2 + v'^2 + f^2v^2 \approx \text{const},$$

where $x = \rho/\rho_0$ is a dimensionless radius, and ρ_0, ϕ_1 are constants. A numerical study of eqs. (4) was carried out in ref. [4], and it was found that $A_t^1 = f(\rho)$ was a confining, power law potential and $A_z^2 = v(\rho)$ was a strongly oscillating potential. Depending on the relationship between the initial conditions $v(0)$ and $f(0)$ the energy density near $\rho = 0$ was either a hollow (*i.e.* an energy density less than the asymptotic value) or a hump (*i.e.* an energy density greater than the asymptotic value). On account of this and the cylindrical symmetry of this solution we called this the “string” solution.

2.2. $SU(3)$ “bunker” solution. – Next we considered the following $SU(3)$ ansatz for the gauge fields [5-7]:

$$(6a) \quad A_0 = \frac{2\varphi(r)}{\mathbf{i}r^2} (\lambda^2x - \lambda^5y + \lambda^7z) + \frac{1}{2}\lambda^a (\lambda_{ij}^a + \lambda_{ji}^a) \frac{x^i x^j}{r^2} w(r),$$

$$(6b) \quad A_i^a = (\lambda_{ij}^a - \lambda_{ji}^a) \frac{x^j}{\mathbf{i}r^2} (f(r) - 1) + \lambda_{jk}^a (\epsilon_{ilj} x^k + \epsilon_{ilk} x^j) \frac{x^l}{r^3} v(r),$$

here λ^a are the Gell-Mann matrices; $a = 1, 2, \dots, 8$ is a color index; the Latin indices $i, j, k, l = 1, 2, 3$ are space indices; $\mathbf{i}^2 = -1$; r, θ, ϕ are spherically coordinates. Substituting eqs. (6) into the Yang-Mills equations (2) gives the following system of equations for $f(r), v(r), w(r)$ and $\varphi(r)$:

$$(7a) \quad r^2 f'' = f^3 - f + 7fv^2 + 2vw\varphi - f(w^2 + \varphi^2),$$

$$(7b) \quad r^2 v'' = v^3 - v + 7vf^2 + 2fv\varphi - v(w^2 + \varphi^2),$$

$$(7c) \quad r^2 w'' = 6w(f^2 + v^2) - 12fv\varphi,$$

$$(7d) \quad r^2 \varphi'' = 2\varphi(f^2 + v^2) - 4fvw.$$

This set of equations is difficult to solve even numerically so we investigated simplified cases.

First we examined the $f = \varphi = 0$ case, under which eqs. (7) became

$$(8a) \quad r^2 v'' = v^3 - v - vw^2,$$

$$(8b) \quad r^2 w'' = 6wv^2.$$

In the asymptotic limit $r \rightarrow \infty$ the form of the solution to eqs. (8) approaches a form similar to the “string” solution:

$$(9a) \quad v \approx A \sin(x^\alpha + \phi_0),$$

$$(9b) \quad w \approx \pm \left[\alpha x^\alpha + \frac{\alpha - 1 \cos(2x^\alpha + 2\phi_0)}{4} \frac{1}{x^\alpha} \right],$$

$$(9c) \quad 3A^2 = \alpha(\alpha - 1),$$

where $x = r/r_0$ is a dimensionless radius and r_0, ϕ_0 , and A are constants. A numerical investigation of eqs. (8) was done in ref. [4]. Since $v(r)$ was strongly oscillating this resulted in the space part of the gauge field of eq. (6b) being strongly oscillating. The ansatz function $w(r)$ increased as some power of x as $x \rightarrow \infty$, which indicated a classically confining potential.

The “magnetic” and “electric” fields associated with this solution can be found from A_μ^a , and have the following behavior:

$$(10a) \quad H_r^a \propto \frac{v^2 - 1}{r^2}, \quad H_\phi^a \propto v', \quad H_\theta^a \propto v',$$

$$(10b) \quad E_r^a \propto \frac{rw' - w}{r^2}, \quad E_\phi^a \propto \frac{vw}{r}, \quad E_\theta^a \propto \frac{vw}{r},$$

for E_r^a, H_θ^a , and H_ϕ^a the color index is $a = 1, 3, 4, 6, 8$ and for H_r^a, E_θ^a and E_ϕ^a $a = 2, 5, 7$. The asymptotic behavior of H_ϕ^a, H_θ^a and E_ϕ^a, E_θ^a is dominated by the strongly oscillating function $v(r)$. Later we will show that the quantum corrections to this solution tend to smooth out these strongly oscillating fields. From eqs. (10) and the asymptotic form (9) the radial components of the “magnetic” and “electric” have the following asymptotic behavior:

$$(11) \quad H_r^a \propto \frac{1}{r^2}, \quad E_r^a \propto \frac{1}{r^{2-\alpha}}.$$

The radial “electric” field falls off slower than $1/r^2$ (since $\alpha > 1$) indicating the presence of a confining potential. The $1/r^2$ fall off of H_r^a indicates that this solution carries a “magnetic” charge. This was also true for the solutions discussed in refs. [5, 7].

This solution of the classical $SU(3)$ field equations exhibits fields which lead to a classical confining behavior, and which have similarities with certain phenomenological potential models of confinement. The most significant drawback of these solutions is that they have infinite field energy. In the case of the “bunker” solution one finds that the asymptotic form of the energy density goes as

$$(12) \quad \mathcal{E} \propto 4 \frac{v'^2}{r^2} + \frac{2}{3} \left(\frac{w'}{r} - \frac{w}{r^2} \right)^2 + 4 \frac{v^2 w^2}{r^4} + \frac{2}{r^4} (v^2 - 1)^2 \approx \frac{2}{3} \frac{\alpha^2 (\alpha - 1)(3\alpha - 1)}{x^{4-2\alpha}}.$$

As $\alpha > 1$ this energy density will yield an infinite field energy when integrated over all space. This can be compared with the finite field energy monopole and dyon solution [8]. However it has been demonstrated [9] that the finite energy monopole solutions cannot trap a test particle while the infinite energy solutions can.

2.3. The classical f , $\varphi \neq 0$ case. – Finally we examine the $v = w = 0$ case:

$$(13a) \quad r^2 f'' = f^3 - f - f\varphi^2,$$

$$(13b) \quad r^2 \varphi'' = 2\varphi f^2.$$

These equations have well-known, nonsingular monopole solutions, but these monopole solutions require special boundary conditions. If one does not impose special conditions, then a more general solution is

$$(14a) \quad f \approx A \sin(x^\alpha + \phi_0),$$

$$(14b) \quad \varphi \approx \pm \left[\alpha x^\alpha + \frac{\alpha - 1}{4} \frac{\cos(2x^\alpha + 2\phi_0)}{x^\alpha} \right],$$

$$(14c) \quad A^2 = \alpha(\alpha - 1),$$

which is the same as the asymptotic form of the “bunker” solution, and thus shares its bad long-distance behavior. A numerical investigation of the solutions to eq. (13) also indicates a similarity to the “bunker” solution. The “magnetic” and “electric” fields associated with this solution are also similar to those of the “bunker” solution.

$$(15a) \quad H_r^a \propto \frac{f^2 - 1}{r^2}, \quad H_\phi^a \propto f', \quad H_\theta^a \propto f',$$

$$(15b) \quad E_r^a \propto \frac{r\varphi' - \varphi}{r^2}, \quad E_\phi^a \propto \frac{f\varphi}{r}, \quad E_\theta^a \propto \frac{f\varphi}{r};$$

here the color indices $a = 2, 5, 7$ indicate that this is really an $SU(2)$ gauge potential embedded within $SU(3)$. We now review the Heisenberg quantization method in order to apply it to these three solutions.

3. – Heisenberg quantization of strongly interacting fields

Heisenberg’s basic idea is that the n -point Green’s functions can be found from some infinite set of coupled differential equations, which are derived from the field equations for the field operators. As an example we show how this method of quantization works for a spinor field with a nonlinear self-interaction [1-3]. The basic equation for the spinor field is

$$(16) \quad \gamma^\mu \partial_\mu \hat{\psi}(x) - l^2 \Im[\hat{\psi}(x)(\hat{\bar{\psi}}(x)\hat{\psi}(x))] = 0,$$

where γ^μ are Dirac matrices; $\hat{\psi}(x)$, $\hat{\bar{\psi}}(x)$ are the operators of the spinor field and its adjoint, respectively; $\Im[\hat{\psi}(\hat{\bar{\psi}}\hat{\psi})] = \hat{\psi}(\hat{\bar{\psi}}\hat{\psi})$ or $\hat{\psi}\gamma^5(\hat{\bar{\psi}}\gamma^5\hat{\psi})$ or $\hat{\psi}\gamma^\mu(\hat{\bar{\psi}}\gamma_\mu\hat{\psi})$ or $\hat{\psi}\gamma^\mu\gamma^5(\hat{\bar{\psi}}\gamma_\mu\gamma^5\hat{\psi})$. The constant l has units of length, and sets the scale for the strength of the interaction. Heisenberg emphasized that the 2-point Green’s function, $G_2(x_2, x_1)$, in this theory differs strongly from the propagator in a linear theory in its behavior on the light cone:

in the nonlinear theory $G_2(x_2, x_1)$ oscillates strongly on the light cone in contrast to the propagator of the linear theory which has a δ -like singularity. Heisenberg defined τ functions

$$(17) \quad \tau(x_1 x_2 \dots | y_1 y_2 \dots) = \langle 0 | T[\hat{\psi}(x_1) \hat{\psi}(x_2) \dots \hat{\psi}(y_1) \hat{\psi}(y_2) \dots] | \Phi \rangle,$$

where T is the time ordering operator; $|\Phi\rangle$ is a state for the system described by eq. (16). Equation (17) establishes a one-to-one correspondence between the system state, $|\Phi\rangle$, and the set of functions τ . This state can be defined using the infinite function set of eq. (17). Using eqs. (16) and (17) we obtain the following infinite system of equations for the τ 's:

$$(18) \quad l^{-2} \gamma_{(r)}^\mu \frac{\partial}{\partial x_{(r)}^\mu} \tau(x_1 \dots x_n | y_1 \dots y_n) = \Im[\tau(x_1 \dots x_n x_r | y_1 \dots y_n y_r)] + \\ + \delta(x_r - y_1) \tau(x_1 \dots x_{r-1} x_{r+1} \dots x_n | y_2 \dots y_{r-1} y_{r+1} \dots y_n) + \\ + \delta(x_r - y_2) \tau(x_1 \dots x_{r-1} x_{r+1} \dots x_n | y_1 y_2 \dots y_{r-1} y_{r+1} \dots y_n) + \dots$$

Equation (18) represents one of an infinite set of coupled equations which relate various orders (given by the index n) of the τ functions to one another. To make some head way toward solving this infinite set of equations Heisenberg employed the Tamm-Dankoff method whereby he only considered τ functions up to a certain order. This effectively turned the infinite set of coupled equations into a finite set of coupled equations.

Heisenberg used the procedure sketched above to study the Dirac equation with a nonlinear coupling. Here we apply this procedure to nonlinear, bosonic field theories such as QCD in the low-energy limit. In particular by applying this method to the infinite energy, solutions just discussed, and making certain assumptions analogous to the Tamm-Dankoff cut-off, the unphysical asymptotic behavior of these classical Yang-Mills solutions is "smoothed" out. It is also possible that this quantization method could be applied to the recently proposed [10] strongly interacting phonon theory of high- T_c superconductors.

4. – Quantization of singular classical solutions

Before applying the above quantization method we make the following simplifying assumptions:

1) The degrees of freedom relevant for studying the singular solutions (both classically and quantum mechanically) are given entirely by the two ansatz functions f, v for eqs. (4), or v, w for eqs. (8), or f, φ for eqs. (13). No other degrees of freedom arise through the quantization process.

2) From eqs. (5), (9) and (14) one function is a smoothly varying function for large x , while another function is strongly oscillating. We take the smoothly varying function to be an almost classical degree of freedom, while the oscillating function is treated as a fully quantum-mechanical degree of freedom. Naively one might think that in this way only the behavior of the second function would change, while the first function stayed the same. However since both functions are interrelated due to the nonlinear nature of the field equations, we find that both functions are modified.

4.1. *Pedagogical example: Weak anharmonic oscillator.* – As a pedagogical example we repeat some details of Heisenberg’s quantization method as applied to the weak anharmonic oscillator [1]. This section is also intended as a reference for when we apply this same procedure to the various classical Yang-Mills solutions. The Hamiltonian equations for the operators of the weak anharmonic oscillator are

$$(19a) \quad \hat{q} = \frac{\hat{p}}{m},$$

$$(19b) \quad \frac{\hat{p}}{m} = -\omega_0^2 \hat{q} - \lambda \hat{q}^3;$$

here \hat{q} and \hat{p} are the operators of the canonical conjugate variables for the anharmonic oscillator. The τ functions are

$$(20) \quad \tau(k|n-k)e^{i\omega t} = \langle 0|Tq(t_1)q(t_2) \cdots q(t_k)p(t_{k+1})p(t_{k+2}) \cdots p(t_n)|\Phi\rangle|_{t_1=t_2=\cdots=t_n=t}.$$

The Hamiltonian operator equations yield the following system:

$$(21) \quad i\omega\tau(k|l) = \frac{k}{m}\tau(k-1|l+1) - m\omega_0^2 l\tau(k+1|l-1) - \\ - m\lambda l\tau(k+3|l-1) + m\lambda \hbar^2 \frac{l(l-1)(l-2)}{4}\tau(k+1|l-3).$$

To a first approximation this reduces to

$$(22a) \quad i\omega\tau(1|0) = \frac{1}{m}\tau(0|1),$$

$$(22b) \quad \frac{i\omega}{m}\tau(0|1) = -\omega_0^2\tau(1|0) - \lambda\tau(3|0).$$

This is not a closed system, but by making the following approximation:

$$(23) \quad \tau(3|0) = \frac{3}{2} \frac{\hbar}{m\omega_0^2} \tau(1|0),$$

one can solve eqs. (22) to obtain

$$(24) \quad \omega = \pm\omega_0 \left(1 + \frac{3}{4} \frac{\hbar\lambda}{m\omega_0^3} \right),$$

which coincides with the quantum-mechanical solution as an expansion of order λ : $\omega_{1,0} = (E_1 - E_0)/\hbar$.

4.2. *The quantized $SU(2)$ gauge “string”.* – To apply Heisenberg’s quantization scheme to the Yang-Mills system of eq. (4) we replace the ansatz functions by operators $\hat{f}(\rho), \hat{v}(\rho)$ as in [11]:

$$(25a) \quad \hat{f}'' + \frac{\hat{f}'}{x} = \hat{f}\hat{v}^2,$$

$$(25b) \quad \hat{v}'' + \frac{\hat{v}'}{x} = -\hat{v}\hat{f}^2,$$

the primes denote derivatives with respect to the dimensionless radius x . Taking into account assumption (2) we let $\hat{f} \rightarrow f$ become just a classical function again, and replace \hat{v}^2 in eq. (25a) by its expectation value

$$(26a) \quad f'' + \frac{f'}{x} = f\langle v^2 \rangle,$$

$$(26b) \quad \hat{v}'' + \frac{\hat{v}'}{x} = -\hat{v}f^2.$$

Now if we took the expectation value of eq. (26b) and ignored the coupling to f on the right-hand side we would have an equation for determining $\langle v \rangle = \langle \Phi | \hat{v} | \Phi \rangle$. However the two nonlinear terms on the right-hand side of eqs. (26a), (26b) show that a new object, $\langle v^2 \rangle$, enters the picture so that eqs. (26a), (26b) are not closed. To obtain an equation for $\langle v^2 \rangle$ we act on $\hat{v}^2(x)$ with the operator $\left(\frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \right)$ giving

$$(27) \quad (\hat{v}^2)'' + \frac{1}{x}(\hat{v}^2)' = -2\hat{v}^2 f^2 + 2(\hat{v}')^2.$$

Taking the expectation value of this equation gives the desired equation for $\langle v^2 \rangle$

$$(28) \quad \langle v^2 \rangle'' + \frac{1}{x} \langle v^2 \rangle' = -2\langle v^2 \rangle f^2 + 2\langle (v')^2 \rangle.$$

Again this equation is not closed due to the $\langle (v')^2 \rangle$ term. We could again try to find an equation for $\langle (v')^2 \rangle$ by the same procedure we employed for $\langle v^2 \rangle$. This equation would also not be closed. Continuing in this way we would find an infinite set of equations. In order to have some hope of handling this problem we need to make some approximation to cut this process off. For the anharmonic oscillator Heisenberg solved this by assuming that $\tau(\underbrace{xx \cdots x}_m | \underbrace{pp \cdots p}_n) = \underbrace{\langle x(t)x(t) \cdots x(t) \rangle}_m \underbrace{\langle p(t)p(t) \cdots p(t) \rangle}_n \approx 0$ for large m or n . We try

two different approximations for the $\langle (v')^2 \rangle$ term and show that both yield similar large x behavior that fixes the infinite field energy problem of the classical solution. First we assume that $\langle (v')^2 \rangle \approx \pm \langle v^2 \rangle'$. Since $\langle (v')^2 \rangle$ is positive definite one picks the \pm sign so that the right-hand side of this assumption is also positive definite. Under this assumption the equations become

$$(29a) \quad \langle v^2 \rangle'' + \left(\frac{1}{x} \mp 2 \right) \langle v^2 \rangle' = -2\langle v^2 \rangle f^2,$$

$$(29b) \quad f'' + \frac{1}{x} f' = f\langle v^2 \rangle.$$

The approximate solution of eqs. (29a), (29b) is of the form

$$(30a) \quad \langle v^2 \rangle \approx v_0^2 \frac{\exp[-\gamma x]}{\sqrt{x}},$$

$$(30b) \quad f \approx f_\infty + f_0 \frac{\exp[-\gamma x]}{\sqrt{x}},$$

where

$$(31) \quad f_0 \gamma^2 = f_\infty v_0^2, \quad \gamma^2 \pm 2\gamma = -2f_\infty^2.$$

The second relationship can be written (using the first relationship) as

$$(32) \quad \gamma = \mp f_\infty \left(f_\infty + \frac{v_0^2}{2f_0} \right).$$

Although $\langle v'^2 \rangle \approx +\langle v^2 \rangle'$ leads to unphysical exponentially growing solutions, the assumption $\langle v'^2 \rangle \approx -\langle v^2 \rangle'$ leads to exponentially decaying solutions. Thus under this latter assumption we find that the quantum-mechanical treatment of this nonlinear system modifies the bad features of the classical solution. The asymptotic behavior of v goes from being strongly oscillating (see eq. (5b)) to decaying exponentially, while the asymptotic behavior of f goes from being linearly increasing (see eq. (5a)) to also decaying exponentially. In some intermediate region the two solutions should match up with one another. If the asymptotic forms for these ansatz functions are used in the energy density, \mathcal{E} , of eq. (5c) we find that the field energy is now finite. To calculate \mathcal{E} we would replace the classical terms $v'^2, f^2 v^2$ by the appropriate quantum operator and take the expectation value. The $\langle v'^2 \rangle$ term would be handled according to the assumption we used for closing the equations.

4.3. Quantized “bunker” solution. – Although the classical behavior of “bunker” solution is interesting due to its similarity with certain phenomenological confining potentials, the infinite field energy discussed at the end of subsect. 2.2 strongly argues against the physical importance of this solution. One possible escape from this conclusion is if quantum effects weakened or removed this bad long-distance behavior.

Again in eq. (8) we replace the ansatz functions by operators $\hat{w}(x), \hat{v}(x)$ [12]:

$$(33a) \quad x^2 \hat{v}'' = \hat{v}^3 - \hat{v} - \hat{v} \hat{w}^2,$$

$$(33b) \quad x^2 \hat{w}'' = 6\hat{w} \hat{v}^2;$$

here the prime denotes derivatives with respect to the dimensionless radius x . Taking into account assumption (2) we let $\hat{w} \rightarrow w$ become just a classical function again, and replace \hat{v}^2 in eq. (33b) by its expectation value to arrive at

$$(34a) \quad x^2 \hat{v}'' = \hat{v}^3 - \hat{v} - \hat{v} w^2,$$

$$(34b) \quad x^2 w'' = 6w \langle \hat{v}^2 \rangle.$$

If we took the expectation value of eq. (34a) we would almost have a closed system of differential equations relating w and $\langle \hat{v} \rangle$. The $\langle \hat{v}^2 \rangle$ term from eq. (34b) and the $\langle \hat{v}^3 \rangle$ term from eq. (34a) prevent the equations from being closed. Applying the operation $x^2 \partial^2 / \partial x^2$ to the operator \hat{v}^2 and using eq. (34a) yields

$$(35) \quad x^2 (\hat{v}^2)'' = 2\hat{v}^2 (\hat{v}^2 - 1 - w^2) + 2x^2 (\hat{v}')^2.$$

If we took the expectation of the above equation with respect to fluctuations in the ansatz function operator \hat{v}^2 , and combined this with eq. (34b) we would almost have a closed system for determining w and \hat{v}^2 except for the $\langle (\hat{v}')^2 \rangle$ term on the right-hand side of

eq. (35). Continuing in this way one could obtain an infinite set of equations for various powers of the ansatz function operator (*i.e.* \hat{v}^n). As in the previous section we make some assumption that effectively cuts off the system of equations at some finite order. We assume that the mean square of the ϕ component of the magnetic field $\hat{H}_\phi = \hat{v}'$ is small⁽¹⁾

$$(36) \quad \langle (\Delta \hat{H}_\phi)^2 \rangle = \langle (\hat{H}_\phi - \langle \hat{H}_\phi \rangle)^2 \rangle = \langle \hat{H}_\phi^2 \rangle - \langle \hat{H}_\phi \rangle^2 \approx 0,$$

that implies

$$(37) \quad \langle \hat{H}_\phi^2 \rangle \approx \langle \hat{H}_\phi \rangle^2 \quad \text{or} \quad \langle (\hat{v}')^2 \rangle \approx (\langle \hat{v}' \rangle)^2.$$

Then by taking the expectation of eq. (35) we arrive at a closed system of equations from eqs. (34a), (34b), (35):

$$(38a) \quad x^2 \langle \hat{v}^2 \rangle'' = 2 \langle \hat{v}^2 \rangle^2 - 2 \langle \hat{v}^2 \rangle w^2 - 2 \langle \hat{v}^2 \rangle + 2x^2 (\langle \hat{v}' \rangle)^2,$$

$$(38b) \quad x^2 w'' = 6w \langle \hat{v}^2 \rangle,$$

$$(38c) \quad x^2 \langle \hat{v} \rangle'' = \langle \hat{v} \rangle^3 - \langle \hat{v} \rangle - w^2 \langle \hat{v} \rangle;$$

here we have further assumed that $\langle v^4 \rangle \approx \langle v^2 \rangle^2$ and $\langle v^3 \rangle \approx \langle v \rangle^3$. Note that the present situation is slightly more complex than the quantization of the string solution of the previous section since now we have three coupled equations rather than two. It is straightforward to show that in the limit $x \rightarrow \infty$ the closed system given by eqs. (38a)-(38c) is solved by

$$(39a) \quad \langle \hat{v}^2 \rangle \approx 1 + \frac{a^2}{2x^2},$$

$$(39b) \quad w \approx \frac{b}{x^2},$$

$$(39c) \quad \langle \hat{v} \rangle \approx \pm 1 + \frac{a}{x}.$$

Equations (39a)-(39c) provide information about the behavior of the “classical” ansatz function w , and the “quantum” ansatz function, v , via $\langle \hat{v} \rangle$ and $\langle \hat{v}^2 \rangle$. The main point is that after applying the Heisenberg-like quantization procedure to the classical singular solution, the infinite increase of the ansatz function w has changed to an acceptable asymptotic behavior (*i.e.* one that leads to a finite field energy). By replacing $v^2, w, (v')^2$ in eq. (12) with $\langle \hat{v}^2 \rangle, w, (\langle \hat{v}' \rangle)^2$ —from eqs. (39a)-(39c)—we find that the field energy density of the quantized “bunker” solution takes the form

$$(40) \quad \mathcal{E} \propto \frac{1}{r^6}$$

in the limit in which quantum fluctuations become important (*i.e.* for non-Abelian theories which exhibit asymptotic freedom this means in the low energy or $r \rightarrow \infty$

⁽¹⁾ In other words the fluctuations of the \hat{H}_ϕ component of the magnetic field are weak.

range) the energy density goes from the form given in eq. (12) to that given in eq. (40). This gives a finite field energy. In the high-energy or short-distance regime we assume that the fields approach the classical configuration of ref. [4] due to asymptotic freedom. This classical configuration is well behaved at $r = 0$, but has an infinite field energy due to its divergence as $r \rightarrow \infty$.

4.4. *The quantized $f, \varphi \neq 0$ case.* – Finally, we employ the above quantization technique to the singular solutions of eqs. (13) for the $v = w = 0$ case. Again we replace the ansatz functions by operators $\hat{f}(x), \hat{\varphi}(x)$:

$$(41a) \quad x^2 \hat{f}'' = \hat{f}^3 - \hat{f} - \hat{f} \hat{\varphi}^2,$$

$$(41b) \quad x^2 \hat{\varphi}'' = 2\hat{\varphi} \hat{f}^2;$$

using assumptions similar to the previous sections we have the following equations for the quantum function f , and the classical function φ :

$$(42a) \quad x^2 \hat{f}'' = \hat{f}^3 - \hat{f} - \hat{f} \langle \varphi^2 \rangle,$$

$$(42b) \quad x^2 \varphi'' = 2\varphi \langle f^2 \rangle.$$

Using the same assumptions (including the assumption about the weak fluctuation) and procedure as in the previous subsection we can turn eqs. (42) into a closed system of equations for the expectation values of various powers of the ansatz function operators:

$$(43a) \quad x^2 \langle \hat{f}^2 \rangle'' = 2\langle \hat{f}^2 \rangle^2 - 2\langle \hat{f}^2 \rangle \varphi^2 - 2\langle \hat{f}^2 \rangle + 2x^2 (\langle \hat{f} \rangle')^2,$$

$$(43b) \quad x^2 \varphi'' = 2\varphi \langle \hat{f}^2 \rangle,$$

$$(43c) \quad x^2 \langle \hat{f} \rangle'' = \langle \hat{f} \rangle^3 - \langle \hat{f} \rangle - \varphi^2 \langle \hat{f} \rangle.$$

This system has the following asymptotic solution:

$$(44a) \quad \langle f^2 \rangle \approx 1 - \frac{a}{x},$$

$$(44b) \quad \langle f \rangle \approx \pm 1 - \frac{a}{x},$$

$$(44c) \quad \varphi \approx \frac{b}{x}.$$

The field energy density for this solution takes the form

$$(45) \quad \mathcal{E} \propto \frac{1}{r^6}.$$

5. – Operator quantization for strongly interacting fields

In the previous sections we saw that strongly nonlinear fields (*e.g.*, non-Abelian gauge fields) can have nonlocal objects (the classical $SU(2)$ flux tube solution and the $SU(2)$, $SU(3)$ “bunker” solutions) which have good long-distance behavior after an application of the Heisenberg quantization procedure. This result is markedly different from the situation for linear quantum field theories with no or small self-interaction. In these cases the quantum fields cannot form static field configurations such as monopoles, flux

tubes, or “bunker” solutions discussed above. In general one needs strongly interacting, nonlinear theories in order for such objects to exist.

All of this may be an indication that *the n -point Green’s functions in non-Abelian gauge theories can have some pieces which are not simply the sum of Feynman diagrams*. Physically, this would imply that some physical processes in non-Abelian gauge theories cannot be explained solely in terms of the perturbative fluctuations of field quanta. Heisenberg may have had this situation in mind when he indicated [1] that the divergence behavior in a theory with strong interactions will differ from that found in a perturbatively, renormalizable theory like quantum electrodynamics. In this section we want to first present some formal and then some heuristic arguments which support this idea of non-perturbative, non-Feynman contributions to the Green’s functions.

5.1. Quantum field theory with nonassociative field operators. – In this subsection we consider field operators which are nonassociative. There are two reasons for doing this. First, in the Heisenberg quantization scheme one problem (which was dealt with by an approximate Tamm-Dankoff cutoff) was the appearance of an infinite system of differential equations with interrelated Green’s functions. Nonassociative field operators can lead to a quantum theory where the Green’s functions are functionally independent of one another, which would help this problem. Second, a nonassociative quantum theory can lead to amplitudes and processes which cannot be represented simply in a Feynman diagrammatic way by the exchange of virtual quanta. This supports the general picture given in the introduction to this section. In the following we will use the notations and conventions of ref. [13].

In the standard, noncommutative quantum field theories it is impossible to define an n -point Green’s function that is not some polylinear combination of propagators and interaction vertices. This general feature of the Green’s functions is altered if the field operators are nonassociative. If one considers the time ordering of a product of n field operators, then the time ordering will result in a shifting of the positions of the individual field operators. In a noncommutative theory the shifting in the positions of the field operators usually leads to the appearance of a commutator. The time ordering can also lead to a change in the ordering of the brackets which group field operators together. In a nonassociative theory such a change in the ordering of the brackets leads to the appearance of an *associator*, just as in a noncommutative algebra the displacement of an operator to the left or right gives rise to a commutator. A good feature of nonassociative quantum field theories is that one can obtain n -point Green’s functions that are independent of each other.

We denote the product of n field operators in which the brackets are arranged in accordance with some rule P by $M_n(P)$. For example,

$$(46) \quad M_3(P) = \left((\hat{\varphi}_x \hat{\varphi}_y) \hat{\varphi}_z \right),$$

where $\hat{\varphi}_x = \hat{\varphi}(x)$ is the field operator at the point x ; in this case, the rule P is that all opening brackets are at the extreme left-hand position.

We call this ordering of the brackets for any number of operators (*i.e.* with all opening brackets in the extreme left-hand position) the normal position ordering of the brackets, and denote it by colons

$$(47) \quad : M_n(P) := \left(\left(\cdots \left(\hat{\varphi}_1 (\hat{\varphi}_2 \cdots \right) \hat{\varphi}_n \right) \right).$$

On the left there are n opening brackets in succession.

Next, we consider a nonassociative quantum field theory with the following axioms.

1) Two monomials $M_n(P_1)$ and $M_n(P_2)$ that differ only in the placement rule of the brackets differ from each other by a numerical function $A(x_1, x_2, \dots, x_n; P_1, P_2)$, which we call the n -point associator:

$$(48) \quad M_n(P_1) - M_n(P_2) = A(x_1, x_2, \dots, x_n; P_1, P_2).$$

2) The action of the product of two monomials on the quantum state vector $|v\rangle$ is defined as follows:

$$(49a) \quad \left(M_n(P_1)M_k(P_2) \right) |v\rangle = M_n(P_1) \left(M_k(P_2) |v\rangle \right),$$

$$(49b) \quad \langle v | \left(M_n(P_1)M_k(P_2) \right) = \left(\langle v | M_n(P_1) \right) M_k(P_2).$$

3) We have the usual commutation rules

$$(50) \quad [\hat{\varphi}_x, \hat{\varphi}_y] = G(x, y),$$

where $G(x, y)$ is the 2-point Green's function (propagator) which differs from the propagator of a linear theory.

As an example of this formalism we examine the following simplified model, in which the action of any field operator on the vacuum state annihilates it

$$(51) \quad \hat{\varphi}_x |\text{vac}\rangle = \langle \text{vac} | \hat{\varphi}_x = 0.$$

Now the 3-point associator is given by

$$(52) \quad (\hat{\varphi}_x \hat{\varphi}_y) \hat{\varphi}_z = \hat{\varphi}_x (\hat{\varphi}_y \hat{\varphi}_z) + A(x, y, z).$$

Using the rules introduced above for the action of the field operators on the vacuum state, we find that $A(x, y, z) = 0$. There are two 4-point associators given by

$$(53a) \quad \left((\hat{\varphi}_x \hat{\varphi}_y) \hat{\varphi}_z \right) \hat{\varphi}_u = (\hat{\varphi}_x \hat{\varphi}_y) (\hat{\varphi}_z \hat{\varphi}_u) + A_1(x, y|z, u),$$

$$(53b) \quad \hat{\varphi}_x \left(\hat{\varphi}_y (\hat{\varphi}_z \hat{\varphi}_u) \right) = (\hat{\varphi}_x \hat{\varphi}_y) (\hat{\varphi}_z \hat{\varphi}_u) + A_2(x, y|z, u).$$

Acting on each of eqs. (53) from the right and left with the vacuum state and using the condition (51), the 4-point associators $A_{1,2}$ can be expressed in terms of the vacuum expectation value of the monomial $(\hat{\varphi}_x \hat{\varphi}_y)(\hat{\varphi}_z \hat{\varphi}_u)$:

$$(54) \quad A_{1,2}(x, y|z, u) = A(x, y|z, u) = -\langle \text{vac} | (\hat{\varphi}_x \hat{\varphi}_y) (\hat{\varphi}_z \hat{\varphi}_u) | \text{vac} \rangle.$$

$x^\mu \sim y^\mu, z^\mu \sim u^\mu$ (x^μ, y^μ, z^μ and u^μ are 4-vectors). Using the commutation properties of the operators, the property of the vacuum state, and taking the points to be pairwise

spacelike ($x^\mu \sim y^\mu, z^\mu \sim u^\mu$) we can establish the following symmetry properties of the 4-point associator:

$$(55) \quad A(x, y|z, u) = A(y, x|z, u) = A(x, y|u, z).$$

The Green's function can now be defined as

$$(56) \quad T(M_n(P)) = M_n(P) + G(x_1, x_2, \dots, x_n; P),$$

where T denotes the ordinary time ordering of the operators in the monomial $M_n(P)$, and also normal ordering of the brackets in the monomial $M_n(P)$. Thus each n -point Green's function can be expressed in terms of a polylinear combination of n -point associators ($m \leq n$) and propagators.

5.2. Heuristic argument for non-Feynman contributions to the Green's functions. – In carrying out our approximate application of Heisenberg's quantization method to certain classical, singular solutions, we examined only two degrees of freedom (f and v , v and w or f and φ). In each of these pairs one function was taken as a quantum degree of freedom and the other remained a classical degree of freedom. To investigate this method further one could drop this assumption by treating both functions as quantum degrees of freedom. In this section we look at a broader issue of how to include the other frozen degrees of freedom. In other words, the quantization process might lead to more degrees of freedom beyond the original classical degrees (*e.g.*, f and v).

Our main assumption in addressing this question is that for any n -point Green's function, for strongly interacting, nonlinear fields, there is a non-perturbative piece which can be treated by the Heisenberg quantization method⁽²⁾. In the classical region these degrees of freedom region can be associated with classical (possibly singular) field configurations. The others degrees of freedom are then handled using ordinary perturbative methods (*i.e.* using Feynman diagram techniques).

Physically, this means that in a theory with strongly interacting fields one has

1) nonlocal static objects, formed by those quantum degrees of freedom which are connected with (possibly singular) solutions of the classical field equations.

2) The quanta which fluctuate around these nonlocal objects.

In terms of QCD this implies that there are degrees of freedom which play a dominate role in the Feynman path integral to a first approximation. These degrees of freedom are connected with classical (possibly singular) solutions of the Yang-Mills field equations. This is somewhat similar to the saddle-point method of evaluating the Feynman path integral. However, in the saddle-point calculation one approximates the path integral via the value of the action evaluated at a particular solution

$$(57) \quad \int D\varphi e^{S[\varphi]} \approx e^{S[\varphi_0]},$$

here φ is a field degree of freedom (in our case it is the non-Abelian potential A_μ^a), and φ_0 is a classical, nonsingular solution of the field equations. For singular, classical solutions

⁽²⁾ This piece may be connected with the nonassociative properties of fields operators.

this approximation is not valid. Instead one can as a first approximation take

$$(58) \quad \int D\varphi e^{S[\varphi]} \approx \int D\varphi_{\text{cl}} e^{S[\varphi_{\text{cl}}]};$$

here φ_{cl} represents the ansatz functions used in the classical field equation. For the ansatz functions of eq. (7) (f, v, w, φ) the integration would become $D\varphi_{\text{cl}} \rightarrow DfDvDwD\varphi$. This leads to the following expression for the path integral:

$$(59) \quad \int DA_{\mu}^a e^{S[A_{\mu}^a]} = \int D(A_{\mu}^a)_{\text{dev}} \left[\int D(A_{\mu}^a)_{\text{cl}} e^{iS[(A_{\mu}^a)_{\text{cl}} + (A_{\mu}^a)_{\text{dev}}]} \right];$$

here $(A_{\mu}^a)_{\text{cl}}$ are the classical solutions in terms of the ansatz functions, and $(A_{\mu}^a)_{\text{dev}}$ are the perturbative, quantum deviations from the classical solution. The first integral is calculated by some nonperturbative method (lattice gauge calculation, Heisenberg method, etc.). The second path integral can be handled by ordinary perturbative methods (Feynman diagrams).

At this point one can ask how renormalization fits in with the above development where the Green's functions contain a nonperturbative piece. In the standard perturbative treatment of a quantum field theory one must introduce a renormalization prescription. The physical basis for this can to some extent be traced to the point-like nature of the quanta and interaction vertices. Mathematically the divergences come from momentum integrals over closed loops which have propagators (which are distributions in linear field theories) raised to some power. In the developments of the previous sections, where the n -point Green's functions have a non-perturbative part due to the nonlinear nature of the quantized fields, it is no longer necessarily true that the Green's functions need to be distributions. A generalization of the standard Feynman diagram technique would represent any complete n -point Green's functions as the sum of two pieces: one the standard, perturbatively calculated field fluctuations, and the other would be the non-Feynman part (possibly related to nonsingular or weakly singular classical solutions). A similar situation arises in superstring theory where the building blocks for the scattering diagrams are nonlocal strings. In both the present case and in the case of string theory the divergences of the quantum theory are eliminated or decreased⁽³⁾.

6. – Discussion and conclusions

The basic idea proposed here is that *quantum field theories with strongly non-linear fields can have the non-Feynman pieces*. Also the classical, possibly singular solutions can play an important role in the corresponding quantum field theory. After quantization these classical solutions can lead to nonlocal objects with good long-distance behavior. These quantized nonlocal objects are connected with the nonperturbative part of the Green's functions, and they are not the result of the perturbative quantum fluctuations of the field.

⁽³⁾ A related question would be to investigate whether there is a classical, nonlinear field theory which after quantization resulted in n -point Green's functions with nonperturbative pieces that had a structure similar to p -branes. If so then one might consider string theory as a quantum field theory arising from some nonlinear (possible nonlocal) field theory!

This approach to the quantization of non-Abelian field theories is very similar to the attempts to quantize gravity on the basis of the Wheeler-de Wit equation. In the present case there is an important distinction: the operator of canonical momentum is not simply related to the derivative of a canonical coordinate (*i.e.* $\hat{P} \neq \delta/\delta\hat{q}$, where \hat{q} is a field operator). This follows from the fact that the propagator has an unusual behavior on the light cone. Reversing the analogy between the non-Abelian field equations and the Wheeler-de Wit equation one could postulate that the Wheeler-de Wit equation may be more complicated than is normally written, since the momentum operator in gravity is also not simply the ordinary derivative with respect to some corresponding field variable.

In subsects. 4'3 and 4'4 we found two spherically symmetric nonlocal objects with good asymptotic behavior after quantization. It may be possible to forge a connection between these solutions and the QCD picture for the “dressing” of valence quarks by gluons and quark-antiquark pairs. The authors of ref. [14] write: “One views dressing as surrounding the charged particle with a cloud of gauge fields. We ... see that, ... this cloud spreads out over the whole space, resulting in a highly nonlocal structure. This is not unexpected in QED, but in QCD we will see that there is a nonperturbative obstruction to the construction of this dressing.” The Heisenberg quantization method that we have outlined might provide a nonperturbative description of the above “dressing” procedure. If this is correct, then our results give structures which are “dressed” with gluons but not with quark-antiquark pairs since we did not consider quark fermion fields in this work.

6'1. Connection with the maximal Abelian gauge model. – The quantization method of non-Abelian gauge fields given in this work shares some common features with the Abelian Projection (AP) model of gluodynamics proposed by 't Hooft in ref. [15]. The basic idea of the AP model is that in a non-Abelian gauge theory such as QCD there are special degrees of freedom which play an essential role in confinement. These degrees of freedom are connected with monopoles which appear in the maximal Abelian subgroup, H , of the full non-Abelian group G . In the AP model the initial $SU(3)$ field equations are rewritten as an Abelian gauge theory with magnetic monopoles and some matter fields. Roughly speaking, the AP method implies that the off-diagonal gauge fields are excluded from consideration. More precisely one integrates over all the off-diagonal (G/H) gauge fields, leaving only the Abelian degrees of freedom [16].

$SU(3)$ gauge theory has monopole solutions, which at large distances look like $U(1)$ Abelian monopoles (*i.e.* the solution exhibits $1/r^2$ behavior asymptotically). Thus after quantization these Abelian monopole solutions should play a more important role in comparison with the quantized solutions examined in subsects. 4'3 and 4'4 since these quantized solutions have a faster fall off. This is in agreement with the AP model.

6'2. Connection with high- T_c superconductivity. – The quantum theory of solids can be taken as a quantum field theory on some lattice. Thus one could ask if the above formalism has an application to condensed matter systems. For example can the quantized, classical solutions discussed above play a role in condensed matter? In ref. [17] it was shown that a flux tube solution can appear in superconductors in the mixed state. In [10] a model of Cooper pairing was proposed in which the phonons exchanged between Cooper electrons are confined in a “flux” tube similar to the chromoelectric flux tubes which are thought to occur between quarks and anti-quarks in QCD.

One possible application of such a model [18] might be as a mechanism to explain the Cooper pairing in high- T_c superconductors. One could postulate that the Lagrangian for

high- T_c materials had a strong, nonlinear potential term. This could result in a classical, cylindrical solution (either singular or nonsingular), which after quantization via the Heisenberg-like method could give rise to a phonon flux tube that stretched between two Cooper electrons, binding them to much higher temperatures than in ordinary BCS superconductors.

* * *

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