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Absolute Convergence of Multiple Series of Fourier Coefficients with respect to Multiplicative Systems

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KEY WORDS: absolute convergence, multiple series, multiplicative systems.

In this paper, we consider series of the Fourier coefficients of functions of several variables with respect to multiplicative systems generalizing the Walsh system. We present a sufficient condition for the convergence of these series in terms of the complete group modulus of continuity. We prove that this condition cannot be improved in the function classes $H_p^{\omega_1,\omega_2,\dots,\omega_n}$.

Let us present the relevant definitions and notation. Suppose that $\{p_{k_j}^{(j)}\}_{k_j=1}^{\infty}$ $(j=1,2,\dots,n)$ are sequences of integers $p_{k_j}^{(j)} \ge 2$, $j=1,2,\dots,n$, $k_j=1,2,\dots$, while

$$G^{(j)} = \left\{ x = \{x_{k_j}^{(j)}\}_{k_j=1}^{\infty}, 0 \le x_{k_j}^{(j)} \le p_{k_j}^{(j)} - 1 \right\}, \qquad j=1,2,\dots,n,$$

are the corresponding groups of integer sequences with the group operation $(\dot{+})$ of coordinate-wise addition modulo $p_{k_j}^{(j)}$, and $G^n = \prod_{j=1}^{n} G^{(j)}$ is the direct product of the groups $G^{(j)}$.

The sets $G_{\nu_l}^{(j)} = \{x = \{x_{k_j}^{(j)}\}_{k_j=1}^{\infty} \in G^{(j)} : x_{k_j}^{(j)} = 0, 0 \le k_j < \nu_l\}$, $j=1,2,\dots,n$, $\nu_l = 1,2,\dots,n$, are subgroups of $G^{(j)}$, and the system of subgroups $G(\nu_l) = \prod_{j=1}^{n} G_{\nu_l}^{(j)}$ $(\nu_l = 1,2,\dots)$ of the group G^n defines a system of neighborhoods of zero in G^n. The group G^n is a compact Abelian zero-dimensional group with respect to the addition operation and the topology thus introduced (the one-dimensional case was considered in [1]).

Let $m_0^{(j)} = 1$, $m_r^{(j)} = \prod_{k_j=1}^{r} p_{k_j}^{(j)}$, $j=1,2,\dots,n$, $r=1,2,\dots$.

Furthermore, let the Vilenkin–Price multiplicative system [1] be defined on the group $G^{(j)}$ $(1 \le j \le n)$:

$$\psi_0^{(j)} = 1, \qquad x \in G^{(j)},$$

$$\psi_{\nu_j}^{(j)}(x^{(j)}) = \prod_{k_j=0}^{l(\nu_j)} \exp \frac{2\pi i x_{k_j+1}^{(j)}}{p_{k_j+1}^{(j)}} \qquad \text{for} \quad \nu_j = \sum_{k_j=0}^{l(\nu_j)} \alpha_{k_j}^{(\nu_j)} m_{k_j}^{(j)},$$

where the $\alpha_{k_j}^{(\nu_j)}$ are integers, $0 \le \alpha_{k_j}^{(\nu_j)} < p_{k_j+1}^{(j)}$, and $x^j = \{x_{k_j}^{(j)}\}_{k_j=1}^{\infty} \in G^j$.

The system $\{\psi_{k_j}^{(j)}(x^j)\}_{k_j=0}^{\infty}$ $(1 \le j \le n)$ is a complete multiplicative system orthonormal with respect to the Haar measure μ_j on the group $G^{(j)}$ (see [1]). For $p_{k_j}^{(j)} = 2$, $k_j = 1,2,\dots$, the system coincides with the Walsh system in the Paley numbering.

We set $\overline{k} = (k_1, k_2, \dots, k_n)$ and $\overline{x} = (x_1, x_2, \dots, x_n)$. The multiple function system

$$\psi_{\overline{k}}(\overline{x}) = \psi_{k_1,k_2,\dots,k_n}(x^{(1)}, x^{(2)}, \dots, x^{(n)}) = \psi_{k_1}(x^{(1)})\psi_{k_2}(x^{(2)}) \cdots \psi_{k_n}(x^{(n)})$$

is a complete multiplicative system orthonormal with respect to the Haar measure $\mu = \mu_1 \times \mu_2 \times \dots \times \mu_n$ on G^n.

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Let $f(\overline{x}) \in L_p(G^n)$, $1 \le p < \infty$,

$$\|f\|_{L_p(G^n)} = \int_{G^n} |f(\overline{x})|^p \, d\overline{\mu}(\overline{x}) = \int_{G^{(1)}} \int_{G^{(2)}} \cdots \int_{G^{(n)}} \left| f(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \right| d\mu_1 x^{(1)} \, d\mu_2 x^{(2)} \cdots d\mu_n x^{(n)},$$

and let

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \widehat{f}(k_1, k_2, \dots, k_n) \psi_{k_1}(x^{(1)}) \psi_{k_2}(x^{(2)}) \cdots \psi_{k_n}(x^{(n)})$$

be the Fourier series of f with respect to the system $\psi_k(\overline{x})$, where

$$\widehat{f}(\overline{k}) = \int_{G^n} f(\overline{x}) \overline{\psi}_{\overline{k}}(\overline{x}) \, d\mu(\overline{x})$$

are the Fourier coefficients of f.

We denote the complete group modulus of continuity of the functions $f(\overline{x}) \in L_p(G^n)$, $1 \le p < \infty$ by

$$\omega_p(f, \overline{k}) = \omega_p(f; k_1, k_2, \dots, k_n) = \sup_{h^{(j)} \in G^{(j)}_{k_j}} \Big\| f(x^{(1)} \dotplus h^{(1)}, x^{(2)} \dotplus h^{(2)}, \dots, x^{(n)} \dotplus h^{(n)}) - f(x^{(1)}, x^{(2)}, \dots, x^{(n)}) \Big\|_{L_p(G^n)}.$$

The following statements hold.

Theorem 1. *Let* $1 < p \le 2$, $1/p + 1/p' = 1$, $0 < \beta \le p'$, $f(\overline{x}) \in L_p(G^n)$, *and*

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \left(\prod_{j=1}^{n} m^{(j)}_{k_j+1} \right)^{1-\beta/p'} \omega_p^{\beta}(f; k_1, k_2, \dots, k_n) < \infty. \tag{1}$$

Then the series

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} |\widehat{f}(k_1, k_2, \dots, k_m)|^{\beta} < \infty \tag{2}$$

converges.

For the case of functions of one variable this theorem was proved in [2].

In this theorem, the systems $\{\psi_{k_j}(x)\}_{k_j=1}^{\infty}$ $(1 \le j \le n)$ forming the sequences $\{p^{(j)}_{k_j}\}_{k_j=1}^{\infty}$ are arbitrary; in particular, they need not be equibounded. For the case in which $\sup_{1 \le k_j < \infty} p^{(j)}_{k_j} = N_j < \infty$, $j = 1, 2, \dots, n$, Theorem 1 readily implies the following corollary.

Corollary. *Let* $1 < p \le 2$, $1/p + 1/p' = 1$, $f(\overline{x}) \in L_p(G^n)$,

$$\sup_{1 \le k_j < \infty} p^{(j)}_{k_j} = N_j < \infty$$

$(j = 1, 2, \dots, n)$ *and*

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \left(\prod_{j=1}^{n} m^{(j)}_{k_j} \right)^{1-\beta/p'} \omega_p^{\beta}(f; k_1, k_2, \dots, k_n) < \infty. \tag{3}$$

Then the series (2) *converges.*

In the one-dimensional case, this assertion was proved in [3]; it is analogous to the classical Szász theorem on trigonometric series. It turns out that the assertion of the above corollary holds only if all the generating sequences $\{p^{(j)}_{k_j}\}_{k_j=1}^{\infty}$, $j = 1, 2, \dots, n$ are bounded. If at least one of the sequences $\{p^{(j)}_{k_j}\}$ is unbounded, then the convergence of the series (3) does not necessarily imply the convergence of the series (2). Specifically, the following theorem is valid.

Theorem 2. *Suppose that* $1 < p \le 2$, $1/p + 1/p' = 1$, $0 < \beta \le p'$,

$$\sup_{1 \le k'_j < \infty} p_{k_{j'}}^{(j')} = \infty$$

for some j' $(1 \le j' \le n)$, *and* $\sup_{1 \le k_j < \infty} p_{k_j}^{(j)} = N_j < \infty$ *for all* $j \ne j'$ $(1 \le j \le n)$. *Then there exists a function* $f(\overline{x}) \in L_p(G^n)$ *such that the series* (3) *converges but the series* (2) *diverges.*

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Point Perturbation-Invariant Solutions of the Schrödinger Equation with a Magnetic Field

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§1. In the series of papers [1–5], the authors studied the two-dimensional Schrödinger operator with a magnetic field perturbed by point potentials localized at the points of a plane lattice. Interest in operators of this form has sharply increased due to the successful study of transport properties of low-dimensional systems in quantizing magnetic fields and, in particular, due to the discovery of the quantization of Hall conductivity, the experimental detection of Aharonov–Bohm oscillations in mesoscopic systems related to Anderson localization, etc. [6]. The above-mentioned operator is a self-adjoint operator in $L^2(\mathbb{R}^2)$ of the form

$$H = H_0 + \sum_{\gamma \in \Gamma} \alpha_\gamma \delta(x - \gamma). \tag{1}$$

The potential on the right-hand side in (1) is the sum of Dirac δ-functions extended to a discrete set $\Gamma \subset \mathbb{R}^2$. We assume that this set is invariant with respect to translations by vectors of some two-dimensional lattice Λ. The nonperturbed operator $H_0 \equiv H_0(\xi)$ is the Schrödinger operator with a magnetic field normal to the system plane $\mathbb{R}^2$ (this operator is also called the Landau operator):

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