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ABOUT THE SPECTRUM OF THE LAPLACE OPERATOR

B.N. Biyarov

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Abstract. The famous French scientist J. Hadamard constructed the well-known example illustrating the incorrectness of the Cauchy problem for the Laplace equation. Since then, the question arises whether there exists a Volterra problem for the Laplace equation. In this paper we prove a theorem for a wide class of correct restrictions of the maximal operator $\widehat{L}$ and the correct extensions of the minimal operator L_0, generated by the Laplace operator, which are not Volterra problems.

1 Introduction and statement of problem

In a Hilbert space $L_2(\mathbb{B})$, where $\mathbb{B}$ is the m-dimensional unit ball in $\mathbb{R}^m$ with the boundary S, we consider the Laplace operator

$$-\Delta u = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_m^2}\right). \tag{1.1}$$

The minimal operator L_0, corresponding to Laplace operator 1.1 is defined as the closure in the graph norm of the restriction of $-\Delta$ on $C_0^\infty(\mathbb{B})$ as an operator from $L_2(\mathbb{B})$ to $L_2(\mathbb{B})$. Denote by $\widehat{L} = L_0^*$ the maximal operator. This definition is the definition of the weak maximal operator. It means, that $u \in D(\widehat{L})$ and $\widehat{L}u = f$ if and only if u and f belong to $L_2(\mathbb{B})$ and for all $v \in C_0^\infty(\mathbb{B})$

$$(f, v) = (u, \widehat{L}v).$$

The operator $\widehat{L}$ is called a strong maximal operator if it coincides with the closure of its restriction on $D(\widehat{L}) \cap C^\infty(\mathbb{B})$. In our case the weak and strong definition of the maximal operator coincide, because $\widehat{L}$ is an operator of local type (see [4, p. 118]).

Thus, we have defined the operators L_0 with the domain $D(L_0) = \overset{\circ}{W}{}_2^2(\mathbb{B})$ and $\widehat{L}$ with the domain $D(\widehat{L}) = \{u \in L_2(\mathbb{B}) : \ \widehat{L}u \in L_2(\mathbb{B})\}$.

Denote by L_D the operator, corresponding to the Dirichlet problem, that is, with the domain

$$D(L_D) = \{u \in W_2^2(\mathbb{B}) : \ u|_S = 0\}.$$

Definition 1. A linear operator L in a Hilbert space H is called *correct* if L has a bounded inverse defined on H.

Definition 2. A linear operator A in a Hilbert space H is called a *Volterra operator* if A compact and quasinilpotent.

We denote by $\mathfrak{S}_\infty(H, H_1)$ the set of all linear compact operators acting from a Hilbert space H to a Hilbert space H_1. If $T \in \mathfrak{S}_\infty(H, H_1)$, then T^*T is a non-negative self-adjoint operator in $\mathfrak{S}_\infty(H) \equiv \mathfrak{S}_\infty(H, H)$ and therefore has a non-negative self-adjoint square root $|T| = (T^*T)^{1/2}$. They will be written in the form of a sequence $\{s_j(T)\}_{j\in\mathbb{N}}$ where $s_1(T) \geq s_2(T) \geq \ldots \geq 0$ and each singular number is repeated as many times as its geometric multiplicity. If T (and hence $|T|$) is of finite rank n we assume that $s_j(T) = 0$ for $j > n$.

We denote by $\mathfrak{S}_p(H, H_1)$ the set of all compact operators $T \in \mathfrak{S}_\infty(H, H_1)$, for which

$$\sum_{j=1}^{\infty} s_j^p(T) < \infty \quad (0 < p < \infty).$$

The inverse operators L^{-1} to all possible correct restrictions L of the maximal operator $\widehat{L}$ corresponding to Laplace operator(1.1) can be described (see [5]) in the following form

$$u \equiv L^{-1}f = L_D^{-1}f + Kf, \tag{1.2}$$

where K is an arbitrary linear operator bounded in $L_2(\mathbb{B})$ satisfying $R(K) \subset Ker\widehat{L}$.

Then the direct operator L is defined by:

$$Lu = \widehat{L}u = f, \quad u \in D(L), \quad f \in L_2(\mathbb{B}), \tag{1.3}$$

$$D(L) = \{u \in D(\widehat{L}) : \ (I - K\widehat{L})u|_S = 0\}, \tag{1.4}$$

where I is the identity operator in $L_2(\mathbb{B})$.

The operators $(L^*)^{-1}$ are defined by

$$(L^*)^{-1}g = L_D^{-1}g + K^*g, \quad g \in L_2(\mathbb{B}). \tag{1.5}$$

They describe the inverses to all possible correct extensions of the minimal operators L_0 if and only if K satisfies the condition (see [1]):

$$Ker(L_D^{-1} + K^*) = \{0\}.$$

If the operator K in (1.2), satisfies one more additional condition

$$KR(L_0) = \{0\},$$

then an operator L corresponding to problem (1.3) - (1.4) will be simultaneously a correct restriction of the maximal operator $\widehat{L}$ and a correct extension of the minimal operator L_0, that is, $L_0 \subset L \subset \widehat{L}$. Such operators L will be called the *boundary correct extensions* of the minimal operator L_0 (with respect to the maximal operator $\widehat{L}$).

Definition 3. A correct restriction, a correct extension, and a boundary correct extension L is called *Volterra* if the inverse operator L^{-1} is a Volterra operator.

By Poisson's formula we have

$$u(x) = (P\varphi)(x) = \frac{1}{|S|}\int_S \varphi(\Theta')\frac{1-\rho^2}{r^m} d_{\Theta'}S; \qquad |x| = \rho < 1, \tag{1.6}$$

where $r^2 = 1 - 2\rho\cos\gamma + \rho^2 = |\xi - x|^2$, $|\xi| = 1$, $\Theta' \in S$, γ is the angle between vectors x and ξ. Note that $u(x)$ is a harmonic function in the ball $\mathbb{B}$ such that $u_{|_S} = \varphi$. The function u can be written in the form of a series (see [6, p. 249])

$$u(x) = \sum_{n=0}^{\infty}\rho^n \sum_{k=1}^{k_{n,m}} a_n^{(k)} Y_{n,m}^{(k)}(\Theta), \qquad a_n^{(k)} = \int_S \varphi(\Theta) Y_{n,m}^{(k)}(\Theta) dS,$$

where $0 < \rho < 1$, $\Theta \in S$, with respect to the orthonormal system $\{Y_{n,m}^{(k)}\}$ of spherical functions. We denote by P the operator, corresponding to (1.6). The operator P acts from $L_2(S)$ to $L_2(\mathbb{B})$. Then the operator P^* acts from $L_2(\mathbb{B})$ to $L_2(S)$ and has the form

$$P^*\psi = \frac{1}{|S|}\int_0^1 \rho d\rho \int_S \psi(\Theta)\frac{1-\rho^2}{r^m} dS.$$

It is easy to see that

$$P^*PY_{n,m}^{(k)}(\Theta) = P^*\rho^n Y_{n,m}^{(k)}(\Theta) = \int_0^1 \rho^{2n} Y_{n,m}^{(k)}(\Theta)\rho d\rho = \frac{1}{2(n+1)} Y_{n,m}^{(k)}(\Theta).$$

This shows that the s-numbers of the operator P are of the form

$$s_n(P) = \frac{1}{\sqrt{2(n+1)}}, \qquad n = 0, 1, 2, \ldots.$$

Therefore

$$P \in \mathfrak{S}_p(L_2(S),\ L_2(\mathbb{B})), \quad \text{for each} \quad p > 2. \tag{1.7}$$

2 Main results

Theorem. *Let $m \geq 2$ and the domain $D(L)$ of a correct restriction L of the maximal operator $\widehat{L}$, corresponding to Laplace operator* (1.1)*, is such that the following smoothness conditions are satisfied*

$$D(L) \subset W_2^l(\mathbb{B}), \quad l > 1, \quad \text{in the case} \quad m = 2, \tag{2.1}$$

or

$$D(L) \subset W_2^1(\mathbb{B}), \quad \text{in the case} \quad m \geq 3. \tag{2.2}$$

Then the operator L^{-1} is not quasinilpotent, hence the correct restriction L is not Volterra.

Idea of the proof. By (1.7) it follows (see [2, p.256]) and (see [7]) that

$$K \in \mathfrak{S}_\alpha(L_2(\mathbb{B}),\ L_2(\mathbb{B})), \quad \forall\alpha > \frac{2}{l+1}, \quad l > 1,$$

in the case (2.1), and

$$K \in \mathfrak{S}_\beta(L_2(\mathbb{B}),\ L_2(\mathbb{B})), \quad \forall\beta > \frac{2(m-1)}{m},$$

in the case (2.2). Suppose to the contrary that L^{-1} is quasinilpotent. Since $L_D^{-1} \geq 0$ and by virtue of V.I. Macaev's theorems (see here [3, p. 269] in the case $\frac{1}{2} < \alpha \leq 1$, [3, p. 273] in the case $0 < \alpha \leq \frac{1}{2}$ and [3, p. 267] in the case $1 < \alpha < +\infty$), we arrive at a contradiction.

Corollary 1. *If the conditions of the theorem are satisfied, then correct extension* (1.5) *of the minimal operator L_0 is not a Volterra operator.*

Corollary 2. *If the operator K is a finite-dimensional operator in $L_2(\mathbb{B})$, then there are no Volterra restrictions of the maximal operator $\widehat{L}$ and there are no Volterra extensions of the minimal operator L_0 independently of the smoothness of $D(L)$.*

Corollary 3. *If $K = K_1 + K_2$, where K_1 satisfies the conditions of the theorem, and K_2 is finite-dimensional, then there are no Volterra restrictions of the maximal operator $\widehat{L}$ and there are no Volterra extensions of the minimal operator L_0.*

Corollary 4. *If the operator K satisfies the conditions of the theorem and $KR(L_0) =$* $\{0\}$ *(that is, $L_0 \subset L \subset \widehat{L}$), then there are no Volterra boundary correct extensions.*

Remark 1. In the one-dimensional ($m = 1$) case the statement of the theorem is not true.

Remark 2. The above results are easily generalised to the case of any bounded domain with the boundary of class C^1.

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Bazarkan Nuroldinovich Biyarov
Faculty of Mechanics and Mathematics
L.N. Gumilyov Eurasian National University
5 Munaitpasov St,
010008 Astana, Kazakhstan
E-mail: splinekz@mail.ru

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