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Informativeness of Linear Functionals

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Abstract—In this paper, we study the informativeness of linear functionals in reconstruction problems and obtain exact orders of the informativeness of linear functionals in the Besov and Sobolev classes W and SW.

Key words: *linear functional, reconstruction problem, Besov class, Sobolev class, numerical integration, reconstruction of functions, Fourier coefficients.*

1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

Let us state the general reconstruction problem (in the version of [1]). Let there be given normed spaces X and Y of complex-valued functions defined on the sets Ω and Ω_1, respectively. Suppose that $F \subset X$ and $Tf = u(y\,;f)$ is the mapping of F into Y. For each positive integer N, let $\{(l^{(N)}\,;\varphi_N)\}$ denote the set of all possible pairs $(l^{(N)}\,;\varphi_N)$, where $l^{(N)} = (l_1,\dots,l_N)$ is a set of functionals

$$l_j(\,\cdot\,)\colon F \to \mathbb{C}, \qquad j = 1,\dots,N,$$

and the function $\varphi_N = \varphi_N(\tau_1,\dots,\tau_N\,;y)$ acts from $\mathbb{C}^N \times \Omega_1$ to $\mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers. We also suppose that for arbitrary fixed $\tau_1,\dots,\tau_N$ the function φ_N regarded as a function of y belongs to Y. For $(l^{(N)}\,;\varphi_N) \in \{(l^{(N)}\,;\varphi_N)\}$, we set

$$\delta_N\big((l^{(N)}\,;\varphi_N)\,;T,F\big)_Y = \sup_{f\in F}\big\|u(\,\cdot\,;f) - \varphi_N(l_1(f),\dots,l_N(f)\,;\,\cdot\,)\big\|_Y \tag{1}$$

and for $D_N \subset \{(l^{(N)}\,;\varphi_N)\}$,

$$\delta_N(D_N\,;T,F)_Y = \inf_{(l^{(N)}\,;\varphi_N)\in D_N} \delta_N\big((l^{(N)}\,;\varphi_N)\,;T,F\big)_Y. \tag{2}$$

Our aim is to derive upper and lower bounds for the quantity (2) (coinciding up to constants, if possible) and to determine the pair $(l^{(N)}\,;\varphi_N)$ yielding the upper bound.

Specifying the spaces X and Y, the classes F, $F\subset X$, the operators T, and the sets D_N in (1), (2), we obtain various statements of the problem (see, for example, [1–12] and the bibliography contained there). Let us present some of them. Suppose that $a = (a_1,\dots,a_N) \in \mathbb{R}^N$. Then, setting

$$\begin{gathered} Tf = \int_\Omega f(x)\,dx, \qquad l_k(f) = a_k f(\xi_k), \\ \varphi_N(\tau_1,\dots,\tau_N\,;y) \equiv \varphi_N(\tau_1,\dots,\tau_N) = \sum_{k=1}^N \tau_k, \end{gathered} \tag{3}$$

we obtain the statement of the approximate integration problem.

For $X = Y$ and $Tf = f(x)$, problem (1), (2) is the problem of the reconstruction of functions of class F. For an appropriate choice of T in problem (1), (2), we have the reconstruction problem for solutions of a partial differential equation when $Tf = u(y\,;\,f)$ is the solution of this equation with boundary or initial condition or with right-hand side $f(x)$ (see, for example, [2, pp. 185–190; 3, pp. 229–234], [1], and [12]).

The choice in D_N of various sets $l^{(N)}$ of functionals and of various algorithms φ_N of discrete information processing also generates numerous statements of problems.

Here the most studied case is, apparently, that of functionals of the form

$$l_j(f) = f(\xi_j), \qquad j = 1, \dots, N,$$

which are used for stating numerical integration problems (3) and also problems of reconstructing functions from their values at the nodes of the grid; then the problem consists in the construction of grids optimal in some sense. Such are, for example, the grids proposed by Korobov [2, 3], Smolyak [4] and Frolov [5].

Sherniyazov [12] defined grids as linear combinations of Korobov grids; for a wide range of classes F and metrics of Y, the errors of reconstruction of functions from their values at the nodes of such grids are optimal or optimal on the power scale.

Of course, in addition to the values of functions at the points, we can also consider other functionals, such as the values of the Fourier coefficients with respect to various orthonormal systems:

$$l_k(f) = \int_\Omega f(x)\phi_k(x)\,dx,$$

where $\{\phi_k(x)\}$ is an orthonormal system on Ω.

However, to make the reconstruction problem meaningful, we need to impose certain constraints on the set of functionals. Namely, if $\Omega = \Omega_1 = [0,1]^s$, the classes X and Y, where $X = Y$, are contained in the Lebesgue space $L(0,1)^s$ and D_N, $N = 1, 2, \dots$, is the set of all possible pairs $\{(l^{(N)}\,;\,\varphi_N)\}$, then $\delta_N(D_N, X)_Y = 0$. Indeed, the cardinality of the set $X \subset L(0,1)^s$ of all functions to be reconstructed is not greater than that of the continuum; therefore, we can establish a one-to-one correspondence $X \ni f \;\leftrightarrow\; c_f \in \{c_f\} \subset \mathbb{R}^1$. We define the functionals

$$l_1(f) = c_f, \qquad l_2(f) \equiv \dots \equiv l_N(f) \equiv 0$$

and the function $\overline{\varphi}_N(\tau_1, \dots, \tau_N\,;\,x)$ which is identically zero if $\tau_1 \notin \{c_f\}$, and equal to $f(x)$ if $\tau_1 = c_f$. Then on X we have

$$\|f(x) - \overline{\varphi}_N(l_1(f), 0, \dots, 0\,;\,x)\|_Y \equiv 0,$$

and the reconstruction problem becomes trivial: each function f of class X can be approximated optimally by itself.

Therefore, the following new problem on the *informativeness* of a particular class Λ of sets of functionals $l^{(N)}$ naturally arises (see [1]):

$$\delta_N(\Lambda\,;\,T, F)_Y = \inf_{(l^{(N)}\,;\,\varphi_N) \in \Lambda \times \{\varphi_N\}} \delta_N\big((l^{(N)}\,;\,\varphi_N)\,;\,T, F\big)_Y. \tag{4}$$

Here we study the following specialization of problem (4):

$$\Omega = \Omega_1 = [0,1]^s, \qquad Y = L^q(0,1)^s, \quad 1 \le q \le \infty, \qquad Tf = f,$$

and F represents the Sobolev classes $W_p^r(0,1)^s$, the Nikol′skii–Besov classes $B_{p,\theta}^r(0,1)^s$ (for definitions of these classes see, for example, [13]), and Sobolev classes with dominant mixed derivative $SW_2^r(0,1)^s$ (for the definition, see, for example, [4]), and $\Lambda = L^N$, where L is the set of all linear functionals on the linear hull F.

We have proved the following theorems.

Theorem 1. *Let s be a positive integer.*

(a) *Suppose that $2 \le p \le q \le \infty$ and $r > 0$ is an integer such that*

$$\frac{r}{s} - \left(\frac{1}{p} - \frac{1}{q}\right) > 0.$$

Then the following relation holds:

$$\delta_N(L^N, W_p^r(0,1)^s)_{L^q(0,1)^s} \asymp N^{-(r/s-(1/p-1/q))}, \qquad N = 1, 2, \dots .$$

(b) *Suppose that $2 \le p \le q \le \infty$, $1 \le \theta \le \infty$, and $r > 0$ satisfy*

$$\frac{r}{s} - \left(\frac{1}{p} - \frac{1}{q}\right) > 0.$$

Then the following relation holds:

$$\delta_N(L^N, B_{p,\theta}^r(0,1)^s)_{L^q(0,1)^s} \asymp N^{-(r/s-(1/p-1/q))}, \qquad N = 1, 2, \dots .$$

Theorem 2. *Suppose that s is a positive integer.*

(a) *Let $r > 1/2$ be a real number. Then the following relation holds:*

$$\delta_N(L^N, SW_2^r(0,1)^s)_{L^\infty(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^{r-1/2}}, \qquad N = 2, 3, \dots .$$

(b) *Let r be a positive number. Then the following relation holds:*

$$\delta_N(L^N, SW_2^r(0,1)^s)_{L^2(0,1)^s} \asymp \frac{\ln^{r(s-1)} N}{N^{r}}, \qquad N = 2, 3, \dots .$$

Theorem 3. *Suppose that s is a positive integer, $1 \le q \le p \le 2$, and r is a positive integer. Then the following relation holds:*

$$\delta_N(L^N, W_p^r(0,1)^s)_{L^q(0,1)^s} \asymp N^{-r/s}, \qquad N = 1, 2, \dots .$$

Remark 1. Upper bounds in all the theorems are attained on functionals, namely, on trigonometric Fourier coefficients (a particular set of coefficients for each class).

Remark 2. The lower bound in Theorem 1 (a) is a generalization of the lower bound obtained by Kudryavtsev [10] in the sense that this lower bound obtained only for functionals identically equal to the values of a function at the points is extended to all possible linear functionals. Also note that the result in Theorem 2 (a) allows us to generalize and improve the lower bound given in [12] (to less than $N^{-(r-1/2)}$) in the problem of reconstructing functions of class $SW_2^r(0,1)^s$ from their values at finitely many points: if s is a positive integer and $r > 1/2$, then we have the relation

$$\begin{aligned}\frac{\ln^{r(s-1)} N}{N^{r-1/2}} &\ll \delta_N(L^N, SW_2^r(0,1)^s)_{L^\infty(0,1)^s} \\ &\le \delta_N(P^N, SW_2^r(0,1)^s)_{L^\infty(0,1)^s} \ll \frac{\ln^{(r+1/2)(s-1)} N}{N^{r-1/2}}, \qquad N = 2, 3, \dots .\end{aligned}$$

Here by P we denote the set of all functionals identically equal to the values at the points.

2. AUXILIARY ASSERTIONS

Let us present the notation and several assertions that we shall use. For a function $f(x)$ which is 1-periodic in each of its s variables and integrable on the cube $[0,1]^s$, we use the standard notation $\widehat{f}(m)$, $m \in \mathbb{Z}^s$, for the trigonometric Fourier–Lebesgue coefficients.

For a positive integer ρ, by $I_\rho \equiv I_\rho^{(s)}$ everywhere below we mean the set $I_\rho = [-\rho, \rho]^s \cap \mathbb{Z}^s$ and by $\Gamma_\rho \equiv \Gamma_\rho^{(s)}$, the set

$$\Gamma_\rho = \{m = (m_1, \dots, m_s) \in \mathbb{Z}^s : \overline{\overline{m}} \le 2^\rho\},$$

where

$$\overline{y}_j = \max\{1\,;\,|y_j|\}, \qquad \overline{\overline{y}} = \prod_{j=1}^{s} \overline{y}_j$$

for any $y = (y_1, \dots, y_s) \in \mathbb{Z}^s$. For a finite set B, the number of its elements is denoted by $|B|$.

Lemma A [13, p. 281]. *Let there be given a positive integer s, a nonnegative integer l, a positive number r, $1 \le p$, $\theta \le \infty$ a positive integer ρ, and let $T_\rho(x)$ be a trigonometric polynomial of degree at most ρ in each of its s variables. Then the following relations are valid*:

$$\|T_\rho\|_{W_p^l} \ll \rho^l \|T_\rho\|_{L^p}, \qquad \|T_\rho\|_{B_{p,\theta}^r} \ll \rho^r \|T_\rho\|_{L^p}.$$

Note that the constants in these inequalities are independent of ρ.

Lemma B. *Let s be a positive integer. Then for each positive integer N we have: for an orthogonal system of functions $\psi \equiv \{\psi_k(x)\}_{k=1}^{N'}$ on $[0,1]^s$ and for a set $G \equiv \{m^{(1)}, \dots, m^{(N')}\} \subset \mathbb{Z}^s$ such that $N' = |G| \ge 3N$ and $|G| \asymp N$ and for arbitrary linear functionals $l_1, \dots, l_N$ defined, at least, on the set of all polynomials with respect to the system ψ and on the set of all trigonometric polynomials with spectrum in G, there exist complex numbers $\{c_k\}_{k=1}^{N'}$ satisfying the condition*

$$\sum_{k=1}^{N'} |c_k| \ge N, \qquad \sum_{k=1}^{N'} |c_k|^2 = N\,;$$

moreover, if

$$\chi(x) = \sum_{k=1}^{N'} c_k e^{2\pi i (m^{(k)}, x)}, \qquad \kappa(x) = \sum_{k=1}^{N'} c_k \psi_k(x),$$

then

$$l_1(\chi) = \dots = l_N(\chi) = l_1(\kappa) = \dots = l_N(\kappa) = 0, \tag{5}$$

$$\|\chi\|_{L^\infty} \ge N, \tag{6}$$

$$\|\chi\|_{L^2} = N^{1/2}. \tag{7}$$

Proof. We define the complex numbers $\{c_k\}_{k=1}^{N'}$ so that the functions

$$g(x) = \sum_{k=1}^{N'} c_k e^{2\pi i (m^{(k)}, x)}, \qquad h(x) = \sum_{k=1}^{N'} c_k \psi_k(x),$$

satisfy the relations $l_1(g) = \dots = l_N(g) = l_1(h) = \dots = l_N(h) = 0$.

By the linearity of the functionals $l_1, \dots, l_N$, this problem can be reduced to the following system of homogeneous linear equations with $2N$ equations in $N' \ge 3N$ unknowns:

$$\begin{aligned} 0 = l_j(g) &= \sum_{k=1}^{N'} c_k l_j(e^{2\pi i(m^{(k)},x)}) = \sum_{k=1}^{N'} c_k l_{j,m^{(k)}} = l_j(h) \\ &= \sum_{k=1}^{N'} c_k l_j(\psi_k(x)) = \sum_{k=1}^{N'} c_k l_{j,\psi_k}, \quad j = 1, \dots, N. \end{aligned} \tag{8}$$

Since the matrix of the system of linear homogeneous equations (8) consists of $2N$ rows and N' columns, its rank r satisfies the relation $r \le 2N$, while $N' \ge 3N$; therefore, the fundamental system of solutions of system (8) consists of $(N' - r) \ge N$ vectors, i.e., the set of all possible solutions $(c_1, \dots, c_{N'})$ forms a linear space of dimension $(N' - r) \ge N$ (see, for example, [14, p. 85]). Hence there exist vectors

$$(c_1^{(1)}, \dots, c_{N'}^{(1)}), \quad \dots, \quad (c_1^{(N)}, \dots, c_{N'}^{(N)}),$$

constituting a linearly independent system of solutions of the original system (8).

We can easily see that the functions

$$g_1(x) = \sum_{k=1}^{N'} c_k^{(1)} e^{2\pi i(m^{(k)},x)}, \quad \dots, \quad g_N(x) = \sum_{k=1}^{N'} c_k^{(N)} e^{2\pi i(m^{(k)},x)}$$

are linearly independent on $[0,1]^s$.

By the theorem on the orthogonalization of a system of independent functions, there exist numbers $\{b_{k,j}\}_{k,j=1}^N$ such that the functions

$$\phi_j(x) = \sum_{k=1}^{N} b_{k,j} g_k(x), \qquad j = 1, \dots, N,$$

form an orthonormal sequence on $[0,1]^s$.

Further, applying the mean-value theorem to the function

$$\Phi(x) = \sum_{j=1}^{N} |\phi_j(x)|^2,$$

we find that there exists a point $\xi \in [0,1]^s$ such that

$$N = \int_{[0,1]^s} \Phi(x)\,dx = \Phi(\xi) = \sum_{j=1}^{N} |\phi_j(\xi)|^2. \tag{9}$$

In that case, if we set

$$\chi(x) = \sum_{j=1}^{N} \overline{\phi_j(\xi)} \phi_j(x) = \sum_{k=1}^{N'} c_k e^{2\pi i(m^{(k)},x)},$$

then the numbers $\{c_k\}_{k=1}^{N'}$ are the desired quantities (as usual, by $\overline{\omega}$ we denote the complex conjugate of the complex number ω). Indeed, relations (5) obviously hold. Since $\phi_1(x), \dots, \phi_N(x)$ form an orthonormal sequence on $[0,1]^s$, from relation (9), we obtain (7). To prove inequalities (6), it suffices to use the definition of the norm on $L^\infty(0,1)^s$, the definition of the function $\chi(x)$, and relation (9). Further, we have

$$\sum_{k=1}^{N'} |c_k| \ge \|\chi\|_{L^\infty} \ge N \qquad \text{and} \qquad \sum_{k=1}^{N'} |c_k|^2 = \|\chi\|_{L^2}^2 = N.$$

The lemma is proved. $\square$

Remark 3. If $\psi \equiv \{e^{2\pi i(m,x)}\}_{m\in G}$, then, as is seen from the proof of the lemma, we can assume that $N' \geq 2N$.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1. Since the following embeddings [13, p. 230] are valid:

$$W_p^r(0,1)^s \subset H_p^r(0,1)^s, \qquad B_{p,1}^r(0,1)^s \subset B_{p,\theta}^r(0,1)^s \subset B_{p,\infty}^r(0,1)^s \equiv H_p^r(0,1)^s,$$

it follows that in both cases it suffices to prove the upper bound for the class $H_p^r(0,1)^s$ and the lower bound for the class $B_{p,1}^r(0,1)^s$.

Upper bound. Suppose we are given a positive integer N. Without loss of generality, we can assume that $N = |I_{2^n}| \asymp 2^{sn}$. We define the linear functionals

$$l_1(f) = \widehat{f}(m^{(1)}), \dots, l_N(f) = \widehat{f}(m^{(N)}),$$

where $\{m^{(1)}, \dots, m^{(N)}\}$ is some ordering of the sets I_{2^n}. We define the function

$$\varphi_N(\tau_1, \dots, \tau_N\,;\, x)$$

so that for each function $g \in L^1(0,1)^s$ the following relation is valid:

$$\varphi_N(l_1(g), \dots, l_N(g)\,;\, x) = V_{2^n}(g\,;\, x) = \sum_{k=0}^{n} q_k(g\,;\, x),$$

where [13, pp. 295–300] $V_{2^k}(g\,;\, x)$ is the Vallée-Poussin means of the function $g(x)$ of order 2^k in each variable and $q_0 = V_{2^0}$, $q_k = V_{2^k} - V_{2^{k-1}}$, $k \geq 1$.

Suppose that $f \in H_p^r(0,1)^s$. Then

$$\|f - \varphi_N\|_{L^q} \equiv \|f(x) - \varphi_N\big(l_1(f), \dots, l_N(f)\,;\, x\big)\|_{L^q} = \left\| \sum_{k=n+1}^{\infty} q_k \right\|_{L^q} \leq \sum_{k=n+1}^{\infty} \|q_k\|_{L^q}. \tag{10}$$

Let us apply to q_k the inequality of different metrics [13, p. 133]:

$$\|q_k\|_{L^q} \ll 2^{sk(1/p-1/q)} \|q_k\|_{L^p}. \tag{11}$$

The Vallée-Poussin means of functions of class $H_p^r(0,1)^s$ satisfy the inequality (see, for example, [13, p. 305])

$$\sup_{k=0,1,\dots} 2^{kr} \|q_k\|_{L^p} \leq c(s,p,r). \tag{12}$$

In view of (11) and (12), we extend the estimate (10):

$$\|f - \varphi_N\|_{L^q} \ll \sum_{k=n+1}^{\infty} 2^{-k(r-s(1/p-1/q))}.$$

Hence we have obtained the required upper bound.

Lower bound. Suppose we are given a positive integer N, N linear functionals $l_1, \dots, l_N$, and a function $\varphi_N(\tau_1, \dots, \tau_N; x)$. We define a positive integer $n = n(s, N)$ from the conditions $|I_n| \ge 2N$ and $|I_n| \asymp N$.

Suppose that the function $\chi(x)$ is defined as in Lemma B for $G = I_n$ (see also Remark 3 to Lemma B) and for the given functionals $l_1, \dots, l_N$. Let us estimate the norms $\|\chi\|_{W_p^r}$ and $\|\chi\|_{B_{p,1}^r}$ for $p \ge 2$. To do this, let us successively apply to $\chi(x)$ Lemma A and the inequality of different metrics:

$$\|\chi\|_{W_p^r} \ll n^r \|\chi\|_{L^p} \ll n^r n^{s(1/2-1/p)} N^{1/2} \ll N^{r/s+1-1/p},$$
$$\|\chi\|_{B_{p,1}^r} \ll n^r \|\chi\|_{L^p} \ll n^r n^{s(1/2-1/p)} N^{1/2} \ll N^{r/s+1-1/p}.$$

Therefore, there exists a number $c = c(s, p, r)$ such that

$$f(x) = f_{l_1,\dots,l_N}(x) = cN^{-r/s-1+1/p}\chi(x) \in W_p^r(0,1)^s \cap B_{p,1}^r(0,1)^s.$$

Let us find a lower bound for $\|f\|_{L^q}$ with $q \ge p$. To do this, we apply the inequality of different metrics to $f(x)$ and use the definition of the norm on L^∞, the definition of the function $f(x)$, and property (6) of the function $\chi(x)$:

$$\|f\|_{L^q} \gg (n^s)^{-1/q} \|f\|_{L^\infty} = n^{-s/q} c N^{-r/s-1+1/p} \|\chi\|_{L^\infty} \gg N^{-r/s+1/p-1/q}.$$

Note that the relations $l_1(f) = \dots = l_N(f) = 0$ are satisfied. Therefore, since $l_1, \dots, l_N$ and φ_N are arbitrary, we obtain the required upper bound.

Theorem 1 is proved. □

Remark 4. As is seen from the proof, the upper bound is valid for all $1 \le p \le q \le \infty$.

Proof of Theorem 2. *Upper bound.* Suppose we are given a positive integer N. Without loss of generality, we can assume that $N = |\Gamma_n|$. It follows that $N \asymp 2^n n^{s-1}$. We define the linear functionals

$$l_1(f) = \widehat{f}(m^{(1)}), \dots, l_N(f) = \widehat{f}(m^{(N)}),$$
$$\text{where } \{m^{(1)}, \dots, m^{(N)}\} \text{ is some ordering of the set } \Gamma_n.$$

We define the function $\varphi_N(\tau_1, \dots, \tau_N; x)$ as follows:

$$\varphi_N(\tau_1, \dots, \tau_N; x) = \sum_{k=1}^{N} \tau_k e^{2\pi i (m^{(k)}, x)}.$$

Suppose that $f \in SW_2^r(0,1)^s$.

(a) Under the condition $r > 1/2$, we have

$$\|f(x) - \varphi_N(l_1(f), \dots, l_N(f); x)\|_{L^\infty} = \Big\| \sum_{m \in \mathbb{Z}^s \setminus \Gamma_n} \widehat{f}(m) e^{2\pi i (m,x)} \Big\|_{L^\infty} \le \sum_{m \in \mathbb{Z}^s \setminus \Gamma_n} |\widehat{f}(m)|.$$

Applying Hölder's inequality, we obtain the required upper bound:

$$\|f(x) - \varphi_N(l_1(f), \dots, l_N(f); x)\|_{L^\infty} \ll \sqrt{\sum_{m \in \mathbb{Z}^s \setminus \Gamma_n} \frac{1}{(\overline{\overline{m}})^{2r}}} \ll \frac{\ln^{r(s-1)} N}{N^{r-1/2}}.$$

(b) Using the definition of the function $\varphi_N(l_1(f), \dots, l_N(f)\,;\,x)$, by Parseval's relation we have

$$\|f(x) - \varphi_N(l_1(f), \dots, l_N(f)\,;\,x)\|_{L^2} = \sqrt{\sum_{m \in Z^s \setminus \Gamma_n} |\widehat{f}(m)|^2} \ll \frac{1}{2^{nr}}\,.$$

Hence we have obtained an upper bound in this case.

Lower bound. Suppose we are given a positive integer N, N linear functionals $l_1, \dots, l_N$, and a function $\varphi_N(\tau_1, \dots, \tau_N\,;\,x)$. We define a positive integer $n = n(s, N)$ from the conditions $|\Gamma_n| \geq 2N$ and $|\Gamma_n| \asymp N$.

Suppose that the function $\chi(x)$ is defined as in Lemma B for $G = \Gamma_n$ (see also Remark 3 to Lemma B) and for the given functionals $l_1, \dots, l_N$.

Let us show that for some $c = c(s, r) > 0$,

$$f(x) = \frac{c}{2^{nr}\|\chi\|_{L^2}}\chi(x) \in SW_2^r(0,1)^s.$$

Indeed,

$$\sum_{m \in \mathbb{Z}^s} |\widehat{f}(m)|^2 (\overline{\overline{m}})^{2r} \leq 2^{2nr} \sum_{m \in \Gamma_n} |\widehat{f}(m)|^2 = 2^{2nr}\|f\|_{L^2}^2 = c^2.$$

Further, using (6), we obtain

$$\|f\|_{L^\infty} = \frac{c}{2^{nr}\|\chi\|_{L^2}}\|\chi\|_{L^\infty} \gg \frac{\ln^{r(s-1)} N}{N^{r-1/2}}$$

and

$$\|f\|_{L^2} = \frac{c}{2^{nr}} \gg \frac{\ln^{r(s-1)} N}{N^r}\,.$$

Further, note that $l_1(f) = \dots = l_N(f) = 0$. Finally, since $l_1, \dots, l_N$ and φ_N are arbitrary, we obtain the required lower bounds. $\square$

Proof of Theorem 3. *Upper bound.* The upper bound readily follows from Remark 4 to the proof of Theorem 1.

Lower bound. Suppose we are given a positive integer N, N linear functionals $l_1, \dots, l_N$, and a function $\varphi_N(\tau_1, \dots, \tau_N\,;\,x)$. We define a positive integer $n = n(s, N)$ from the conditions $3N \leq n^s$ and $n^s \asymp N$.

Suppose that $\omega(z) \in C^\infty(-\infty, +\infty)$ is a function satisfying $\operatorname{supp}\omega \subset (0,1)$. For

$$k = (k_1, \dots, k_s) \in \mathbb{Z}^s, \qquad 0 \leq k_j \leq (n-1), \quad j = 1, \dots, s,$$

we set

$$\psi_k(x) = \omega(nx_1 - k_1) \cdots \omega(nx_s - k_s).$$

For a given system $\{\psi_k\}$ and functionals $l_1, \dots, l_N$ (and for an arbitrary set $G \in \mathbb{Z}^s$, to be definite, we can take $G = [0, n-1]^s \cap \mathbb{Z}^s$), applying Lemma B we find that there exists a function

$$\kappa(x) = \sum_{0 \leq k_j \leq n-1} c_k \psi_k(x)$$

such that

$$\sum_{0\le k_j\le(n-1)} |c_k| \ge N, \tag{13}$$

$$\sum_{0\le k_j\le(n-1)} |c_k|^2 = N. \tag{14}$$

For $l = 0, 1, \dots,\ t = 1, \dots, s$, and $1 \le \nu < \infty$, we find that (for any function g, we assume $g^{(0)} \equiv g$)

$$\begin{aligned}\left\|\frac{\partial^l\kappa}{\partial x_t^l}\right\|_{L^\nu}^\nu &= n^{l\nu}\int_{[0,1]^s}\left|\sum_{0\le k_j\le(n-1)} c_k\omega(nx_1-k_1)\cdots\omega^{(l)}(nx_t-k_t)\cdots\omega(nx_s-k_s)\right|^\nu dx\\ &= n^{l\nu}\sum_{0\le m_j\le(n-1)}\int_{\left[\frac{m_1}{n},\frac{m_1+1}{n}\right]\times\cdots\times\left[\frac{m_s}{n},\frac{m_s+1}{n}\right]}\\ &\qquad\left|\sum_{0\le k_j\le(n-1)} c_k\omega(nx_1-k_1)\cdots\omega^{(l)}(nx_t-k_t)\cdots\omega(nx_s-k_s)\right|^\nu dx\\ &= n^{l\nu}\sum_{0\le m_j\le(n-1)}\int_{\left[\frac{m_1}{n},\frac{m_1+1}{n}\right]\times\cdots\times\left[\frac{m_s}{n},\frac{m_s+1}{n}\right]}\\ &\qquad\left|c_m\omega(nx_1-m_1)\cdots\omega^{(l)}(nx_t-m_t)\cdots\omega(nx_s-m_s)\right|^\nu dx\\ &= n^{l\nu}\sum_{0\le m_j\le(n-1)}|c_m|^\nu\prod_{j=1}^s\int_{m_j/n}^{(m_j+1)/n}|\omega^{(\lambda_j)}(nx_j-m_j)|^\nu\,dx_j,\end{aligned}$$

where $\lambda_j = 0$ if $j \ne t$ and $\lambda_j = l$ if $j = t$.

Let us evaluate

$$\int_{m_j/n}^{(m_j+1)/n}|\omega^{(\lambda_j)}(nx_j-m_j)|^\nu\,dx_j = n^{-1}\int_0^1|\omega^{(\lambda_j)}(z)|^\nu\,dz = n^{-1}c_1(\lambda_j,\nu,\omega)$$

separately. Hence

$$\left\|\frac{\partial^l\kappa}{\partial x_t^l}\right\|_{L^\nu} = c_2(s,l,\nu,\omega)n^{l-s/\nu}\left(\sum_{0\le m_j\le n-1}|c_m|^\nu\right)^{1/\nu}. \tag{15}$$

Therefore, for some $c = c(s, r, \omega)$ we have

$$f(x) = \frac{cn^{-r+s/2}}{\left(\sum_{0\le m_j\le(n-1)}|c_m|^2\right)^{1/2}}\kappa(x) \in W_2^r(0,1)^s \subset W_p^r(0,1)^s.$$

On the other hand, in view of (13) and (14), using relation (15) for $l = 0$ and $\nu = 1$, we find that

$$\|f\|_{L^q} \ge \|f\|_{L^1} \asymp n^{-r-s/2}N^{1/2} \asymp N^{-r/s}.$$

The lower bound, and hence Theorem 3, is proved. □

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