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Spectral Stability of the Robin Laplacian[1]

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We consider the Robin Laplacian in two bounded regions Ω_1 and Ω_2 of $\mathbb{R}^N$ with Lipschitz boundaries and such that $\Omega_2 \subset \Omega_1$, and we obtain two-sided estimates for the eigenvalues $\lambda_{n,2}$ of the Robin Laplacian in Ω_2 via the eigenvalues $\lambda_{n,1}$ of the Robin Laplacian in Ω_1. Our estimates depend on the measure of the set difference $\Omega_1 \backslash \Omega_2$ and on suitably defined characteristics of vicinity of the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, and of the functions defined on $\partial\Omega_1$ and on $\partial\Omega_2$ that enter the Robin boundary conditions.

1. INTRODUCTION

In this paper, we study the stability properties of the spectrum of the Laplace operator with Robin boundary conditions under a small variation of the domain of definition of the problem.

Let Ω be a region of $\mathbb{R}^N$, i.e., an open connected subset of $\mathbb{R}^N$. It is assumed that Ω is bounded and that its boundary $\partial\Omega$ is of Lipschitz class. We consider an essentially bounded nonnegative measurable function h on $\partial\Omega$ and the quadratic form on $L^2(\Omega)$ defined by

$$Q_{\Omega,h}[f] \equiv \begin{cases} \int\limits_{\Omega} |\nabla f|^2\,dx + \int\limits_{\partial\Omega} h|\mathrm{tr}\, f|^2\,d\sigma & \text{if } \ f \in W^{1,2}(\Omega), \\ +\infty & \text{if } \ f \in L^2(\Omega) \setminus W^{1,2}(\Omega), \end{cases}$$

where $d\sigma$ denotes the usual surface measure on $\partial\Omega$ and $\mathrm{tr}\, f$ denotes the trace on $\partial\Omega$ of the function f of the Sobolev space $W^{1,2}(\Omega)$. The Robin Laplacian in Ω corresponding to h is defined to be the nonnegative self-adjoint operator $-\Delta_{\Omega,h}$ acting in $L^2(\Omega)$ and associated with the quadratic form $Q_{\Omega,h}$. We consider the eigenvalue problem

$$-\Delta_{\Omega,h}[u] = \lambda u. \tag{1.1}$$

We are interested in understanding what happens with the eigenvalues of problem (1.1) if we perturb Ω and h. More precisely, we consider Ω_1, h_1 and Ω_2, h_2 as Ω, h above, with $\Omega_2 \subset \Omega_1$. We denote by $\{\lambda_{n,i}\}_{n\in\mathbb{N}_0}$ the sequence of eigenvalues in Ω_i, for $i = 1, 2$, and establish a two-sided estimate for $\lambda_{n,2}$ via $\lambda_{n,1}$.

We note that the classical formulation of problem (1.1) in a region with a smooth boundary is

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} + hu = 0 \quad \text{on } \partial\Omega,$$

where ν is the exterior unit normal to $\partial\Omega$.

For $h = 0$, one obtains the Neumann problem, for which the two-sided estimates mentioned above have been obtained by Burenkov and Davies [2]. In this respect, we also mention the results obtained by Lamberti and Lanza de Cristoforis [9] with a different method, which also concern corresponding estimates for the eigenfunctions. For further spectral stability results for a wide class of Neumann-type operators, see Burenkov and Lamberti [3, 4].

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As in [2], we proceed in two steps. In both steps we apply the variational formulas for the eigenvalues of the Robin Laplacian. In the first step we estimate $\lambda_{n,2}$ from above via $\lambda_{n,1}$ by applying L^∞-estimates from above for the normalized eigenfunctions of $-\Delta_{\Omega_1,h_1}$ and the characteristics of vicinity $G(\partial\Omega_1,\partial\Omega_2)$ and $L(h_1,h_2)$ of $\partial\Omega_1$ and $\partial\Omega_2$ and of h_1 and h_2, respectively, introduced in Definition 3.1 below. In the second step we consider the map of Ω_1 onto a region $\Omega_3\subset\Omega_2$ introduced in [2] (see Lemma 3.4 below). By applying a kind of a scaling argument, we estimate from below the eigenvalues $\lambda_{n,3}$ of the Robin Laplacian in Ω_3 with suitable data on $\partial\Omega_3$ via $\lambda_{n,1}$. Then we estimate from above $\lambda_{n,3}$ via $\lambda_{n,2}$ by using the first step. Hence, we obtain an estimate from below for $\lambda_{n,2}$ via $\lambda_{n,1}$.

Compared with the Neumann problem, the main distinction is that for the regions Ω_1 and Ω_2 under consideration, the estimates for the Neumann problem depend only on the measure $|\Omega_1\setminus\Omega_2|$ of the set difference $\Omega_1\setminus\Omega_2$, while for the Robin problem, the estimates also depend substantially on the characteristics of vicinity $G(\partial\Omega_1,\partial\Omega_2)$ and $L(h_1,h_2)$.

Some of the statements of this paper were formulated without proof in the survey paper by Burenkov, Lamberti, and Lanza de Cristoforis [5].

2. PRELIMINARIES ON THE ROBIN LAPLACIAN

First, we introduce some notation and technical preliminaries. It is known, already in the case of the Neumann problem, that by perturbing a bounded region Ω_1 with a smooth boundary in a neighborhood of a boundary point of $\partial\Omega_1$, one need not obtain a region Ω_2 with a discrete spectrum. Thus we need to introduce a regularity class for the regions we consider.

We denote by $\mathbb{N}$ the set of natural numbers and by $\mathbb{N}_0$ the set $\mathbb{N}\cup\{0\}$. Throughout the paper, N denotes a natural number greater than or equal to 2. By a rotation in $\mathbb{R}^N$, we mean an $N\times N$ orthogonal matrix with real entries, which we identify with the corresponding linear operator acting in $\mathbb{R}^N$. If A is a matrix, A^t denotes the transpose of A. A cuboid denotes a subset of $\mathbb{R}^N$ obtained by applying a rotation to a Cartesian product of N bounded open intervals of $\mathbb{R}$. If V is a proper subset of $\mathbb{R}^N$ and $\delta>0$, we denote by V_δ the set $\{x\in V\colon d(x,\partial V)>\delta\}$, where $d(x,\partial V)$ is the Euclidean distance from x to ∂V. We also set $B(x,\delta)\equiv\{y\in\mathbb{R}^N\colon d(x,y)<\delta\}$.

Definition 2.1. Let $M,\delta\in\,]0,+\infty[$, $s,s'\in\mathbb{N}$, and $s'\le s$. Let $\{a_{jl}\}_{\substack{j=1,\dots,s\\ l=1,\dots,N}}$ and $\{b_{jl}\}_{\substack{j=1,\dots,s\\ l=1,\dots,N}}$ be real numbers such that $a_{jl}<b_{jl}$ for all $j=1,\dots,s$ and $l=1,\dots,N$. Let $\{r_j\}_{j=1}^s$ be a family of rotations. Furthermore, let

$$K_j\equiv\prod_{l=1}^{N}\,]a_{jl},b_{jl}[,\qquad W_j\equiv\prod_{l=1}^{N-1}\,]a_{jl},b_{jl}[,\qquad V_j\equiv r_j^t(K_j)$$

for $j=1,\dots,s$. We say that a bounded region Ω of $\mathbb{R}^N$ is of class $C^{0,1}\big(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\big)$ provided that the following three conditions are satisfied:

(i) $\Omega\subset\bigcup_{j=1}^s(V_j)_\delta$ and $(V_j)_\delta\cap\Omega\ne\varnothing$ for all $j=1,\dots,s$;

(ii) $V_j\cap\partial\Omega\ne\varnothing$ for $j=1,\dots,s'$, and $V_j\subseteq\Omega_\delta$ for $s'<j\le s$;

(iii) if $j=1,\dots,s'$, then there exists a map φ_j of W_j to $]a_{jN}+\delta,b_{jN}-\delta[$ such that

$$|\varphi_j(\overline{x})-\varphi_j(\overline{y})|\le M|\overline{x}-\overline{y}|\qquad\forall\,\overline{x},\overline{y}\in W_j,$$

$$r_j(\Omega\cap V_j)=\big\{(\overline{x},x_N)\in K_j\colon\ a_{jN}<x_N<\varphi_j(\overline{x})\big\},$$

where $\overline{x}\equiv(x_1,\dots,x_{N-1})$.

If Ω is a bounded region of $\mathbb{R}^N$, we say that Ω is of class $C^{0,1}$ provided that there exist M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ as above such that Ω is of class $C^{0,1}\big(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\big)$.

Let $0 < \gamma \leq 1$. We say that a bounded region Ω of $\mathbb{R}^N$ is of class $C^{1,\gamma}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ provided that conditions (i)–(iii) hold and that the functions of the family $\{\varphi_j\}_{j=1}^{s'}$ are continuously differentiable and satisfy the inequality

$$|\nabla\varphi_j(\overline{x}) - \nabla\varphi_j(\overline{y})| \leq M|\overline{x} - \overline{y}|^\gamma \qquad \forall\, \overline{x}, \overline{y} \in W_j, \quad j = 1, \dots, s'.$$

Concerning condition (iii), we note that by the Rademacher theorem, the gradient $\nabla\varphi_j$ exists for almost all $\overline{x} \in W_j$, and that for those $\overline{x}$ we have $|\nabla\varphi_j(\overline{x})| \leq M$.

In the sequel, we need to use a partition of unity on the boundary of sets of class $C^{0,1}$. Thus we recall the following statement, which is well known. For all open subsets Ω of $\mathbb{R}^N$, we denote by $\mathcal{D}(\Omega)$ the set of functions in $C^\infty(\Omega)$ with compact support.

Lemma 2.2. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Then there exists a family $\{\zeta_j\}_{j=1}^s$ of functions of $\mathcal{D}(\mathbb{R}^N)$ such that $0 \leq \zeta_j \leq 1$, $\operatorname{supp}\zeta_j \subset V_j$ for all $j = 1, \dots, s$, and $\sum_{j=1}^s \zeta_j = 1$ on $\bigcup_{j=1}^s (V_j)_\delta$. In particular, if Ω is a bounded region of $\mathbb{R}^N$ of class $C^{0,1}\big(M, \delta, s, s'$, $\{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$, then $\sum_{j=1}^{s'} \zeta_j = 1$ on $\partial\Omega$.*

If Ω is an open subset of $\mathbb{R}^N$, then we denote by $W^{1,2}(\Omega)$ the Sobolev space of real-valued functions u in $L^2(\Omega)$ such that the weak derivatives $\frac{\partial u}{\partial x_j}$ belong to $L^2(\Omega)$ for all $j = 1, \dots, N$. Also, we set

$$\|u\|_{W^{1,2}(\Omega)} \equiv \left(\int_\Omega \big(|u|^2 + |\nabla u|^2\big)\, dx\right)^{1/2} \qquad \forall u \in W^{1,2}(\Omega).$$

In order to handle the boundary term of the Robin Laplacian, we need the following lemma on traces.

Lemma 2.3. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1.

If $N \geq 3$, then there exists a constant $c_{\mathrm{tr}} > 0$ such that

$$\|\operatorname{tr} f\|_{L^q(\partial\Omega)} \leq c_{\mathrm{tr}} \|f\|_{W^{1,2}(\Omega)} \qquad \forall f \in W^{1,2}(\Omega) \tag{2.1}$$

for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ and for all $2 \leq q \leq \frac{2(N-1)}{N-2}$.

If $N = 2$ and $2 \leq q < \infty$, then there exists a constant $c_{\mathrm{tr}} > 0$ such that (2.1) *holds for all bounded regions Ω of $\mathbb{R}^2$ of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$.*

For a proof of Lemma 2.3, we refer the reader, say, to Nečas [12, théorème 4.2, p. 84, théorème 4.6, p. 86].

Actually, under the additional assumptions that f is essentially bounded in Ω, the trace of f is also essentially bounded, as the following lemma shows.

Lemma 2.4. *Let Ω be a bounded region of $\mathbb{R}^N$ of class $C^{0,1}$. If $f \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, then the following two statements hold.*

(i) *There exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ in $C^\infty(\overline{\Omega})$ such that $\lim_{j\to\infty} f_j = f$ in $W^{1,2}(\Omega)$ and such that $\|f_j\|_{L^\infty(\Omega)} \leq \|f\|_{L^\infty(\Omega)}$ for all $j \in \mathbb{N}$.*

(ii) *$\operatorname{tr} f \in L^\infty(\partial\Omega)$ and*

$$\|\operatorname{tr} f\|_{L^\infty(\partial\Omega)} \leq \|f\|_{L^\infty(\Omega)}. \tag{2.2}$$

Proof. Since $f \in W^{1,2}(\Omega)$ and Ω is of class $C^{0,1}$, f admits an extension $\widetilde{f} \in W^{1,2}(\mathbb{R}^N)$. Now let G be the function of $\mathbb{R}$ to itself defined by $G(t) = t$ if $|t| \leq \|f\|_{L^\infty(\Omega)}$, $G(t) = \|f\|_{L^\infty(\Omega)}$ if $t \geq \|f\|_{L^\infty(\Omega)}$, and $G(t) = -\|f\|_{L^\infty(\Omega)}$ if $t \leq -\|f\|_{L^\infty(\Omega)}$. Clearly, G is Lipschitz continuous and thus $F \equiv G \circ \widetilde{f} \in W^{1,2}(\mathbb{R}^N)$ (see Marcus and Mizel [11, p. 298, 301].) Obviously, $F_{|\Omega} = f$ and $\|F\|_{L^\infty(\mathbb{R}^N)} = \|f\|_{L^\infty(\Omega)}$. Now let $\{\eta_j\}_{j\in\mathbb{N}}$ be a standard family of mollifiers, with $0 \le \eta_j \in \mathcal{D}(\mathbb{R}^N)$, $\operatorname{supp}\eta_j = \overline{B(0,2^{-j})}$, and $\int_{\mathbb{R}^N} \eta_j(x)\,dx = 1$. Then we have $\lim_{j\to\infty} \eta_j * F = F$ in $W^{1,2}(\mathbb{R}^N)$ and $\|\eta_j*F\|_{L^\infty(\mathbb{R}^N)} \le \|\eta_j\|_{L^1(\mathbb{R}^N)}\|F\|_{L^\infty(\mathbb{R}^N)} = \|F\|_{L^\infty(\mathbb{R}^N)}$. Thus, by setting $f_j \equiv (\eta_j*F)_{|\Omega}$, we conclude that statement (i) holds. We now prove statement (ii). Since $\eta_j * F$ is continuous, we also have $\sup_{\mathbb{R}^N} |\eta_j * F| \le \|F\|_{L^\infty(\mathbb{R}^N)}$. By the continuity of the restriction map of $W^{1,2}(\mathbb{R}^N)$ to $W^{1,2}(\Omega)$ and by the continuity of the trace operator of $W^{1,2}(\Omega)$ to $L^1(\partial\Omega)$, we have $\lim_{j\to\infty}(\eta_j*F)_{|\partial\Omega} = \operatorname{tr}(F_{|\Omega}) = \operatorname{tr} f$ in $L^1(\partial\Omega)$. Possibly extracting a subsequence, we can assume that $\lim_{j\to\infty}(\eta_j * F)_{|\partial\Omega} = \operatorname{tr} F$ almost everywhere on $\partial\Omega$, and thus

$$|\operatorname{tr} f| = \lim_{j\to\infty}\big|(\eta_j * F)_{|\partial\Omega}\big| \le \|F\|_{L^\infty(\mathbb{R}^N)} = \|f\|_{L^\infty(\Omega)}$$

almost everywhere on $\partial\Omega$. □

We now wish to show that the quadratic form $Q_{\Omega,h}$ is associated with a self-adjoint operator. We start with the following.

Lemma 2.5. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1.
If $N \ge 3$, then there exists a constant $c_Q > 0$ such that

$$\|f\|_{W^{1,2}(\Omega)} \le \|f\|_{Q_{\Omega,h}} \equiv \big(Q_{\Omega,h}[f] + \|f\|^2_{L^2(\Omega)}\big)^{1/2}$$
$$\le c_Q\big(1 + \|h\|_{L^{N-1}(\partial\Omega)}\big)^{1/2}\|f\|_{W^{1,2}(\Omega)} \qquad \forall f \in W^{1,2}(\Omega)$$

for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{0,1}\big(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\big)$ and for all nonnegative $h \in L^{N-1}(\partial\Omega)$.
If $N = 2$ and $p \in \,]1,+\infty[$, then there exists a constant $c_Q > 0$ such that

$$\|f\|_{W^{1,2}(\Omega)} \le \|f\|_{Q_{\Omega,h}} \le c_Q\big(1 + \|h\|_{L^p(\partial\Omega)}\big)^{1/2}\|f\|_{W^{1,2}(\Omega)} \qquad \forall f \in W^{1,2}(\Omega)$$

for all bounded regions Ω of $\mathbb{R}^2$ of class $C^{0,1}\big(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\big)$ and for all nonnegative $h \in L^p(\partial\Omega)$.

Proof. If $N \ge 3$, Hölder's inequality and Lemma 2.3 with $q = \frac{2(N-1)}{N-2}$ imply that

$$\int\limits_{\partial\Omega} h|\operatorname{tr} f|^2\,d\sigma \le \|h\|_{L^{N-1}(\partial\Omega)}\|\operatorname{tr} f\|^2_{L^{\frac{2(N-1)}{N-2}}(\partial\Omega)}$$
$$\le c_{\mathrm{tr}}^2\|f\|^2_{W^{1,2}(\Omega)}\|h\|_{L^{N-1}(\partial\Omega)} \qquad \forall f \in W^{1,2}(\Omega). \tag{2.3}$$

If $N = 2$, Hölder's inequality and Lemma 2.3 with $q = \frac{2p}{p-1}$ imply that

$$\int\limits_{\partial\Omega} h|\operatorname{tr} f|^2\,d\sigma \le \|h\|_{L^p(\partial\Omega)}\|\operatorname{tr} f\|^2_{L^{\frac{2p}{p-1}}(\partial\Omega)} \le c_{\mathrm{tr}}^2\|h\|_{L^p(\partial\Omega)}\|f\|^2_{W^{1,2}(\Omega)} \qquad \forall f \in W^{1,2}(\Omega). \tag{2.4}$$

The desired inequalities follow immediately from inequalities (2.3) and (2.4). □

Combining Lemma 2.5 with Davies [6, Lemma 4.4.1, Theorem 4.4.2, p. 81–84], we can deduce the following.

Theorem 2.6. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Let Ω be a bounded region of $\mathbb{R}^N$ of class $C^{0,1}\big(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\big)$. Let h be a nonnegative function, with $h \in L^{N-1}(\partial\Omega)$ for $N \ge 3$ and $h \in L^p(\partial\Omega)$ for some $p \in \,]1,+\infty[$ for $N = 2$. The space*

$W^{1,2}(\Omega)$ with the norm $\|\cdot\|_{Q_{\Omega,h}}$ is complete, and there exists a self-adjoint linear operator $H_{\Omega,h}$ from a linear subset $\mathrm{Dom}(H_{\Omega,h})$ *of $W^{1,2}(\Omega)$ with the scalar product of $L^2(\Omega)$ to $L^2(\Omega)$ such that*

$$\mathrm{Dom}\big(H_{\Omega,h}^{1/2}\big) = W^{1,2}(\Omega) \tag{2.5}$$

and

$$\begin{aligned} B_{\Omega,h}[u_1,u_2] &\equiv \int_\Omega \nabla u_1 \nabla u_2\,dx + \int_{\partial\Omega} h(\mathrm{tr}\,u_1)(\mathrm{tr}\,u_2)\,d\sigma \\ &= \int_\Omega \big(H_{\Omega,h}^{1/2}u_1\big)\big(H_{\Omega,h}^{1/2}u_2\big)\,dx \qquad \forall u_1,u_2\in W^{1,2}(\Omega). \end{aligned} \tag{2.6}$$

A function f of $W^{1,2}(\Omega)$ belongs to $\mathrm{Dom}(H_{\Omega,h})$ *if and only if $H_{\Omega,h}^{1/2}f$ belongs to $W^{1,2}(\Omega)$. Finally,*

$$B_{\Omega,h}[u_1,u_2] = \int_\Omega (H_{\Omega,h}u_1)u_2\,dx \qquad \forall u_1,u_2\in \mathrm{Dom}(H_{\Omega,h}).$$

In particular,

$$\begin{aligned} Q_{\Omega,h}[f] &= \int_\Omega (H_{\Omega,h}f)f\,dx \qquad \forall f\in \mathrm{Dom}(H_{\Omega,h}), \\ Q_{\Omega,h}[f] &= \int_\Omega \big(H_{\Omega,h}^{1/2}f\big)^2\,dx \qquad \forall f\in W^{1,2}(\Omega). \end{aligned} \tag{2.7}$$

We will call $H_{\Omega,h}$ the *Robin Laplacian.* We now introduce some spectral properties of the Robin Laplacian, which we deduce from facts on Dirichlet forms. We start with the following.

Lemma 2.7. *Let Ω and h be as in Theorem* 2.6. *Then $\{e^{-H_{\Omega,h}t}\}_{t>0}$ is a symmetric Markov semigroup.*

Proof. For the definition of a Markov semigroup, we refer the reader to Davies [7, § 1.4]. By Davies [7, Theorems 1.3.2, 1.3.3], it suffices to show that

(i) if $f\in\mathrm{Dom}\big(H_{\Omega,h}^{1/2}\big)$, then $|f|\in\mathrm{Dom}\big(H_{\Omega,h}^{1/2}\big)$ and $Q_{\Omega,h}[|f|]\le Q_{\Omega,h}[f]$;

(ii) if $0\le f\in\mathrm{Dom}\big(H_{\Omega,h}^{1/2}\big)$, then $\min\{f,1\}\in\mathrm{Dom}\big(H_{\Omega,h}^{1/2}\big)$ and $Q_{\Omega,h}[\min\{f,1\}]\le Q_{\Omega,h}[f]$.

Since $\mathrm{Dom}\big(H_{\Omega,h}^{1/2}\big) = W^{1,2}(\Omega)$, properties (i) and (ii) follow by the definition of $Q_{\Omega,h}$ and by the well-known properties of $W^{1,2}(\Omega)$. Indeed, if $f\in W^{1,2}(\Omega)$, then we have $|f|,\min\{f,1\}\in W^{1,2}(\Omega)$, $\int_\Omega|\nabla f|^2\,dx=\int_\Omega|\nabla|f||^2\,dx$, and $\int_\Omega|\nabla(\min\{f,1\})|^2\,dx\le\int_\Omega|\nabla f|^2\,dx$ (see, e.g., Lieb and Loss [10, § 6.17, p. 152]). □

Since $\{e^{-H_{\Omega,h}t}\}_{t>0}$ is a symmetric Markov semigroup, we will apply the following general statement, which we report from Davies [7, Lemma 2.1.2, Corollary 2.4.3].

Theorem 2.8. *Let Ω be a region of $\mathbb{R}^N$. Let H be a nonnegative self-adjoint linear operator from the linear dense subset* $\mathrm{Dom}(H)$ *of $L^2(\Omega)$ to $L^2(\Omega)$. Let $\{e^{-Ht}\}_{t>0}$ be a symmetric Markov semigroup in $L^2(\Omega)$. Let $\theta>2$. Then the following statements are equivalent.*

(i) *There exists $c_\mathrm{e}>0$ such that*

$$\|f\|_{L^{\frac{2\theta}{\theta-2}}(\Omega)} \le c_\mathrm{e}\left(\int_\Omega \big(|f|^2+|H^{1/2}f|^2\big)\,dx\right)^{1/2} \qquad \forall f\in\mathrm{Dom}(H^{1/2}).$$

(ii) *There exists* $c_{\mathrm{c}} > 0$ *such that*

$$\|e^{-Ht}f\|_{L^\infty(\Omega)} \le c_{\mathrm{c}} t^{-\theta/4} \|f\|_{L^2(\Omega)} \qquad \forall f \in L^2(\Omega), \quad \forall t \in \,]0,1].$$

(iii) *There exists a function* K *of* $]0,+\infty[\times\Omega^2$ *to* $[0,+\infty[$ *such that* $K(t,\cdot,\cdot)$ *is measurable on* Ω^2 *for all* $t \in \,]0,+\infty[$ *and*

$$e^{-Ht}f(x) = \int_\Omega K(t,x,y) f(y)\,dy \qquad \forall f \in L^2(\Omega)$$

for all $t \in \,]0,+\infty[$, *and there exists a constant* $c_{\mathrm{ker}} > 0$ *such that for each* $t \in \,]0,1]$

$$0 \le K(t,x,y) \le c_{\mathrm{ker}} t^{-\theta/2}$$

for almost all $(x,y) \in \Omega^2$.

Moreover, if (i) *holds for some* $c_{\mathrm{e}} > 0$, *then* c_{c} *and* c_{ker} *can be chosen to depend only on* c_{e} *and* θ. *If* (iii) *holds, then* c_{c} *can be chosen to depend only on* c_{ker} *and* θ. *Finally, the equivalence of* (ii) *and* (iii) *holds for all* $\theta > 0$.

Then we have the following for the Robin Laplacian.

Theorem 2.9. *Let* M, δ, s, s', $\{V_j\}_{j=1}^s$, *and* $\{r_j\}_{j=1}^s$ *be as in Definition* 2.1. *Let* $\tau = N$ *if* $N \ge 3$ *and* $2 < \tau < \infty$ *if* $N = 2$. *There exists a constant* $c'_{\mathrm{ker}} > 0$ *such that for all bounded regions* Ω *of* $\mathbb{R}^N$ *of class* $C^{0,1}\big(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\big)$ *and for all* h *as in Theorem* 2.6, *there exists a nonnegative kernel* $K_{\Omega,h}$ *defined in* $]0,+\infty[\,\times\Omega^2$ *for the symmetric Markov semigroup* $\{e^{-H_{\Omega,h}t}\}_{t>0}$ *such that for each* $t \in \,]0,1]$ *the function* $K_{\Omega,h}(t,\cdot,\cdot)$ *is measurable in* Ω^2 *and*

$$0 \le K_{\Omega,h}(t,x,y) \le c'_{\mathrm{ker}} t^{-\tau/2} \tag{2.8}$$

for almost all $(x,y) \in \Omega^2$.

Proof. For $h = 0$, the statement was proved in Davies [7, p. 77]. For the sake of completeness, we now prove the statement by imitating the proof for the case $h = 0$. We first consider the case $N \ge 3$. As is well known (see, e.g., Burenkov [1, p. 186–192]), there exists a constant $c_{\mathrm{S}} > 0$ such that the Sobolev inequality

$$\|f\|_{L^{\frac{2N}{N-2}}(\Omega)} \le c_{\mathrm{S}} \|f\|_{W^{1,2}(\Omega)} \qquad \forall f \in W^{1,2}(\Omega) \tag{2.9}$$

holds for all Ω as in the statement. Hence, Lemma 2.5 and the second equality of (2.7) imply that

$$\|f\|_{L^{\frac{2N}{N-2}}(\Omega)} \le c_{\mathrm{S}} \left(\int_\Omega \big(|f|^2 + |H_{\Omega,h}^{1/2} f|^2\big)\,dx \right)^{1/2} \qquad \forall f \in W^{1,2}(\Omega).$$

Then Theorem 2.8 implies the validity of (2.8).

Now let $N = 2$. Then we choose $q \in \,]2,+\infty[$ such that $\tau = \frac{2q}{q-2}$. As is well known (see, e.g., Burenkov [1, p. 186–192]), there exists $c_{\mathrm{S},q} > 0$ such that

$$\|f\|_{L^q(\Omega)} \le c_{\mathrm{S},q} \|f\|_{W^{1,2}(\Omega)} \qquad \forall f \in W^{1,2}(\Omega).$$

By applying Lemma 2.5, the second equality of (2.7), and Theorem 2.8, we conclude that $\{e^{-H_{\Omega,h}t}\}_{t>0}$ has a nonnegative kernel $K_{\Omega,h}(t,x,y)$ defined on $]0,+\infty[\,\times\Omega^2$ and that there exists $c'_{\mathrm{ker}} > 0$ such that for each $t \in \,]0,1]$ inequality (2.8) holds for almost all $(x,y) \in \Omega^2$ and for all Ω and h as in the statement. □

Then we proceed by stating the following classical result about eigenvalues.

Theorem 2.10. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Then the following statements hold.*

(i) *Let Ω and h be as in Theorem* 2.6. *Then the eigenvalues of the problem*

$$-\Delta_{\Omega,h}u = \lambda u$$

for $\lambda \in \mathbb{R}$ and $u \in L^2(\Omega)$ form an increasing unbounded sequence $\{\lambda_n[\Omega,h]\}_{n\in\mathbb{N}_0}$ of eigenvalues, which we write as

$$0 \le \lambda_0[\Omega,h] \le \lambda_1[\Omega,h] \le \ldots \le \lambda_n[\Omega,h] \le \ldots .$$

For each $n \in \mathbb{N}_0$, the eigenspace $\{u \in L^2(\Omega)\colon -\Delta_{\Omega,h}u = \lambda_n[\Omega,h]u\}$ is contained in $W^{1,2}(\Omega)$ and has a finite dimension, the multiplicity of $\lambda_n[\Omega,h]$. In the above sequence, we write each eigenvalue as many times as its multiplicity.

(ii) *Let Ω and h be as in Theorem* 2.6. *Let $\mathcal{E}_n(\Omega)$ denote the collection of subspaces E of $W^{1,2}(\Omega)$ of dimension $n+1$ for all $n \in \mathbb{N}_0$. Let*

$$\mu[E,\Omega,h] \equiv \sup\left\{\frac{Q_{\Omega,h}[u]}{\|u\|^2_{L^2(\Omega)}}\colon\ u \in E\setminus\{0\}\right\},$$

$$\mu_n[\Omega,h] \equiv \inf\bigl\{\mu[E,\Omega,h]\colon\ E \in \mathcal{E}_n(\Omega)\bigr\} \qquad \forall n \in \mathbb{N}_0.$$

Then

$$\lambda_n[\Omega,h] = \mu_n[\Omega,h] \qquad \forall n \in \mathbb{N}_0.$$

(iii) *Let $\tau = N$ if $N \ge 3$ and $2 < \tau < \infty$ if $N = 2$. Let f_n denote an eigenfunction of $-\Delta_{\Omega,h}$ corresponding to the eigenvalue $\lambda_n[\Omega,h]$ for each $n \in \mathbb{N}_0$. There exists $c > 0$ such that*

$$\|\mathrm{tr}\, f_n\|_{L^\infty(\partial\Omega)} \le \|f_n\|_{L^\infty(\Omega)} \le c(1+\lambda_n[\Omega,h])^{\tau/4}\|f_n\|_{L^2(\Omega)} \tag{2.10}$$

for all $n \in \mathbb{N}_0$, for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{0,1}\bigl(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\bigr)$, and for all nonnegative $h \in L^\infty(\partial\Omega)$.

Proof. Since $H_{\Omega,h}$ is associated with the nonnegative form $Q_{\Omega,h}$, the eigenvalues are nonnegative. The compactness of the embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ guarantees that $H_{\Omega,h}$ has a compact resolvent. Then statements (i) and (ii) follow by the classical Riesz theory for compact self-adjoint operators (see also Davies [6, Theorem 4.5.3, p. 91]). By combining Theorems 2.8 and 2.9, we deduce the existence of $c'_{\mathrm{c}} > 0$ such that

$$\bigl\|e^{-H_{\Omega,h}t}f\bigr\|_{L^\infty(\Omega)} \le c'_{\mathrm{c}}t^{-\tau/4}\|f\|_{L^2(\Omega)} \qquad \forall t \in\,]0,1], \quad \forall f \in L^2(\Omega)$$

for all Ω and h as in the statement. As in Burenkov and Davies [2, Lemma 10], we can deduce that

$$\|f_n\|_{L^\infty(\Omega)} \le \begin{cases} ec'_{\mathrm{c}}\|f_n\|_{L^2(\Omega)} & \text{if } \ 0 \le \lambda_n[\Omega,h] \le 1,\\ ec'_{\mathrm{c}}\lambda_n[\Omega,h]^{\tau/4}\|f_n\|_{L^2(\Omega)} & \text{if } \ 1 < \lambda_n[\Omega,h]. \end{cases}$$

Hence,

$$\|f_n\|_{L^\infty(\Omega)} \le ec'_{\mathrm{c}}(1+\lambda_n[\Omega,h])^{\tau/4}\|f_n\|_{L^2(\Omega)} \qquad \forall n \in \mathbb{N}_0.$$

Finally, inequality $\|\mathrm{tr}\, f_n\|_{L^\infty(\partial\Omega)} \le \|f_n\|_{L^\infty(\Omega)}$ follows by the membership of the eigenfunctions in $W^{1,2}(\Omega)$ and by Lemma 2.4. $\square$

Corollary 2.11. *Let Ω and h be as in Theorem* 2.6. *Let $W_0^{1,2}(\Omega)$ denote the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega)$. Let $\{\lambda_n^{\mathrm{D}}[\Omega]\}_{n\in\mathbb{N}_0}$ and $\{\lambda_n^{\mathrm{N}}[\Omega]\}_{n\in\mathbb{N}_0}$ be the sequences of the Dirichlet and Neumann eigenvalues of the problem $-\Delta u = \lambda u$, respectively. Then*

$$\lambda_n^{\mathrm{N}}[\Omega] = \lambda_n[\Omega, 0] \le \lambda_n[\Omega, h] \le \lambda_n^{\mathrm{D}}[\Omega] \qquad \forall n \in \mathbb{N}_0. \tag{2.11}$$

Proof. Let $\mathcal{E}_n^0(\Omega)$ denote the collection of subspaces E of $W_0^{1,2}(\Omega)$ of dimension $n+1$ for all $n \in \mathbb{N}_0$. As is well known,

$$\lambda_n^{\mathrm{D}}[\Omega] \equiv \inf\bigl\{\mu^{\mathrm{D}}[E,\Omega]\colon\ E \in \mathcal{E}_n^0(\Omega)\bigr\} \qquad \forall n \in \mathbb{N}_0,$$

where

$$\mu^{\mathrm{D}}[E,\Omega] \equiv \sup\biggl\{\frac{\int_\Omega |\nabla u|^2\,dx}{\|u\|^2_{L^2(\Omega)}}\colon\ u \in E\setminus\{0\}\biggr\}.$$

Since $\operatorname{tr} u = 0$ on $\partial\Omega$ for all $u \in W_0^{1,2}(\Omega)$, we have $\mu[E,\Omega,h] \le \mu^{\mathrm{D}}[E,\Omega]$ for all $E \in \mathcal{E}_n^0(\Omega)$. Hence, $\lambda_n[\Omega,h] \le \lambda_n^{\mathrm{D}}[\Omega]$. Since $Q_{\Omega,0} \le Q_{\Omega,h}$, the first inequality in (2.11) is obvious. $\square$

Corollary 2.12. *Let Ω and h be as in Theorem* 2.6. *Then*

$$\lambda_n[\Omega,h] \le \lambda_n^{\mathrm{D}}\bigl[\,]0,\delta[^N\bigr] \qquad \forall n \in \mathbb{N}_0.$$

Proof. By Definition 2.1(ii), Ω contains $r_1^t\bigl(W_1 \times\,]a_{1N}, a_{1N}+\delta[\bigr)$. Moreover, Definition 2.1(i) implies that $0 < 2\delta < \min_{l=1,\dots,N}(b_{1l}-a_{1l})$. Hence,

$$\lambda_n[\Omega,h] \le \lambda_n^{\mathrm{D}}[\Omega] \le \lambda_n^{\mathrm{D}}\bigl[r_1^t\bigl(W_1 \times\,]a_{1N}, a_{1N}+\delta[\bigr)\bigr]$$
$$\le \lambda_n^{\mathrm{D}}\left[r_1^t\left(\prod_{l=1}^{N-1}\right]\frac{a_{1l}+b_{1l}}{2}-\frac{\delta}{2}, \frac{a_{1l}+b_{1l}}{2}+\frac{\delta}{2}\left[\ \times\,]a_{1N}, a_{1N}+\delta[\right)\right] = \lambda_n^{\mathrm{D}}\bigl[\,]0,\delta[^N\bigr]. \qquad \square$$

3. SPECTRAL STABILITY THEOREMS FOR THE ROBIN LAPLACIAN

As we have said, we want to compare the spectra of $-\Delta_{\Omega_1,h_1}$ and $-\Delta_{\Omega_2,h_2}$ for two regions Ω_1 and Ω_2 and for two functions h_1 and h_2 defined on $\partial\Omega_1$ and $\partial\Omega_2$, respectively. To do so, we introduce a characteristic of vicinity for h_1 and h_2 and a characteristic of vicinity for $\partial\Omega_1$ and $\partial\Omega_2$.

Definition 3.1. Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition 2.1. Let Ω_1 and Ω_2 be bounded regions of $\mathbb{R}^N$ of class $C^{0,1}\bigl(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\bigr)$, with corresponding families of functions $\{\varphi_{1,j}\}_{j=1}^{s'}$ and $\{\varphi_{2,j}\}_{j=1}^{s'}$ as in Definition 2.1(iii). Let $h_1 \in L^1(\partial\Omega_1)$ and $h_2 \in L^1(\partial\Omega_2)$. Then we set

$$G(\partial\Omega_1,\partial\Omega_2) \equiv \sum_{j=1}^{s'} \int\limits_{W_j} \bigl||\nabla\varphi_{1,j}(\overline{x})| - |\nabla\varphi_{2,j}(\overline{x})|\bigr|\,d\overline{x},$$

and

$$L(h_1,h_2) \equiv \sum_{j=1}^{s'} \int\limits_{W_j} \bigl|h_1\circ r_j^t(\overline{x},\varphi_{1,j}(\overline{x})) - h_2\circ r_j^t(\overline{x},\varphi_{2,j}(\overline{x}))\bigr|\,d\overline{x}.$$

If instead we assume that $h_1 \in L^\infty(\partial\Omega_1)$ and $h_2 \in L^\infty(\partial\Omega_2)$, then we set

$$a(h_1,h_2) \equiv \max\bigl\{\|h_1\|_{L^\infty(\partial\Omega_1)},\ \|h_2\|_{L^\infty(\partial\Omega_2)},\ \|h_1\|^2_{L^\infty(\partial\Omega_1)},\ \|h_2\|^2_{L^\infty(\partial\Omega_2)}\bigr\}.$$

We note that there exist $k_1, k_2 > 0$ such that

$$k_1\|h_1 - h_2\|_{L^1(\partial\Omega)} \le L(h_1, h_2) \le k_2\|h_1 - h_2\|_{L^1(\partial\Omega)} \tag{3.1}$$

for all $\Omega \equiv \Omega_1 = \Omega_2$ as in Definition 3.1 and $h_1, h_2 \in L^1(\partial\Omega)$.

We are now ready to carry out the first step of the strategy we have announced in the Introduction.

Theorem 3.2. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Then for each $n \in \mathbb{N}_0$ there exists $b_n > 0$ such that*

$$\lambda_n[\Omega_2, h_2] \le \lambda_n[\Omega_1, h_1] + b_n\Big[|\Omega_1\setminus\Omega_2| + a(h_1, h_2)\big(|\Omega_1\setminus\Omega_2| + G(\partial\Omega_1, \partial\Omega_2)\big) + L(h_1, h_2)\Big] \tag{3.2}$$

for all bounded regions Ω_1 and Ω_2 of $\mathbb{R}^N$ that are of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ and satisfy

$$\Omega_2 \subset \Omega_1, \qquad |\Omega_1\setminus\Omega_2| \le b_n^{-1}, \tag{3.3}$$

and for all nonnegative $h_1 \in L^\infty(\partial\Omega_1)$ and $h_2 \in L^\infty(\partial\Omega_2)$.

Proof. Let τ be as in Theorem 2.9. To shorten our notation, we set

$$Q_i \equiv Q_{\Omega_i, h_i} \qquad \text{and} \qquad \lambda_{n,i} \equiv \lambda_n[\Omega_i, h_i] \tag{3.4}$$

for $i = 1, 2$. Let $\{u_n\}_{n=0}^\infty$ denote an orthonormal basis of $L^2(\Omega_1)$ of eigenfunctions $u_n \in W^{1,2}(\Omega_1)$ with $\|u_n\|_{L^2(\Omega_1)} = 1$, $-\Delta_{\Omega_1,h_1}u_n = \lambda_{n,1}u_n$. Let E_n be the subspace of $W^{1,2}(\Omega_1)$ generated by $\{u_0, \dots, u_n\}$. Let P denote the restriction map of $L^2(\Omega_1)$ to $L^2(\Omega_2)$. By inequality (2.10), if $f \in E_n$ with $f = \sum_{l=0}^n \beta_l u_l$, $\beta_l \in \mathbb{R}$, then

$$\begin{aligned}\|f\|_{L^\infty(\Omega_1)} &\le \sum_{l=0}^n |\beta_l|\cdot\|u_l\|_{L^\infty(\Omega_1)} \le c\sum_{l=0}^n |\beta_l|(1+\lambda_{l,1})^{\tau/4}\|u_l\|_{L^2(\Omega_1)} \\ &\le c\left(\sum_{l=0}^n (1+\lambda_{l,1})^{\tau/2}\right)^{1/2}\left(\sum_{l=0}^n \beta_l^2\right)^{1/2} = c\left(\sum_{l=0}^n (1+\lambda_{l,1})^{\tau/2}\right)^{1/2}\|f\|_{L^2(\Omega_1)}.\end{aligned}$$

Now we set

$$c_n \equiv 2c^2\sum_{l=0}^n (1+\lambda_{l,1})^{\tau/2}.$$

Then we have

$$\|f\|_{L^\infty(\Omega_1)} \le \sqrt{\frac{c_n}{2}}\,\|f\|_{L^2(\Omega_1)}. \tag{3.5}$$

Next we note that

$$\|Pf\|^2_{L^2(\Omega_1)} = \|f\|^2_{L^2(\Omega_2)} = \|f\|^2_{L^2(\Omega_1)} - \int\limits_{\Omega_1\setminus\Omega_2} |f|^2\,dx \ge \|f\|^2_{L^2(\Omega_1)}\Big[1 - \frac{c_n}{2}|\Omega_1\setminus\Omega_2|\Big]. \tag{3.6}$$

Assuming that $|\Omega_1\setminus\Omega_2| \le c_n^{-1}$, we have

$$\|Pf\|^{-2}_{L^2(\Omega_1)} \le \|f\|^{-2}_{L^2(\Omega_1)}\big(1 + c_n|\Omega_1\setminus\Omega_2|\big).$$

Indeed, $(1-(t/2))^{-1}\le 1+t$ for $0\le t\le 1$. Now let $f\in E_n$ and $g\equiv Pf$. Then we have

$$\frac{Q_2[g]}{\|g\|^2_{L^2(\Omega_2)}}\le \bigl(1+c_n|\Omega_1\setminus\Omega_2|\bigr)\frac{Q_2[g]}{\|f\|^2_{L^2(\Omega_1)}}. \tag{3.7}$$

We now turn to estimating $Q_2[g]$. We note that

$$Q_2[g]=Q_1[f]-\int\limits_{\Omega_1\setminus\Omega_2}|\nabla f|^2\,dx+\int\limits_{\partial\Omega_2}h_2(\operatorname{tr}f)^2\,d\sigma-\int\limits_{\partial\Omega_1}h_1(\operatorname{tr}f)^2\,d\sigma. \tag{3.8}$$

Thus we now estimate $\int_{\partial\Omega_2}h_2(\operatorname{tr}f)^2\,d\sigma-\int_{\partial\Omega_1}h_1(\operatorname{tr}f)^2\,d\sigma$.

We first do so for f replaced by an arbitrary function w of $C^\infty(\overline{\Omega}_1)$. Let $\{\zeta_j\}_{j=1}^s$ be a partition of unity as in Lemma 2.2. For brevity, we set

$$\widehat{h}_{i,j}(\overline{x})\equiv h_i\circ r_j^t(\overline{x},\varphi_{i,j}(\overline{x})),\qquad \widehat{\operatorname{tr}w}_{i,j}(\overline{x})\equiv(\operatorname{tr}w)\circ r_j^t(\overline{x},\varphi_{i,j}(\overline{x})),$$
$$\widehat{\zeta}_{i,j}(\overline{x})\equiv\zeta_j\circ r_j^t(\overline{x},\varphi_{i,j}(\overline{x}))\qquad\forall\,\overline{x}\in W_j,\quad i=1,2.$$

Then we have

$$\int\limits_{\partial\Omega_2}h_2(\operatorname{tr}w)^2\,d\sigma-\int\limits_{\partial\Omega_1}h_1(\operatorname{tr}w)^2\,d\sigma=\sum_{j=1}^{s'}\Biggl(\int\limits_{\partial\Omega_2}h_2(\operatorname{tr}w)^2\zeta_j\,d\sigma-\int\limits_{\partial\Omega_1}h_1(\operatorname{tr}w)^2\zeta_j\,d\sigma\Biggr). \tag{3.9}$$

Since $\operatorname{supp}\zeta_j\subset V_j$, the right-hand side of (3.9) can be estimated as follows:

$$\begin{aligned}\sum_{j=1}^{s'}&\Biggl(\int\limits_{V_j\cap\partial\Omega_2}h_2(\operatorname{tr}w)^2\zeta_j\,d\sigma-\int\limits_{V_j\cap\partial\Omega_1}h_1(\operatorname{tr}w)^2\zeta_j\,d\sigma\Biggr)\\
&\le\sum_{j=1}^{s'}\Biggl|\int\limits_{W_j}\Bigl\{\widehat{h}_{2,j}(\overline{x})\bigl(\widehat{\operatorname{tr}w}_{2,j}(\overline{x})\bigr)^2\widehat{\zeta}_{2,j}(\overline{x})\sqrt{1+|\nabla\varphi_{2,j}(\overline{x})|^2}\\
&\qquad-\widehat{h}_{1,j}(\overline{x})\bigl(\widehat{\operatorname{tr}w}_{1,j}(\overline{x})\bigr)^2\widehat{\zeta}_{1,j}(\overline{x})\sqrt{1+|\nabla\varphi_{1,j}(\overline{x})|^2}\Bigr\}\,d\overline{x}\Biggr|\equiv\sum_{j=1}^{s'}I_j.\end{aligned} \tag{3.10}$$

By the triangle inequality, we have

$$\begin{aligned}I_j&\le\int\limits_{W_j}\widehat{h}_{2,j}(\overline{x})\Bigl|\bigl(\widehat{\operatorname{tr}w}_{2,j}(\overline{x})\bigr)^2-\bigl(\widehat{\operatorname{tr}w}_{1,j}(\overline{x})\bigr)^2\Bigr|\sqrt{1+|\nabla\varphi_{2,j}(\overline{x})|^2}\,d\overline{x}\\
&\quad+\int\limits_{W_j}\widehat{h}_{2,j}(\overline{x})\bigl(\widehat{\operatorname{tr}w}_{1,j}(\overline{x})\bigr)^2\Bigl|\sqrt{1+|\nabla\varphi_{2,j}(\overline{x})|^2}-\sqrt{1+|\nabla\varphi_{1,j}(\overline{x})|^2}\Bigr|\,d\overline{x}\\
&\quad+\int\limits_{W_j}\bigl|\widehat{h}_{2,j}(\overline{x})-\widehat{h}_{1,j}(\overline{x})\bigr|\bigl(\widehat{\operatorname{tr}w}_{1,j}(\overline{x})\bigr)^2\sqrt{1+|\nabla\varphi_{1,j}(\overline{x})|^2}\,d\overline{x}\\
&\quad+\int\limits_{W_j}\widehat{h}_{1,j}(\overline{x})\bigl(\widehat{\operatorname{tr}w}_{1,j}(\overline{x})\bigr)^2\bigl|\widehat{\zeta}_{2,j}(\overline{x})-\widehat{\zeta}_{1,j}(\overline{x})\bigr|\sqrt{1+|\nabla\varphi_{1,j}(\overline{x})|^2}\,d\overline{x}\\
&\equiv I_{j,1}+I_{j,2}+I_{j,3}+I_{j,4}.\end{aligned}$$

We now estimate $I_{j,1}$:

$$I_{j,1} \le \|h_2\|_{L^\infty(\partial\Omega_2)} \sqrt{1+M^2} \int\limits_{W_j} \left| \left(\widehat{\operatorname{tr} w}_{2,j}(\overline{x})\right)^2 - \left(\widehat{\operatorname{tr} w}_{1,j}(\overline{x})\right)^2 \right| d\overline{x}$$

$$\le 2\|h_2\|_{L^\infty(\partial\Omega_2)} \sqrt{1+M^2} \int\limits_{W_j} \int\limits_{\varphi_{2,j}(\overline{x})}^{\varphi_{1,j}(\overline{x})} \left| w \circ r_j^t(\overline{x}, x_N)\right| \left| \frac{\partial}{\partial x_N} \left(w \circ r_j^t(\overline{x}, x_N)\right) \right| dx_N \, d\overline{x}$$

$$\le 2\|h_2\|_{L^\infty(\partial\Omega_2)} \sqrt{1+M^2} \int\limits_{(\Omega_1 \setminus \Omega_2)\cap V_j} |w| \cdot |\nabla w| \, dx.$$

Hence,

$$\sum_{j=1}^{s'} I_{j,1} \le 2s' \|h_2\|_{L^\infty(\partial\Omega_2)} \sqrt{1+M^2} \int\limits_{\Omega_1 \setminus \Omega_2} |w| \cdot |\nabla w| \, dx$$

$$\le \int\limits_{\Omega_1 \setminus \Omega_2} |\nabla w|^2 \, dx + \left(s' \|h_2\|_{L^\infty(\partial\Omega_2)} \sqrt{1+M^2}\right)^2 \int\limits_{\Omega_1 \setminus \Omega_2} |w|^2 \, dx$$

$$\le \int\limits_{\Omega_1 \setminus \Omega_2} |\nabla w|^2 \, dx + s'^2 \|h_2\|^2_{L^\infty(\partial\Omega_2)} (1+M^2) \, |\Omega_1 \setminus \Omega_2| \cdot \|w\|^2_{L^\infty(\Omega_1)}. \tag{3.11}$$

Next, we estimate $\sum_{j=1}^{s'} I_{j,2}$. Since $\left|\sqrt{1+a} - \sqrt{1+b}\right| \le |a-b|$ for all $a, b \in [0, +\infty[$, we have

$$\sum_{j=1}^{s'} I_{j,2} \le \|h_2\|_{L^\infty(\partial\Omega_2)} \|w\|^2_{L^\infty(\Omega_1)} \sum_{j=1}^{s'} \int\limits_{W_j} \left| |\nabla \varphi_{1,j}| - |\nabla \varphi_{2,j}| \right| d\overline{x}$$

$$\le \|h_2\|_{L^\infty(\partial\Omega_2)} \|w\|^2_{L^\infty(\Omega_1)} G(\partial\Omega_1, \partial\Omega_2). \tag{3.12}$$

Now we consider $\sum_{j=1}^{s'} I_{j,3}$. Clearly,

$$\sum_{j=1}^{s'} I_{j,3} \le L(h_1, h_2) \|w\|^2_{L^\infty(\Omega_1)} \sqrt{1+M^2}. \tag{3.13}$$

We now consider $\sum_{j=1}^{s'} I_{j,4}$. By the mean value inequality, we have

$$\left|\widehat{\zeta}_{2,j}(\overline{x}) - \widehat{\zeta}_{1,j}(\overline{x})\right| \le \beta \left| r_j^t(\overline{x}, \varphi_{2,j}(\overline{x})) - r_j^t(\overline{x}, \varphi_{1,j}(\overline{x}))\right| = \beta |\varphi_{2,j}(\overline{x}) - \varphi_{1,j}(\overline{x})|$$

for all $\overline{x} \in \overline{W_j}$, where

$$\beta \equiv \max_{j=1,\dots,s'} \max_{\mathbb{R}^N} |\nabla \zeta_j|.$$

We note that β depends only on N, M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$. Then we have

$$\sum_{j=1}^{s'} I_{j,4} \le \|h_1\|_{L^\infty(\partial\Omega_1)} \beta \|w\|^2_{L^\infty(\Omega_1)} \sqrt{1+M^2} \, |\Omega_1 \setminus \Omega_2| \, s'. \tag{3.14}$$

Then, by Definition 3.1 and by (3.9)–(3.14), we deduce that

$$\int_{\partial\Omega_2} h_2(\operatorname{tr} w)^2\,d\sigma - \int_{\partial\Omega_1} h_1(\operatorname{tr} w)^2\,d\sigma \le \int_{\Omega_1\setminus\Omega_2} |\nabla w|^2\,dx + s'^2\|h_2\|^2_{L^\infty(\partial\Omega_2)}(1+M^2)|\Omega_1\setminus\Omega_2|\cdot\|w\|^2_{L^\infty(\Omega_1)}$$
$$+\|h_2\|_{L^\infty(\partial\Omega_2)}G(\partial\Omega_1,\partial\Omega_2)\|w\|^2_{L^\infty(\Omega_1)} + \sqrt{1+M^2}\,L(h_1,h_2)\|w\|^2_{L^\infty(\Omega_1)}$$
$$+ s'\beta\|h_1\|_{L^\infty(\partial\Omega_1)}\sqrt{1+M^2}\,|\Omega_1\setminus\Omega_2|\cdot\|w\|^2_{L^\infty(\Omega_1)}. \tag{3.15}$$

By Lemmas 2.3 and 2.4(i), we immediately deduce that inequality (3.15) also holds for $w = f$. Then equality (3.8) and inequality (3.5) imply that

$$Q_2[g] \le Q_1[f] + \Big\{a(h_1,h_2)\Big(s'^2(1+M^2) + s'\beta\sqrt{1+M^2}\Big)|\Omega_1\setminus\Omega_2|$$
$$+ a(h_1,h_2)G(\partial\Omega_1,\partial\Omega_2) + \sqrt{1+M^2}\,L(h_1,h_2)\Big\}\|f\|^2_{L^\infty(\Omega_1)}$$
$$\le Q_1[f] + \Big\{a(h_1,h_2)\Big(s'^2(1+M^2) + s'\beta\sqrt{1+M^2}\Big)|\Omega_1\setminus\Omega_2|$$
$$+ a(h_1,h_2)G(\partial\Omega_1,\partial\Omega_2) + \sqrt{1+M^2}\,L(h_1,h_2)\Big\}\frac{c_n}{2}\|f\|^2_{L^2(\Omega_1)}$$
$$\le Q_1[f] + b'_n\Big\{a(h_1,h_2)\big(|\Omega_1\setminus\Omega_2| + G(\partial\Omega_1,\partial\Omega_2)\big) + L(h_1,h_2)\Big\}\|f\|^2_{L^2(\Omega_1)}, \tag{3.16}$$

where

$$b'_n \equiv \frac{bc_n}{2}, \qquad b \equiv s'^2(1+M^2) + s'\beta\sqrt{1+M^2}.$$

By inequality (3.6) and by assumption $|\Omega_1\setminus\Omega_2| \le c_n^{-1}$, the restriction operator P is injective from E_n onto PE_n. Thus we have $\dim(PE_n) = \dim(E_n) = n+1$. Then, by the definition of $\lambda_{n,2}$, by Theorem 2.10(ii), by (3.7) and (3.16), and by the condition $|\Omega_1\setminus\Omega_2| \le c_n^{-1}$, we have

$$\lambda_{n,2} \le \sup_{0\ne g\in PE_n} \frac{Q_2[g]}{\|g\|_{L^2(\Omega_2)}}$$
$$\le (1 + c_n|\Omega_2\setminus\Omega_1|)\bigg\{\sup_{0\ne f\in E_n}\frac{Q_1[f]}{\|f\|_{L^2(\Omega_1)}} + b'_n\Big[a(h_1,h_2)\big(|\Omega_1\setminus\Omega_2| + G(\partial\Omega_1,\partial\Omega_2)\big) + L(h_1,h_2)\Big]\bigg\}$$
$$\le (1 + c_n|\Omega_1\setminus\Omega_2|)\sup_{0\ne f\in E_n}\frac{Q_1[f]}{\|f\|_{L^2(\Omega_1)}} + 2b'_n\Big[a(h_1,h_2)\big(|\Omega_1\setminus\Omega_2| + G(\partial\Omega_1,\partial\Omega_2)\big) + L(h_1,h_2)\Big].$$

Hence, inequality (3.2) follows with b_n replaced by

$$b''_n \equiv \max\{c_n\lambda_{n,1}, bc_n\}$$

and for $|\Omega_1\setminus\Omega_2| \le c_n^{-1}$. We now note that Corollary 2.12 implies that

$$b''_n \le 2c^2(1+\lambda_{n,1})\sum_{l=0}^{n}(1+\lambda_{l,1})^{\tau/2}\max\{1,b\} \le 2c^2\big(1+\lambda_n^{\mathrm{D}}[\,]0,\delta[^N]\big)\sum_{l=0}^{n}\big(1+\lambda_l^{\mathrm{D}}[\,]0,\delta[^N]\big)^{\tau/2}b \equiv b_n.$$

Hence, inequality (3.2) follows for $|\Omega_1\setminus\Omega_2| \le c_n^{-1}$. Since $c_n \le bc_n \le b''_n \le b_n$, inequality (3.2) holds for $|\Omega_1\setminus\Omega_2| \le b_n^{-1}$. □

In case $\Omega \equiv \Omega_1 = \Omega_2$, one could apply Theorem 3.2, and by interchanging the roles of h_1 and h_2 and applying inequality (3.1), one could obtain an estimate from above and below for $\lambda_n[\Omega, h_2]$ in terms of $\lambda_n[\Omega, h_1]$ and $\|h_1 - h_2\|_{L^1(\partial\Omega)}$ for all nonnegative $h_1, h_2 \in L^\infty(\partial\Omega)$. However, by arguing directly, we can obtain the following slightly sharper statement.

Theorem 3.3. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Then for each $n \in \mathbb{N}_0$ there exists $\beta_n > 0$ such that*

$$\lambda_n[\Omega, h_1] - \beta_n \|h_1 - h_2\|_{L^1(\partial\Omega)} \le \lambda_n[\Omega, h_2] \le \lambda_n[\Omega, h_1] + \beta_n \|h_1 - h_2\|_{L^1(\partial\Omega)} \tag{3.17}$$

for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ and for all nonnegative h_1 and h_2 as in Theorem 2.9.

Proof. Let f be as in the beginning of the proof of Theorem 3.2. Then we have

$$Q_2[f] = Q_1[f] + \int\limits_{\partial\Omega} h_2 (\operatorname{tr} f)^2\, d\sigma - \int\limits_{\partial\Omega} h_1 (\operatorname{tr} f)^2\, d\sigma.$$

Then, by inequalities (2.10) and (3.5), we can deduce that $\lambda_{n,2} \le \lambda_{n,1} + \frac{c_n}{2}\|h_2 - h_1\|_{L^1(\partial\Omega)}$. Then, by interchanging the roles of h_1 and of h_2, we can obtain (3.17). $\square$

Inequality (3.17) implies the (certainly known) continuous dependence of the eigenvalues of the Robin Laplacian upon perturbation of the function entering the boundary conditions, and gives an estimate for $\lambda_n[\Omega, h_1]$ and $\lambda_n[\Omega, h_2]$ in terms of the vicinity of h_1 and h_2 and of a constant that depends solely on the Lipschitz class and not on the specific domain Ω.

We now turn to the second step of the strategy explained in the Introduction. We set

$$\partial_\epsilon \Omega \equiv \big\{x \in \Omega\colon\ \operatorname{dist}(x, \partial\Omega) \le \epsilon\big\}$$

for all subsets Ω of $\mathbb{R}^N$ and $\epsilon > 0$, and we introduce a family of diffeomorphisms of $\mathbb{R}^N$ depending on a parameter ϵ that map a region Ω into a subset of $\Omega \setminus \partial_\epsilon\Omega$. Namely, we introduce the following technical lemma of Burenkov and Davies [2, p. 488, Lemma 18, p. 499].

Lemma 3.4. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Then there exist ϵ_0, $A > 0$, and a one-parameter family $\{\Lambda_\epsilon\}_{\epsilon\in[0,\epsilon_0[}$ of diffeomorphisms of class C^∞ of $\mathbb{R}^N$ onto $\mathbb{R}^N$ such that*

(i) *the map of $\mathbb{R}^N \times [0, \epsilon_0[$ to $\mathbb{R}^N$ that takes (p, ϵ) to $\Lambda_\epsilon(p)$ is continuous*;

(ii) $|x - \Lambda_\epsilon(x)| \le A\epsilon$ *and* $\big|\frac{\partial(\Lambda_\epsilon)_l}{\partial x_j}(x) - \delta_{jl}\big| \le A\epsilon$ *for all $(x, \epsilon) \in \mathbb{R}^N \times [0, \epsilon_0[$, where $\delta_{jl} = 1$ for $j = l$ and $\delta_{jl} = 0$ for $j \ne l$, $j, l = 1, \dots, N$*;

(iii) $2^{-1} \le 1 - A\epsilon \le \det(\nabla\Lambda_\epsilon(x)) \le 1 + A\epsilon$ *for all $(x, \epsilon) \in \mathbb{R}^N \times [0, \epsilon_0[$, where $\nabla\Lambda_\epsilon(x)$ denotes the Jacobian matrix of Λ_ϵ at x*;

and such that for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ the following conditions hold:

(iv) $\Lambda_\epsilon(\Omega) \subset \Omega \setminus \partial_\epsilon\Omega$ *for all $\epsilon \in [0, \epsilon_0[$*;

(v) $|\partial_\epsilon\Omega| \le |\Omega \setminus \Lambda_\epsilon(\Omega)| \le A\epsilon$ *for all $\epsilon \in [0, \epsilon_0[$*.

However, in order to prove our final statement, we need some extra properties of the family of maps $\{\Lambda_\epsilon\}_{\epsilon\in[0,\epsilon_0[}$, which follow from (i)–(v) above. Thus we now introduce some extra notation and the following two technical lemmas.

For a given cube

$$K_j \equiv \prod_{l=1}^N]a_{jl}, b_{jl}[\equiv W_j \times]a_{jN}, b_{jN}[$$

and for $0 < \nu < \frac{1}{2}\min_{l=1,\dots,N}|b_{jl} - a_{jl}|$, we clearly have

$$(W_j)_\nu = \prod_{l=1}^{N-1}]a_{jl} + \nu, b_{jl} - \nu[, \qquad (K_j)_\nu = \prod_{l=1}^{N}]a_{jl} + \nu, b_{jl} - \nu[.$$

Then we have the following.

Lemma 3.5. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Let $\{\Lambda_\epsilon\}_{\epsilon\in[0,\epsilon_0[}$ be as in Lemma* 3.4. *Then there exists $0 < \epsilon_1 < \epsilon_0$ such that, for all $\epsilon \in [0, \epsilon_1[$ and for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$, the set $\Lambda_\epsilon(\Omega)$ is a bounded region of $\mathbb{R}^N$ of class $C^{0,1}\big(M+1, \delta/4, s, s', \{(V_j)_{\delta/2}\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$.*

Proof. We first prove that if ϵ belongs to the interval $[0, \min\{\epsilon_0, \delta/(4A), (AN(M+1))^{-1}\}[$ and j is fixed in $\{1,\dots,s'\}$, then $r_j\big((\partial\Lambda_\epsilon(\Omega)) \cap (V_j)_{\delta/2}\big)$ is a graph of a function of $(W_j)_{\delta/2}$ to $]a_{jN}, b_{jN}[$. By Lemma 3.4(ii), we have

$$|\Lambda_\epsilon(x) - x| \le A\epsilon \qquad \forall (x,\epsilon) \in \partial\Omega \times [0, \epsilon_0[. \tag{3.18}$$

Since $\epsilon < \delta/(2A)$, for each $P \in \Lambda_\epsilon(\partial\Omega) \cap (V_j)_{\delta/2}$ there exists a unique $p' \in W_j$ such that

$$P = \Lambda_\epsilon r_j^t(p', \varphi_j(p')).$$

We denote by π_1 and π_2 the projection of $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$ onto $\mathbb{R}^{N-1}$ and $\mathbb{R}$, respectively. Now let $p \in (W_j)_{\delta/2}$. We now show that for ϵ sufficiently small, the equation

$$p = \pi_1 r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p')) \tag{3.19}$$

has one and only one solution $p' \in W_j$. Now inequality (3.18) implies that

$$\begin{aligned}\big|\pi_1 r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p')) - p'\big| &\le \big|r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p')) - (p', \varphi_j(p'))\big| \\ &= \big|\Lambda_\epsilon r_j^t(p', \varphi_j(p')) - r_j^t(p', \varphi_j(p'))\big| \le A\epsilon \end{aligned} \tag{3.20}$$

for all $p' \in \overline{W_j}$. By our assumptions on φ_j, the function φ_j can be extended by continuity to $\overline{W_j}$. Since $p \in (W_j)_{\delta/2}$, we have

$$\big|p - \pi_1 r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p'))\big| \ge |p - p'| - \big|p' - \pi_1 r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p'))\big| \ge \frac{\delta}{2} - A\epsilon > 0 \qquad \forall \epsilon \in [0, \delta/(2A)[$$

for all $p' \in \partial W_j$. Moreover, the right-hand side of equation (3.19) reduces to the identity map in W_j for $\epsilon = 0$, and the map $\pi_1 r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p'))$ is continuous in the variable $(p', \epsilon) \in \overline{W_j} \times [0, \epsilon_0[$. Then an application of the topological degree to equation (3.19) (see, e.g., Deimling [8]) yields the existence of at least one solution $p' \in W_j$. To prove the uniqueness, we observe that equation (3.19) can be rewritten as

$$p = p' + \pi_1 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p', \varphi_j(p')), \tag{3.21}$$

where id denotes the identity operator in $\mathbb{R}^N$. By Lemma 3.4(ii) and by the fundamental theorem of calculus, we have

$$|(\Lambda_\epsilon - \mathrm{id})(P_1) - (\Lambda_\epsilon - \mathrm{id})(P_2)| \le \epsilon N A |P_1 - P_2| \tag{3.22}$$

for all $P_1, P_2 \in \mathbb{R}^N$. Thus, if $p_1', p_2' \in W_j$ solve equation (3.21), then we have

$$\begin{aligned}|p_1' - p_2'| &\le \big|\pi_1 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_1', \varphi_j(p_1')) - \pi_1 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_2', \varphi_j(p_2'))\big| \\ &\le \epsilon N A \big|(p_1', \varphi_j(p_1')) - (p_2', \varphi_j(p_2'))\big| \le \epsilon N A (1+M) |p_1' - p_2'|. \end{aligned} \tag{3.23}$$

Since $\epsilon < (NA(1+M))^{-1}$, we have $p_1' = p_2'$ and thus the uniqueness. For each $p \in (W_j)_{\delta/2}$, we set

$$\psi_j^\epsilon(p) \equiv \pi_2 r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p')),$$

where $p' \in W_j$ is the only solution of (3.19). By (3.18), we have

$$|\psi_j^\epsilon(p) - \varphi_j(p')| \le \big|\Lambda_\epsilon r_j^t(p', \varphi_j(p')) - r_j^t(p', \varphi_j(p'))\big| \le A\epsilon \le \frac{\delta}{4}.$$

Consequently,

$$a_{jN} + \frac{3\delta}{4} \le \psi_j^\epsilon \le b_{jN} - \frac{3\delta}{4}.$$

By the definition of ψ_j^ϵ, we have

$$\operatorname{graph}(\psi_j^\epsilon) \subset r_j\big(\Lambda_\epsilon(\partial\Omega) \cap (V_j)_{\delta/2}\big).$$

Conversely, if $P \equiv (p, y) \in r_j\big(\Lambda_\epsilon(\partial\Omega) \cap (V_j)_{\delta/2}\big)$, then there exists a unique $P' \in \partial\Omega$ such that $P = r_j(\Lambda_\epsilon(P'))$. By inequality (3.18) and the inequality $\epsilon < \delta/(2A)$, we have $P' \in V_j$. Now let $p' \in W_j$ with $r_j P' \equiv (p', \varphi_j(p'))$. Clearly, equation (3.19) must hold. By the definition of ψ_j^ϵ, we must have

$$(p, \psi_j^\epsilon(p)) = r_j \Lambda_\epsilon r_j^t(p', \varphi_j(p')),$$

and thus $\psi_j^\epsilon(p) = y$, i.e., $P \in \operatorname{graph}(\psi_j^\epsilon)$. Thus we conclude that

$$\operatorname{graph}(\psi_j^\epsilon) = r_j\big(\Lambda_\epsilon(\partial\Omega) \cap (V_j)_{\delta/2}\big).$$

We now show the Lipschitz continuity of ψ_j^ϵ. Let $p_1', p_2' \in W_j$ be the unique solutions of equation (3.19) corresponding to the points p_1 and p_2 of $(W_j)_{\delta/2}$, respectively. Then (3.22) implies that

$$\begin{aligned}|\psi_j^\epsilon(p_1) - \psi_j^\epsilon(p_2)| &\le |\varphi_j(p_1') - \varphi_j(p_2')| + \big|\pi_2 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_1', \varphi_j(p_1')) - \pi_2 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_2', \varphi_j(p_2'))\big| \\ &\le |\varphi_j(p_1') - \varphi_j(p_2')| + \epsilon AN(M+1)|p_1' - p_2'| \le (M + \epsilon AN(M+1))|p_1' - p_2'|.\end{aligned}$$

Now, by (3.21) and by the argument used to prove (3.23), we have

$$\begin{aligned}|p_1' - p_2'| &\le |p_1 - p_2| + \big|\pi_1 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_1', \varphi_j(p_1')) - \pi_1 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_2', \varphi_j(p_2'))\big| \\ &\le |p_1 - p_2| + \epsilon AN(M+1)|p_1' - p_2'|.\end{aligned} \tag{3.24}$$

Since $\epsilon AN(M+1) < 1$, we conclude that

$$|\psi_j^\epsilon(p_1) - \psi_j^\epsilon(p_2)| \le \frac{M + \epsilon AN(M+1)}{1 - \epsilon AN(M+1)}|p_1 - p_2|.$$

By setting

$$\epsilon_1 \equiv \min\big\{\epsilon_0,\ \delta/(4A),\ [AN(M+1)(M+2)]^{-1}\big\},$$

we also have

$$|\psi_j^\epsilon(p_1) - \psi_j^\epsilon(p_2)| \le (M+1)|p_1 - p_2| \qquad \forall \epsilon \in [0, \epsilon_1[.$$

By the continuity of $\Lambda_\epsilon(p)$ in $(p,\epsilon)\in\overline{\Omega}\times[0,\epsilon_0[$, by inequality (3.18), and by use of topological degree, it follows that all points of $\Omega_{\delta/4}$ are contained in $\Lambda_\epsilon(\Omega)$ and that all points of $\mathbb{R}^N$ with distance to $\overline{\Omega}$ greater than or equal to $\delta/4$ must be in $\mathbb{R}^N\setminus\overline{\Lambda_\epsilon(\Omega)}$. Accordingly,

$$(W_j)_{\delta/2}\times\left\{b_{jN}-\frac{3\delta}{4}\right\}\subset\mathbb{R}^N\setminus r_j\bigl(\overline{\Lambda_\epsilon(\Omega)}\bigr),\qquad (W_j)_{\delta/2}\times\left\{a_{jN}+\frac{3\delta}{4}\right\}\subset r_j(\Lambda_\epsilon(\Omega)).\tag{3.25}$$

Then a simple topological argument, together with the equality

$$r_j\bigl(\Lambda_\epsilon(\partial\Omega)\cap(V_j)_{\delta/2}\bigr)=\operatorname{graph}(\psi_j^\epsilon),$$

shows that

$$r_j\bigl(\Lambda_\epsilon(\Omega)\cap(V_j)_{\delta/2}\bigr)=\Bigl\{(\overline{x},x_N)\in(W_j)_{\delta/2}\times]a_{jN}+(\delta/2),b_{jN}-(\delta/2)[:\ a_{jN}+(\delta/2)<x_N<\psi_j^\epsilon(\overline{x})\Bigr\}$$

for all $\epsilon\in[0,\epsilon_1[$. We also have

$$\Lambda_\epsilon(\Omega)\subset\Omega\subset\bigcup_{j=1}^{s}(V_j)_\delta\subset\bigcup_{j=1}^{s}\bigl((V_j)_{\delta/2}\bigr)_{\delta/4}.$$

By the argument used to prove (3.25) and by Definition 2.1(ii), we have

$$(V_j)_{\delta/2}\subset V_j\subset\Omega_\delta\subset\Omega_{\delta/4}\subset\Lambda_\epsilon(\Omega)$$

for all $\epsilon\in[0,\epsilon_0[$ and $s'<j\le s$. Moreover, if $x\in(V_j)_{\delta/2}$ and $p\in\partial\Omega$, the inclusion $V_j\subset\Omega_\delta$ implies that $d(x,\partial\Omega)\ge\frac{3}{2}\delta$. Since $\epsilon<\delta/(2A)$, inequality (3.18) implies that

$$|x-\Lambda_\epsilon(p)|\ge|x-p|-|p-\Lambda_\epsilon(p)|\ge\frac{3}{2}\delta-A\epsilon>\delta$$

for all $x\in(V_j)_{\delta/2}$, $p\in\partial\Omega$, and $0<\epsilon<\epsilon_0$. Hence,

$$(V_j)_{\delta/2}\subset(\Lambda_\epsilon(\Omega))_{\delta/4}$$

for $s'<j\le s$ and $\epsilon\in[0,\epsilon_0[$. In particular, $((V_j)_{\delta/2})_{\delta/4}\cap\Lambda_\epsilon(\Omega)\ne\varnothing$ for $s'<j\le s$. We now show that $((V_j)_{\delta/2})_{\delta/4}\cap\Lambda_\epsilon(\Omega)\ne\varnothing$ for $1\le j\le s'$. If $1\le j\le s'$, then by (3.25) we have $(W_j)_{\delta/2}\times\{a_{jN}+(3\delta/4)\}\subset r_j(\Lambda_\epsilon(\Omega))$. Since $(W_j)_{\delta/2}\times\{a_{jN}+(3\delta/4)\}\cap\partial\bigl\{((V_j)_{\delta/2})_{\delta/4}\bigr\}\ne\varnothing$ and $r_j(\Lambda_\epsilon(\Omega))$ is open, we conclude that $((V_j)_{\delta/2})_{\delta/4}\cap\Lambda_\epsilon(\Omega)\ne\varnothing$. Thus the proof is complete. □

Lemma 3.6. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Let $\{\Lambda_\epsilon\}_{\epsilon\in[0,\epsilon_1[}$ be as in Lemma* 3.5. *Let $0<\gamma\le1$. Then there exist $\epsilon_2\in]0,\epsilon_1[$ and $A_1>0$ such that the following statements hold.*

(i) *Let G' denote the characteristic of vicinity introduced in Definition* 3.1 *relative to the class $C^{0,1}\bigl(M+1,\delta/4,s,s',\{(V_j)_{\delta/2}\}_{j=1}^s,\{r_j\}_{j=1}^s\bigr)$. Then*

$$G'(\partial\Lambda_\epsilon(\Omega),\partial\Omega)\le A_1\epsilon^\gamma\qquad\forall\epsilon\in[0,\epsilon_2[\tag{3.26}$$

for all bounded regions Ω of $\mathbb{R}^N$ of class $C^{1,\gamma}\bigl(M,\delta,s,s',\{V_j\}_{j=1}^s,\{r_j\}_{j=1}^s\bigr)$.

(ii) *Let L' denote the characteristic of vicinity introduced in Definition* 3.1 *relative to the class $C^{0,1}\bigl(M+1,\delta/4,s,s',\{(V_j)_{\delta/2}\}_{j=1}^s,\{r_j\}_{j=1}^s\bigr)$. Then*

$$L'(h_\epsilon,h)\le A_1\operatorname{Lip}_\gamma[h]\epsilon^\gamma\qquad\forall\epsilon\in[0,\epsilon_2[,\tag{3.27}$$

where $h_\epsilon(x) \equiv h \circ \Lambda_\epsilon^{(-1)}(x)$ *for all* $x \in \partial\Lambda_\epsilon(\Omega)$, *for all bounded regions* Ω *of* $\mathbb{R}^N$ *of class* $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ *and for all* $h \in L^\infty(\partial\Omega)$ *such that the number*

$$\mathrm{Lip}_\gamma[h] \equiv \sup\left\{\frac{|h(x)-h(y)|}{|x-y|^\gamma}\colon\ x, y \in \partial\Omega, x \neq y\right\} \tag{3.28}$$

is finite.

Proof. We employ the same notation as in the proof of Lemma 3.5. In order to estimate $\nabla(\psi_j^\epsilon - \varphi_j)$, we first estimate the Lipschitz constant of $\psi_j^\epsilon - \varphi_j$. Let $p_1, p_2 \in (W_j)_{\delta/2}$. Let $p_1', p_2' \in W_j$ be the unique solutions of (3.19) corresponding to p_1 and p_2, respectively. Clearly,

$$\psi_j^\epsilon(p_i) - \varphi_j(p_i) = \varphi_j(p_i') + \pi_2 r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t(p_i', \varphi_j(p_i')) - \varphi_j(p_i), \qquad i = 1, 2.$$

Then, by the argument used to prove (3.23), we have

$$\begin{aligned}
&\big|\big(\psi_j^\epsilon(p_1) - \varphi_j(p_1)\big) - \big(\psi_j^\epsilon(p_2) - \varphi_j(p_2)\big)\big| \\
&\qquad \le \big|(\varphi_j(p_1') - \varphi_j(p_2')) - (\varphi_j(p_1) - \varphi_j(p_2))\big| + \epsilon AN(M+1)|p_1' - p_2'| \\
&\qquad = \left|\int_0^1 \nabla\varphi_j(p_2' + t(p_1' - p_2')) \cdot (p_1' - p_2')\,dt - \int_0^1 \nabla\varphi_j(p_2 + t(p_1 - p_2)) \cdot (p_1 - p_2)\,dt\right| \\
&\qquad\quad + \epsilon AN(M+1)|p_1' - p_2'| \\
&\qquad \le \left|\int_0^1 \nabla\varphi_j(p_2' + t(p_1' - p_2')) \cdot \big[(p_1' - p_2') - (p_1 - p_2)\big]\,dt\right| \\
&\qquad\quad + \left|\int_0^1 \big[\nabla\varphi_j(p_2' + t(p_1' - p_2')) - \nabla\varphi_j(p_2 + t(p_1 - p_2))\big]\,dt\right| |p_1 - p_2| \\
&\qquad\quad + \epsilon AN(M+1)|p_1' - p_2'|.
\end{aligned}$$

Hence, equality (3.21) and inequality (3.22) imply that

$$\begin{aligned}
&\big|\big(\psi_j^\epsilon(p_1) - \varphi_j(p_1)\big) - \big(\psi_j^\epsilon(p_2) - \varphi_j(p_2)\big)\big| \\
&\quad \le \|\,|\nabla\varphi_j|\,\|_{L^\infty(W_j)} \Big|\pi_1\Big[r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t\big(p_1', \varphi_j(p_1')\big) - r_j(\Lambda_\epsilon - \mathrm{id}) r_j^t\big(p_2', \varphi_j(p_2')\big)\Big]\Big| \\
&\quad\quad + M\int_0^1 \big|(p_2' - p_2) + t[(p_1' - p_2') - (p_1 - p_2)]\big|^\gamma\,dt|p_1 - p_2| + \epsilon AN(M+1)|p_1' - p_2'| \\
&\quad \le M\epsilon AN(M+1)|p_1' - p_2'| + M\big(2|p_2' - p_2| + |p_1' - p_1|\big)^\gamma |p_1 - p_2| \\
&\quad\quad + \epsilon AN(M+1)|p_1' - p_2'|.
\end{aligned} \tag{3.29}$$

Since $\epsilon AN(M+1) < 1$, inequality (3.24) implies that

$$|p_1' - p_2'| \le \frac{1}{1 - \epsilon AN(M+1)}|p_1 - p_2|.$$

Furthermore, (3.18) and (3.21) imply that

$$|p_i' - p_i| \leq \bigl|r_j(\mathrm{id} - \Lambda_\epsilon)r_j^t\bigl(p_i', \varphi_j(p_i')\bigr)\bigr| \leq A\epsilon, \qquad i = 1, 2.$$

Hence, the right-hand side of (3.29) is less than or equal to

$$\epsilon^\gamma \left\{ \frac{\epsilon_1^{1-\gamma} AN(M+1)^2}{1 - \epsilon AN(M+1)} + M(3A)^\gamma \right\} |p_1 - p_2|.$$

Since $\epsilon_1 \leq AN(M+1)^{-1}$, we have

$$\bigl|\nabla\psi_j^\epsilon(\overline{x}) - \nabla\varphi_j(\overline{x})\bigr| \leq \epsilon^\gamma A_1' \qquad \forall \overline{x} \in (W_j)_{\delta/2}$$

for $A_1' \equiv \epsilon_1^{1-\gamma} \cdot 2AN(M+1)^2 + M(3A)^\gamma$ and for all $0 < \epsilon \leq \frac{\epsilon_1}{2}$. Hence,

$$G'(\partial\Lambda_\epsilon(\Omega), \partial\Omega) \leq \Bigl(\sup_{j=1,\dots,s'} |W_j|\Bigr) \sum_{j=1}^{s'} \sup_{\overline{x} \in (W_j)_{\delta/2}} \bigl|\nabla\psi_j^\epsilon(\overline{x}) - \nabla\varphi_j(\overline{x})\bigr| \leq \Bigl(\sup_{j=1,\dots,s'} |W_j|\Bigr) s' A_1' \epsilon^\gamma$$

for all $0 < \epsilon \leq \epsilon_1/2$. We now prove statement (ii). Let $0 < \epsilon < \epsilon_2 \equiv \epsilon_1/2$. Then

$$L'(h_\epsilon, h) \leq \sum_{j=1}^{s'} \int\limits_{(W_j)_{\delta/2}} \bigl|h \circ \Lambda_\epsilon^{(-1)} r_j^t\bigl(\overline{x}, \psi_j^\epsilon(\overline{x})\bigr) - h \circ r_j^t\bigl(\overline{x}, \varphi_j(\overline{x})\bigr)\bigr|\, d\overline{x}.$$

Now, for each $\overline{x} \in (W_j)_{\delta/2}$, there exists $\overline{x}' \in W_j$ such that

$$r_j^t\bigl(\overline{x}, \psi_j^\epsilon(\overline{x})\bigr) = \Lambda_\epsilon r_j^t\bigl(\overline{x}', \varphi_j(\overline{x}')\bigr).$$

Moreover, inequality (3.18) and equality (3.21) imply that

$$\bigl|(\overline{x}', \varphi_j(\overline{x}')) - (\overline{x}, \varphi_j(\overline{x}))\bigr| \leq (M+1)|\overline{x}' - \overline{x}| \leq A(M+1)\epsilon.$$

Then we have

$$\begin{aligned} L'(h_\epsilon, h) &\leq A^\gamma (M+1)^\gamma \epsilon^\gamma \sum_{j=1}^{s'} \int\limits_{(W_j)_{\delta/2}} \sup_{\overline{y} \in (W_j)_{\delta/2} \setminus \{\overline{x}\}} \frac{\bigl|h \circ r_j^t(\overline{y}, \varphi_j(\overline{y})) - h \circ r_j^t(\overline{x}, \varphi_j(\overline{x}))\bigr|}{|(\overline{y}, \varphi_j(\overline{y})) - (\overline{x}, \varphi_j(\overline{x}))|^\gamma}\, d\overline{x} \\ &\leq s' A^\gamma (M+1)^\gamma \epsilon^\gamma \int\limits_{\partial\Omega} \sup_{y \in \partial\Omega \setminus \{x\}} \frac{|h(y) - h(x)|}{|y - x|^\gamma}\, d\sigma_x \\ &\leq (s')^2 (1+M)^{\gamma+1} \Bigl(\sup_{j=1,\dots,s'} |W_j|\Bigr) \mathrm{Lip}_\gamma[h] A^\gamma \epsilon^\gamma. \quad \square \end{aligned} \tag{3.30}$$

We now prove an inequality from below for the eigenvalues of the Robin Laplacian of the image $\Lambda_\epsilon(\Omega)$ of a set Ω of class $C^{0,1}$.

Theorem 3.7. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Let ϵ_0, A, and $\{\Lambda_\epsilon\}_{\epsilon\in[0,\epsilon_0[}$ be as in Lemma* 3.4, *and ϵ_1 as in Lemma* 3.5. *Then there exists a constant A_0 depending only on N and A such that*

$$\lambda_n\bigl[\Lambda_\epsilon(\Omega_1), h_{1\epsilon}\bigr] \geq (1 - A_0\epsilon)\lambda_n[\Omega_1, h_1],$$

where $h_{1\epsilon}(x) \equiv h_1 \circ \Lambda_\epsilon^{(-1)}(x)$ *for all* $x \in \partial\Lambda_\epsilon(\Omega_1)$, *for all bounded regions* Ω_1 *of* $\mathbb{R}^N$ *of class* $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$, *for all nonnegative* $h_1 \in L^\infty(\partial\Omega_1)$, *for all* $n \in \mathbb{N}_0$, *and for all* $\epsilon \in [0, \epsilon_1[$.

Proof. Let $\epsilon \in [0, \epsilon_1[$. By Lemma 3.5, $\Lambda_\epsilon(\Omega_1)$ is of class $C^{0,1}$. Since Λ_ϵ is a diffeomorphism of $\overline{\Omega}_1$ onto $\overline{\Lambda_\epsilon(\Omega_1)}$, the set

$$\widetilde{E} \equiv \big\{f \circ \Lambda_\epsilon^{(-1)} :\ f \in E\big\}$$

is an $(n+1)$-dimensional subspace of $W^{1,2}(\Lambda_\epsilon(\Omega_1))$ for all $(n+1)$-dimensional subspaces E of $W^{1,2}(\Omega_1)$. Moreover, the correspondence that takes $E \in \mathcal{E}_n(\Omega_1)$ to $\widetilde{E}$ in $\mathcal{E}_n(\Lambda_\epsilon(\Omega_1))$ (see Theorem 2.10(ii)) is a global bijection. To shorten our notation, we set $\Omega_{1\epsilon} \equiv \Lambda_\epsilon(\Omega_1)$. Then we have the following:

$$\mu_n[\Omega_{1\epsilon}, h_{1\epsilon}] = \inf_{\widetilde{E}\in\mathcal{E}_n(\Omega_{1\epsilon})} \sup_{g\in\widetilde{E}} \frac{\int_{\Omega_{1\epsilon}} |\nabla g|^2\,dy + \int_{\partial\Omega_{1\epsilon}} h_{1\epsilon}|\mathrm{tr}\, g|^2\,d\sigma}{\int_{\Omega_{1\epsilon}} |g|^2\,dx}$$

$$= \inf_{E\in\mathcal{E}_n(\Omega)} \sup_{f\in E} \frac{\int_{\Omega_{1\epsilon}} \big|\nabla\big(f\circ\Lambda_\epsilon^{(-1)}\big)\big|^2\,dy + \int_{\partial\Omega_{1\epsilon}} h_{1\epsilon}\big|\mathrm{tr}\big(f\circ\Lambda_\epsilon^{(-1)}\big)\big|^2\,d\sigma}{\int_{\Omega_{1\epsilon}}\big|f\circ\Lambda_\epsilon^{(-1)}\big|^2\,dx}$$

$$= \inf_{E\in\mathcal{E}_n(\Omega_1)} \sup_{f\in E} \left\{ \int_{\Omega_1} \big|\nabla f(\nabla\Lambda_\epsilon)^{-1}\big|^2\,|\det(\nabla\Lambda_\epsilon)|\,dx \right.$$

$$\left. + \int_{\partial\Omega_1} h_1|\mathrm{tr}\, f|^2\,|\det(\nabla\Lambda_\epsilon)|\cdot\big|(\nabla\Lambda_\epsilon)^{-t}\nu_{\Omega_1}\big|\,d\sigma \right\} \left\{ \int_{\Omega_1} |f|^2\,|\det(\nabla\Lambda_\epsilon)|\,dx \right\}^{-1}$$

for all $n \in \mathbb{N}_0$, where ν_{Ω_1} denotes the almost everywhere defined exterior normal to $\partial\Omega_1$. Next we note that

$$|\nabla f| \le \big|\nabla f(\nabla\Lambda_\epsilon)^{-1}\big| \sup_{a\in\mathbb{R}^N\setminus\{0\}} \frac{|a|}{|a(\nabla\Lambda_\epsilon)^{-1}|} = \big|\nabla f(\nabla\Lambda_\epsilon)^{-1}\big| \sup_{b\in\mathbb{R}^N\setminus\{0\}} \frac{|b\nabla\Lambda_\epsilon|}{|b|}$$

$$= \big|\nabla f(\nabla\Lambda_\epsilon)^{-1}\big| \cdot \|\nabla\Lambda_\epsilon\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)},$$

where $\|\cdot\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)}$ denotes the operator norm in the space $\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)$ of linear and continuous operators in $\mathbb{R}^N$; that

$$\big|(\nabla\Lambda_\epsilon)^{-t}\nu_{\Omega_1}\big| \ge \inf_{a\in\mathbb{R}^N\setminus\{0\}} \frac{|(\nabla\Lambda_\epsilon)^{-t}a^t|}{|a|} \ge \|\nabla\Lambda_\epsilon\|^{-1}_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)};$$

and that

$$\|\nabla\Lambda_\epsilon\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)} \le \|I\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)} + \|I-\nabla\Lambda_\epsilon\|_{\mathcal{L}(\mathbb{R}^N,\mathbb{R}^N)} \le 1 + \epsilon AN$$

(see Lemma 3.4(ii)). Thus we conclude that

$$\mu_n[\Omega_{1\epsilon}, h_{1\epsilon}] \ge \frac{1-\epsilon A}{(1+\epsilon AN)^2(1+\epsilon A)} \inf_{E\in\mathcal{E}_n(\Omega_1)} \sup_{f\in E} \frac{\int_{\Omega_1}|\nabla f|^2\,dx + \int_{\partial\Omega_1}(\mathrm{tr}\, f)^2 h_1\,d\sigma}{\int_{\Omega_1}|f|^2\,dx}$$

for all $n \in \mathbb{N}_0$ (see Lemma 3.4(ii)). Clearly, there exists $A_0 > 0$ such that

$$\frac{1-\epsilon A}{(1+\epsilon AN)^2(1+\epsilon A)} \ge 1 - \epsilon A_0 \qquad \forall \epsilon \in [0, +\infty[.$$

Thus we have $\mu_n[\Omega_{1\epsilon}, h_{1\epsilon}] \ge (1 - A_0\epsilon)\mu_n[\Omega_1, h_1]$ for all $n \in \mathbb{N}_0$ and for all $0 \le \epsilon < \epsilon_1$. □

We now turn to our final statement.

Theorem 3.8. *Let M, δ, s, s', $\{V_j\}_{j=1}^s$, and $\{r_j\}_{j=1}^s$ be as in Definition* 2.1. *Let $\alpha > 0$ and $0 < \gamma \leq 1$. Then, for each $n \in \mathbb{N}_0$, there exists $\xi_n > 0$ such that*

$$\lambda_n[\Omega_1, h_1] - \xi_n \epsilon^\gamma \leq \lambda_n[\Omega_2, h_2] \leq \lambda_n[\Omega_1, h_1] + \xi_n \epsilon \tag{3.31}$$

for all $0 < \epsilon < \xi_n^{-1}$, for all bounded regions Ω_1 of $\mathbb{R}^N$ of class $C^{1,\gamma}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$, for all nonnegative $h_1 \in L^\infty(\partial\Omega_1)$ such that

$$\|h_1\|_{L^\infty(\partial\Omega_1)} \leq \alpha \qquad \text{and} \qquad \mathrm{Lip}_\gamma[h_1] \leq \alpha$$

(*see* (3.28)), *for all bounded regions Ω_2 of $\mathbb{R}^N$ of class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$ satisfying*

$$\Omega_1 \setminus \partial_\epsilon \Omega_1 \subset \Omega_2 \subset \Omega_1, \qquad G(\partial\Omega_1, \partial\Omega_2) \leq \epsilon,$$

and for all nonnegative $h_2 \in L^\infty(\partial\Omega_2)$ satisfying

$$\|h_2\|_{L^\infty(\partial\Omega_2)} \leq \alpha \qquad \text{and} \qquad L(h_1, h_2) \leq \epsilon.$$

Proof. Let $\{\Lambda_\epsilon\}_{\epsilon \in [0, \epsilon_1[}$ be a family of diffeomorphisms as in Lemmas 3.4 and 3.5. Let A and A_1 be the corresponding constants. Let $n \in \mathbb{N}_0$. Let b_n be as in Theorem 3.2 for the class $C^{0,1}\big(M, \delta, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$. Then, by Theorem 3.2, by the assumption $|\Omega_1 \setminus \Omega_2| \leq |\partial_\epsilon \Omega_1|$, and by the inequality $|\partial_\epsilon \Omega_1| \leq A\epsilon$ (see Lemma 3.4(v)), we have

$$\begin{aligned} \lambda_n[\Omega_2, h_2] &\leq \lambda_n[\Omega_1, h_1] + b_n\Big[|\Omega_1 \setminus \Omega_2| + a(h_1, h_2)\big(|\Omega_1 \setminus \Omega_2| + G(\partial\Omega_1, \partial\Omega_2)\big) + L(h_1, h_2)\Big] \\ &\leq \lambda_n[\Omega_1, h_1] + b_n\big[A + (\alpha + \alpha^2)(A+1) + 1\big]\epsilon \leq \lambda_n[\Omega_1, h_1] + \xi_n' \epsilon, \end{aligned} \tag{3.32}$$

where

$$\xi_n' \equiv b_n (A+1)(\alpha^2 + \alpha + 1),$$

provided that $|\Omega_1 \setminus \Omega_2| \leq b_n^{-1}$ and thus for $0 \leq \epsilon < \min\{\epsilon_1, (Ab_n)^{-1}\}$. By Theorem 3.7, there exists A_0 depending only on N and A such that

$$\lambda_n[\Lambda_\epsilon(\Omega_1), h_{1\epsilon}] \geq (1 - A_0\epsilon)\lambda_n[\Omega_1, h_1] \tag{3.33}$$

for all $0 \leq \epsilon < \epsilon_1$. Obviously, both sets Ω_1 and Ω_2 belong to the class $C^{0,1}\big(M+1, \delta/4, s, s', \{(V_j)_{\delta/2}\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$, no matter how we choose $\epsilon \in [0, \epsilon_1[$. Since $\Omega_1 \setminus \partial_\epsilon \Omega_1 \subset \Omega_2 \subset \Omega_1$, we have $\Lambda_\epsilon(\Omega_1) \subset \Omega_2$. Let b_n' be as in Theorem 3.2 for the class $C^{0,1}\big(M+1, \delta/4, s, s', \{(V_j)_{\delta/2}\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$. Then by Theorem 3.2 we have

$$\begin{aligned} \lambda_n[\Lambda_\epsilon(\Omega_1), h_{1\epsilon}] &\leq \lambda_n[\Omega_2, h_2] + b_n'\Big[|\Omega_2 \setminus \Lambda_\epsilon(\Omega_1)| \\ &\quad + a(h_{1\epsilon}, h_2)\big(|\Omega_2 \setminus \Lambda_\epsilon(\Omega_1)| + G'(\partial\Omega_2, \partial\Lambda_\epsilon(\Omega_1))\big) + L'(h_{1\epsilon}, h_2)\Big] \end{aligned} \tag{3.34}$$

for all $0 \leq \epsilon < \epsilon_1$ such that $|\Omega_2 \setminus \Lambda_\epsilon(\Omega_1)| < (b_n')^{-1}$, where L' and G' are the characteristics L and G of vicinity from Definition 3.1 corresponding to the class $C^{0,1}\big(M+1, \delta/4, s, s', \{(V_j)_{\delta/2}\}_{j=1}^s, \{r_j\}_{j=1}^s\big)$. By Lemma 3.4(v), we have

$$|\Omega_2 \setminus \Lambda_\epsilon(\Omega_1)| \leq |\Omega_1 \setminus \Lambda_\epsilon(\Omega_1)| \leq A\epsilon.$$

Thus (3.34) holds for all $0 \le \epsilon < \min\{\epsilon_1, (Ab'_n)^{-1}\}$. By Lemma 3.6(i), (ii), we have

$$G'(\partial\Omega_2, \partial\Lambda_\epsilon(\Omega_1)) \le G'(\partial\Omega_2, \partial\Omega_1) + G'(\partial\Omega_1, \partial\Lambda_\epsilon(\Omega_1)) \le G(\partial\Omega_2, \partial\Omega_1) + A_1\epsilon^\gamma \le \big(\epsilon_2^{1-\gamma} + A_1\big)\epsilon^\gamma$$

and

$$L'(h_{1\epsilon}, h_2) \le L'(h_{1\epsilon}, h_1) + L'(h_1, h_2) \le A_1 \operatorname{Lip}_\gamma[h_1]\epsilon^\gamma + L(h_1, h_2) \le \big(A_1 \operatorname{Lip}_\gamma[h_1] + \epsilon_2^{1-\gamma}\big)\epsilon^\gamma$$

for all $0 \le \epsilon < \epsilon_2$. Thus we have

$$\lambda_n[\Lambda_\epsilon(\Omega_1), h_{1\epsilon}] \le \lambda_n[\Omega_2, h_2] + b'_n A_2 \epsilon^\gamma,$$

where

$$A_2 \equiv A\epsilon_2^{1-\gamma} + (\alpha^2 + \alpha)\big(A\epsilon_2^{1-\gamma} + (\epsilon_2^{1-\gamma} + A_1)\big) + \big(A_1\alpha + \epsilon_2^{1-\gamma}\big)$$

for all $0 \le \epsilon < \min\{(Ab'_n)^{-1}, \epsilon_2\}$. By inequality (3.33), we obtain

$$(1 - A_0\epsilon)\lambda_n[\Omega_1, h_1] \le \lambda_n[\Omega_2, h_2] + b'_n A_2 \epsilon^\gamma$$

for all $0 \le \epsilon < \min\{(Ab'_n)^{-1}, \epsilon_2\}$. Hence,

$$\lambda_n[\Omega_1, h_1] - \big(A_0\epsilon_2^{1-\gamma}\lambda_n[\Omega_1, h_1] + b'_n A_2\big)\epsilon^\gamma \le \lambda_n[\Omega_2, h_2], \tag{3.35}$$

and thus inequality (3.31) follows with ξ_n replaced by

$$\xi''_n \equiv \max\big\{\xi'_n,\ A_0\epsilon_2^{1-\gamma}\lambda_n[\Omega_1, h_1] + b'_n A_2\big\}$$

and for all $0 \le \epsilon < \min\{(Ab_n)^{-1}, (Ab'_n)^{-1}, \epsilon_2\}$. We note that by Corollary 2.12 we have

$$\xi''_n \le \xi_n \equiv \max\big\{\xi'_n,\ A_0\epsilon_2^{1-\gamma}\lambda_n^{\mathrm{D}}\big[]0, \delta[^N\big] + b'_n A_2,\ Ab_n,\ Ab'_n,\ \epsilon_2^{-1}\big\}.$$

Hence inequality (3.31) follows for $0 \le \epsilon < \xi_n^{-1}$. □

Remark 3.9. By inspecting the proof of Lemma 3.6 (see (3.30)) and the proof of Theorem 3.8, one can easily see that the condition $\operatorname{Lip}_\gamma[h_1] \le \alpha$ in the statement of Theorem 3.8 can be replaced by the weaker assumption

$$\int\limits_{\partial\Omega} \sup_{y \in \partial\Omega \setminus \{x\}} \frac{|h(y) - h(x)|}{|y - x|^\gamma}\, d\sigma_x \le \alpha.$$

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