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CONDITIONS FOR EXISTENCE OF A GLOBAL STRONG SOLUTION TO ONE CLASS OF NONLINEAR EVOLUTION EQUATIONS IN HILBERT SPACE

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Abstract: We obtain a criterion of global strong solvability for one class of nonlinear evolution equations in Hilbert space.

Keywords: Navier–Stokes equation, global strong solution

1. Introduction. Let H be a separable real Hilbert space. We denote the norm on H by $|\cdot|$ and the inner product, by $\langle\cdot,\cdot\rangle$. Let $C^\infty(H;0,a)$ be the set of infinitely smooth H-valued functions on $[0,a]$ $(a>0)$. We denote the completion of $C^\infty(H;0,a)$ in the metric of the inner product

$$\langle x,y\rangle_{H_2}=\int_0^a\langle x(t),y(t)\rangle\,dt,\quad x(t),y(t)\in C^\infty(H;0,a),$$

by $H_2=H_2[0,a]$.

Suppose that A is a nonnegative selfadjoint operator with a compact inverse and $D(A)$ is the domain of A. Consider the following Cauchy problem in the space of H-valued functions:

$$u_t'+Au+B(u,u)=f(t),\quad u(0)=0,\ 0<t<a, \tag{1}$$

where $B(u,g)$ is a bilinear operator and $f(t)$ is an H-valued function.

The Navier–Stokes equations can be written in the form (1) and the well-known problem of existence of a global strong solution reduces to a similar problem for the abstract equation (1).

In this article we consider the question of global strong solvability of (1), on assuming that A and $B(\cdot,\cdot)$ are connected by some conditions that $B(\cdot,\cdot)$ is subordinate to A. Namely, we require that

$$|(A+E)^{-\gamma}B(u,g)|\le c[|(A+E)^{\gamma_0}u||(A+E)^{\gamma_0+1/2}g|+|(A+E)^{\gamma_0}g||(A+E)^{\gamma_0+1/2}u|] \tag{2}$$

for all γ and γ_0 satisfying the relations

$$\gamma_0=\delta_0-\gamma/2,\quad -\infty<\gamma<3/4,\quad 0<\delta_0<1/2, \tag{3}$$

where δ_0 is a constant and c depends on γ. Conditions (2) and (3) mean that $B(\cdot,\cdot)$ is subordinate to A. In the case of the three-dimensional Navier–Stokes equations these conditions are satisfied for $\delta_0=3/8$. We choose (2) and (3) according to the inequalities valid for the nonlinear term in the Navier–Stokes system. In fact, we can replace them by many other conditions, in particular, requiring that (3) hold only for some γ rather than for all $\gamma\in(-\infty,3/4)$.

Definition 1. We say that *problem* (1) is *globally strongly solvable* if for every $a>0$ the condition $f(t)\in H_2[0,a]$ implies existence of a solution $u(t)$ to (1) in $(0,a)$ such that $u'+Au\in H_2[0,a]$.

The main result of the article is

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Theorem A. *Suppose that $A \geq E$, A^{-1} is compact, and conditions* (2) *and* (3) *are satisfied. Then* (1) *is globally strongly solvable if and only if the only solution $(w(t), s(t))$ to the equations*

$$\begin{aligned} -s'(t) + As(t) + B_w^* s(t) = 0, \quad s(a) = 0, \quad 0 < t < a, \\ w' + Aw(t) + B(w, w) = s(t), \quad w(0) = 0, \quad 0 < t < a, \end{aligned} \tag{4}$$

is the zero solution; i.e., $s(t) \equiv 0$ and $w(t) \equiv 0$.

Here B_w^* is the adjoint of B_w and B_w is defined by the formula $B_w g = B(w, g) + B(g, w)$. This theorem will be proven in Section 3. Below, we explain the meaning of the word "solution" used in this theorem.

Definition 2. A pair $\{w_1, w_2\}$ of infinitely smooth H-valued functions with range in $D(A)$ such that

(a) $w_1 = 0$ in a neighborhood of $t = 0$ and $w_2 = 0$ in a neighborhood of $t = a$;

(b) $Aw_j \in H_2[0, a]$ $(j = 1, 2)$

is called a *test function*. Denote the set of test functions by D_a.

Definition 3. A pair of vector-functions $(w(t), s(t))$ is called a *solution* to (4) if

(c) $w(t), s(t) \in H_2[0, a]$, $w' + Aw \in H_2[0, a - \delta]$ for every $\delta \in (0, a)$;

(d) if $(w_1, w_2) \in D_a$ and $B_w w_1 \in H_2[0, a]$ then

$$\langle s, w_1' + Aw_1 + B_w w_1 \rangle_{H_2[0,a]} = 0;$$

$$\langle -w_2' + Aw_2 + (1/2) B_w^* w_2, w \rangle_{H_2[0,a]} - \langle w_2, s \rangle_{H_2[0,a]} = 0.$$

Thus, in Theorem A we speak of a weak solution.

Observe that the equation in $s(t)$ is the backward parabolic equation and the Cauchy condition is given at the right endpoint. Moreover, it is linear homogeneous. Therefore, if $w(t)$ in a neighborhood of $t = a$ such that $B_w^* s(t)$ is subordinate to $-s'(t) + As(t)$ then the problem has only the zero solution. But since the vector-function $w(t)$ can have singularities at $t = a$ the problem for $s(t)$ may have a nonzero solution. Therefore, Theorem A does not solve the problem of existence of a globally strong solution but reduces it to another problem.

We can exclude $s(t)$ from (4). Then from (4) we obtain the following two-point problem for $w(t)$:

$$\begin{gathered} \left(-\frac{d^2}{dt^2} + A^2\right) w + \left(-\frac{d}{dt} + A\right) B(w, w) + B_w^* \left(\frac{d}{dt} w + Aw + B(w, w)\right) = 0, \\ (w' + Aw(t) + B(w, w))|_{t=a} = 0, \quad w(0) = 0. \end{gathered} \tag{4'}$$

The linear part of this problem (which "looks like" the principal part) is "elliptic"; however we failed to prove that the only solution is the zero solution. For the Navier–Stokes equations we have the weak (energy) estimate. We do not use this estimate and our main results are valid without weak a priori estimates. Therefore, our principal result remains valuable if even somebody manages to solve the problem of existence of a strong solution to the Navier–Stokes equations. Moreover, Theorem A admits generalization to a rather wide class of parabolic equations. Observe that there are examples in which (4) has a nontrivial weak solution; for instance, the following system of two (scalar) equations:

$$\begin{gathered} -s'(t) + s(t) + 2w(t)s(t) = 0, \quad w' + w(t) + w^2(t) - s(t) = 0, \\ w(0) = 0, \quad s(1) = 0, \quad 0 < t < 1. \end{gathered} \tag{4''}$$

A brief exposition of the results of this article is given in [1] (where condition (d) of Definition 3 should be replaced with (d) of Definition 3 of the present article). The results of this paper can be applied to the Navier–Stokes equations and the equation of magnetic gas dynamics. Some applications of this kind will be presented in one of the forthcoming articles.

2. Estimates for solutions to one nonlinear parabolic equation in Hilbert space. Put

$$T(\lambda)g = \int_0^t e^{-(A+\lambda E)(t-\xi)} g(\xi)\, d\xi,$$

where E is the identity operator and $\exp(-A-\lambda E)(t-\xi)$ is understood in the sense of the spectral decomposition.

Alongside $T(\lambda)$ we introduce the operator $T(\lambda, \alpha)$ by the formula

$$T(\lambda,\alpha)v = \frac{1}{\Gamma(\alpha)} \int_0^t e^{-(A+\lambda)(t-\xi)} (t-\xi)^{\alpha-1} v(\xi)\, d\xi, \tag{5}$$

where $\Gamma(\alpha)$ is the gamma function, $\alpha > 0$.

Lemma 1. *If $\alpha, \beta > 0$ then $T(\lambda,\alpha)T(\lambda,\beta) = T(\lambda, \alpha+\beta)$ and $T(\lambda, 1) = T(\lambda)$.*

Proof. We have

$$T(\lambda,\alpha)T(\lambda,\beta)v = (\Gamma(\alpha)\Gamma(\beta))^{-1} \int_0^t e^{-(A+\lambda)(t-\xi)} (t-\xi)^{\alpha-1}$$

$$\times \left(\int_0^\xi e^{-(A+\lambda)(\xi-\eta)} (\xi-\eta)^{\beta-1} v(\eta) d\eta \right) d\xi$$

$$= (\Gamma(\alpha)\Gamma(\beta))^{-1} \int_0^t e^{-(A+\lambda)(t-\eta)} v(\eta) \left(\int_\eta^t (t-\xi)^{\alpha-1} (\xi-\eta)^{\beta-1}\, d\xi \right) d\eta,$$

but

$$\int_\eta^t (t-\xi)^{\alpha-1}(\xi-\eta)^{\beta-1}\, d\xi = \int_0^{t-\eta} (t-\eta-\tau)^{\alpha-1} \tau^{\beta-1}\, d\tau$$

$$= \int_0^{t-\eta} (t-\eta)^{\alpha-1} \left(1 - \frac{\tau}{t-\eta}\right)^{\alpha-1} \left(\frac{\tau}{t-\eta}\right)^{\beta-1} (t-\eta)^\beta\, d\left(\frac{\tau}{t-\eta}\right)$$

$$= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} (t-\eta)^{\alpha+\beta-1};$$

therefore,

$$T(\lambda,\alpha)T(\lambda,\beta)v = T(\lambda,\alpha+\beta)v.$$

Since $\Gamma(1) = 1$, we obtain $T(\lambda, 1) = T(\lambda)$. The lemma is proven.

Observe that, by Lemma 1, instead of $T(\lambda,\alpha)$ we can write

$$T(\lambda,\alpha) = T^\alpha(\lambda)$$

and understand $T(\lambda,\alpha)$ to be the fractional power of the operator $T(\lambda)$. For $\alpha = 0$ we can take $T^\alpha(\lambda)$ to be E, since the strong limit of $T^\alpha(\lambda)$ as $\alpha \to 0$ is exactly E.

From Lemma 1 we obtain the equality ($0 \le \alpha \le 1$)

$$T(\lambda) = (A+\lambda E)^b T^\alpha(\lambda) (A+\lambda E)^{-b} T^{1-\alpha}(\lambda).$$

Now, denote

$$T(\lambda,\alpha,b) = (A+\lambda E)^b T^\alpha(\lambda). \tag{6}$$

Lemma 2. *The following estimate is valid for* $0 \le \alpha \le 1$, $\gamma - \theta + b \ge 0$, $\alpha + \theta - \gamma - b > 1/2$, *and* $\lambda \ge 1$:

$$\int_0^a |(A+\lambda)^{\gamma+1/2} T(\lambda,\alpha,b)v|^2\,dt < C\int_0^a |(A+\lambda)^\theta v|^2\,dt,$$

where $T(\lambda,\alpha,b)$ *is taken from* (6).

Proof. We have

$$\begin{aligned} M &\equiv \int_0^a |(A+\lambda)^{\gamma+1/2} T(\lambda,\alpha,b)v|^2\,dt \\ &= \frac{1}{\Gamma^2(\alpha)}\int_0^a \left|\int_0^t (A+\lambda)^{\gamma+b+1/2}(t-\xi)^{\alpha-1}e^{-(A+\lambda)(t-\xi)}v(\xi)d\xi\right|^2 dt \\ &= C\int_0^a\left|\int_0^t (A+\lambda)^{\gamma+b-\theta+1/2}(t-\xi)^{\gamma+b-\theta+1/2}e^{-(A+\lambda)(t-\xi)}\right. \\ &\qquad \left.\times(t-\xi)^{\alpha+\theta-\gamma-b-1/2-1}(A+\lambda)^\theta v(\xi)\,d\xi\right|^2 dt. \end{aligned}$$

Here and in the sequel C denotes (generally different) constants whose exact values are unimportant. Now, for $\gamma+b-\theta+1/2 \ge 0$ we obtain

$$\|(A+\lambda)^{\gamma+b-\theta+1/2}(t-\xi)^{\gamma+b-\theta+1/2}e^{-(A+\lambda)(t-\xi)}\| \le \sup_{x\ge 0} x^{\gamma+b-\theta+1/2}e^{-x} < \infty,$$

where $\|\cdot\|$ is the operator norm. This inequality is a consequence of the spectral decomposition. Hence,

$$M \le C\int_0^a\left(\int_0^t (t-\xi)^{\alpha+\theta-\gamma-b-\frac{3}{2}}|(A+\lambda)^\theta v(\xi)|\,d\xi\right)^2 dt.$$

Extend the function $|(A+\lambda)^\theta v(\xi)|$ by zero beyond $[0,a]$. Then it follows from the last inequality that

$$M \le C\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} |t-\xi|^{\alpha+\theta-\gamma-b-\frac{3}{2}}e^{-\delta|t-\xi|}\chi_a(\xi)|(A+\lambda)^\theta v(\xi)|d\xi\right)^2 dt,$$

where $\chi_a(\xi)$ is the characteristic function of $[0,a]$ and C and $\delta > 0$ are constants.

Using the estimates for translation-invariant operators (see [2]), we obtain

$$\begin{aligned} M &\le C\left[\int_{-\infty}^{\infty}|\xi|^{\alpha+\theta-\gamma-b-\frac{3}{2}}e^{-\delta|\xi|}d\xi\right]^2\int_{-\infty}^{\infty}\chi_a(\xi)|(A+\lambda)^\theta v(\xi)|^2\,d\xi \\ &= C\left[\int_{-\infty}^{\infty}|\xi|^{\alpha+\theta-\gamma-b-\frac{3}{2}}e^{-\delta|\xi|}d\xi\right]^2\int_0^a |(A+\lambda)^\theta v(\xi)|^2\,d\xi. \end{aligned}$$

Since $\alpha+\theta-\gamma-b > 1/2$, the lemma is proven.

Lemma 3. *Let* $0 \le \alpha \le 1$, $\gamma + b - \theta \ge 0$, $\alpha - \gamma - b + \theta > 1/2$, *and* $\lambda \ge 1$. *Then*

$$\sup_{0<t<a} |(A+\lambda)^{\gamma} T(\lambda,\alpha,b)v(t)|^2 \le C \int_0^a |(A+\lambda)^{\theta} v(t)|^2\,dt.$$

Proof. We have

$$|(A+\lambda)^{\gamma} T(\lambda,\alpha,b)v|^2(t)$$

$$= \frac{1}{\Gamma^2(\alpha)} \left| \int_0^t (A+\lambda)^{\gamma+b-\theta} (t-\xi)^{\gamma+b-\theta} e^{-(A+\lambda)(t-\xi)} (t-\xi)^{\alpha-1-\gamma-b+\theta} (A+\lambda)^{\theta} v(\xi)\,d\xi \right|^2$$

$$\le C\left(\int_0^t |(t-\xi)^{\alpha-1-\gamma-b+\theta} (A+\lambda)^{\theta} v(\xi)|\,d\xi \right)^2 \le C \int_0^t |(A+\lambda)^{\theta} v(\xi)|^2\,d\xi.$$

Here we have used Cauchy's inequality. Hence, in view of the arbitrariness of $t \in (0,a)$, we obtain the claim of the lemma.

Introduce the norm

$$|v|^2_{\gamma,\lambda} = \sup_{0<t<a} |(A+\lambda)^{\gamma} v(t)|^2 + \int_0^a |(A+\lambda)^{\gamma+1/2} v(t)|^2\,dt.$$

From Lemmas 2 and 3 we derive

Corollary 1. *If* $0 \le \alpha \le 1$, $\gamma + b - \theta \ge 0$, $\alpha - \gamma - b + \theta > 1/2$, *and* $\lambda \ge 1$ *then*

$$|T(\lambda,\alpha,b)v|^2_{\gamma,\lambda} \le C \int_0^a |(A+\lambda)^{\theta} v|^2\,dt.$$

Proof. Apply Lemma 3 to the first term in the definition of $|\cdot|_{\gamma,\lambda}$ and Lemma 2, to the second.

Lemma 4. *Let* $0 \le \alpha \le 1$, $\gamma + b - \theta \ge 0$, $\alpha - \gamma - b + \theta > 1/2$, *and* $\lambda \ge 1$. *If* $\varepsilon > 0$ *then*

$$|T(\lambda,\alpha,b)v|^2_{\gamma-\varepsilon,\lambda} \le C\lambda^{-2\varepsilon} \int_0^a |(A+\lambda)^{\theta} v(t)|^2\,dt.$$

Proof. Using Corollary 1, we obtain

$$|T(\lambda,\alpha,b)v|^2_{\gamma-\varepsilon,\lambda} \le \left(\frac{1}{\lambda}\right)^{2\varepsilon} |T(\lambda,\alpha,b)v|^2_{\gamma,\lambda} \le C\lambda^{-2\varepsilon} \int_0^a |(A+\lambda)^{\theta} v(t)|^2\,dt.$$

The lemma is proven.

Lemma 5. $|T(\lambda)v|^2_{l,\lambda} \le 2\int_0^a |(A+\lambda)^{l-\frac{1}{2}} v|^2\,dt$, *where* $l \in \mathbb{R}$ *and* $\lambda \ge 0$.

Proof. We have $(Tv)' + (A+\lambda)Tv = v$ and $Tv|_{t=0} = 0$, where $T = T(\lambda)$. Multiply this equation scalarly by $(A+\lambda)^{2l} Tv$ and integrate from 0 to t. Then we obtain

$$\frac{1}{2}|(A+\lambda)^{l} Tv|^2(t) + \int_0^t |(A+\lambda)^{l+1/2} Tv|^2\,dt = \int_0^t \langle (A+\lambda)^{l-1/2} v, (A+\lambda)^{l+1/2} Tv \rangle\,dt$$

$$\le 1/2 \int_0^t |(A+\lambda)^{l-1/2} v|^2\,dt + 1/2 \int_0^t |(A+\lambda)^{l+1/2} Tv|^2\,dt.$$

Hence, we easily derive the claim of the lemma.

Lemma 6. *Let $b \geq 0$, $\alpha - b > -1$, and $\lambda \geq 1$. Then $T(\lambda, \alpha+1, b)$ is a compact operator from $H_2[0,a]$ to $H_2[0,a]$.*

PROOF. Let $v(t) = \sum_{n=1}^{\infty} c_n \varphi_n$, where $\{\varphi_n\}$ is a complete orthonormal system of eigenvectors of the selfadjoint operator $A \geq 0$ with eigenvalues $\{\lambda_n\}$. The eigenvalues are enumerated in nondecreasing order; moreover, since the resolvent of A is compact, $\lambda_n \to +\infty$ as $n \to \infty$.

For $\lambda \geq 1$

$$T(\lambda, \alpha+1, b)v = \frac{1}{\Gamma(\alpha+1)} \sum_{n=1}^{\infty} \varphi_n (\lambda_n + \lambda)^b \int_0^t (t-\xi)^{\alpha} e^{-(\lambda_n+\lambda)(t-\xi)} \langle v(\xi), \varphi_n \rangle \, d\xi$$

$$= \frac{1}{\Gamma(\alpha+1)} \sum_{n=1}^{N} \varphi_n (\lambda_n + \lambda)^b \int_0^t (t-\xi)^{\alpha} e^{-(\lambda_n+\lambda)(t-\xi)} \langle v(\xi), \varphi_n \rangle \, d\xi$$

$$+ \frac{1}{\Gamma(\alpha+1)} \sum_{n=N+1}^{\infty} \varphi_n (\lambda_n + \lambda)^b \int_0^t (t-\xi)^{\alpha} e^{-(\lambda_n+\lambda)(t-\xi)} \langle v(\xi), \varphi_n \rangle \, d\xi = S_N v + M_N v;$$

i.e., $T(\lambda, \alpha, b) = S_N + M_N$.

The operator S_N is a finite sum of compact operators, since their kernels have a weak singularity. Therefore, it suffices to prove that

$$\|M_N\| \to 0 \quad \text{as } N \to \infty. \tag{7}$$

But

$$\int_0^a |M_N v|^2(t) \, dt = C \sum_{n=N+1}^{\infty} \int_0^a (\lambda_n + \lambda)^{2b} \left| \int_0^t (t-\xi)^{\alpha} e^{-(\lambda_n+\lambda)(t-\xi)} \langle v(\xi), \varphi_n \rangle \, d\xi \right|^2 dt$$

$$\leq C \int_0^a \frac{1}{(\lambda_N + \lambda)^{2\varepsilon}} \sum_{n=N+1}^{\infty} \left(\int_0^t (\lambda_n + \lambda)^{b+\varepsilon} (t-\xi)^{b+\varepsilon} e^{-(\lambda_n+\lambda)(t-\xi)} \langle v(\xi), \varphi_n \rangle (t-\xi)^{\alpha-b-\varepsilon} \, d\xi \right)^2 dt.$$

Take $\varepsilon > 0$ so that $\alpha - b - \varepsilon > -1$. This is possible, since $\alpha - b > -1$. Use the well-known inequality

$$\sup_{x>0} x^{b+\varepsilon} e^{-x} < \infty,$$

which is valid for $b + \varepsilon > 0$ to obtain

$$\int_0^a |M_N v|^2 dt \leq \frac{C}{(\lambda_N + \lambda)^{2\varepsilon}} \int_0^a \sum_{n=N+1}^{\infty} \left(\int_0^t |\langle v(\xi), \varphi_n \rangle| (t-\xi)^{\alpha-b-\varepsilon} d\xi \right)^2 dt.$$

We now use the method of the proof of Lemma 2. Then

$$\int_0^a |M_N v|^2 \, dt \leq \frac{C}{(\lambda_N + \lambda)^{2\varepsilon}} \sum_{n=N+1}^{\infty} \int_0^a |\langle v(\xi), \varphi_n \rangle|^2 \, d\xi \leq \frac{C}{(\lambda_N + \lambda)^{2\varepsilon}} \int_0^a |v(\xi)|^2 \, d\xi.$$

By arbitrariness of $v \in H_2[0,a]$, we arrive at (7), since $\lambda_N \to +\infty$ and $\varepsilon > 0$. The lemma is proven.

Lemma 7. *Let $0 \le \alpha \le 1$, $\gamma + b - \theta \ge 0$, $\alpha - \gamma - b + \theta > 1/2$, $\lambda \ge 1$, $0 < -\theta - \varepsilon < 3/4$, and $\varepsilon \ge 0$. Then*

$$|T(\lambda, \alpha, b)B(u, v)|^2_{\gamma,\lambda} \le C\lambda^{-2\varepsilon}|u|^2_{\frac{3}{8}+\frac{\theta+\varepsilon}{2},1}|v|^2_{\frac{3}{8}+\frac{\theta+\varepsilon}{2},1},$$

where C is independent of $\varepsilon \ge 0$ and $\lambda \ge 1$.

PROOF. By Corollary 1,

$$|T(\lambda, \alpha, b)B(u, v)|^2_{\gamma,\lambda} \le C\int_0^a |(A+\lambda)^\theta B(u, v)|^2\, dt \le C\lambda^{-2\varepsilon}\int_0^a |(A+\lambda)^{\theta+\varepsilon}B(u, v)|^2\, dt.$$

Now, use (2) and (3):

$$|T(\lambda, \alpha, b)B(u, v)|^2_{\gamma,\lambda} \le C\lambda^{-2\varepsilon}\int_0^a |(A+E)^{\frac{3}{8}+\frac{\theta+\varepsilon}{2}}u|^2|(A+E)^{\frac{7}{8}+\frac{\theta+\varepsilon}{2}}v|^2(t)$$

$$+|(A+E)^{\frac{3}{8}+\frac{\theta+\varepsilon}{2}}v|^2|(A+E)^{\frac{7}{8}+\frac{\theta+\varepsilon}{2}}u|^2(t)\, dt \le C\lambda^{-2\varepsilon}|u|^2_{\frac{3}{8}+\frac{\theta+\varepsilon}{2},1}|v|^2_{\frac{3}{8}+\frac{\theta+\varepsilon}{2},1}.$$

The lemma is proven.

Lemma 8. *Let $\gamma \ge -1/4$. Then*

$$|T(\lambda)B(u, v)|^2_{\gamma,1} \le C|u|^2_{\gamma_0,1}|v|^2_{\gamma_0,1}, \quad \gamma_0 = 1/8 + \gamma/2.$$

PROOF. Use Lemma 5 and then (2) and (3):

$$|T(\lambda)B(u, v)|^2_{\gamma,1} \le 2\int_0^a |(A+E)^{\gamma-1/2}B(u, v)|^2\, dt$$

$$\le C\int_0^a |(A+E)^{\gamma_0}u|^2(t)|(A+E)^{\gamma_0+1/2}v|^2(t) + |(A+E)^{\gamma_0}v|^2(t)|(A+E)^{\gamma_0+1/2}u|^2(t)\, dt.$$

Hence, the claim of the lemma follows.

Lemma 9. *Suppose that $f(t) \in H_2[0, a]$ and a solution $u(t)$ to (1) satisfies the condition $|u|^2_{\gamma,1} = C < \infty$ for some $\gamma > 1/4$. Then $u' + Au \in H_2[0, a]$.*

PROOF. Denote the solution to (1) by u_0. It is a solution to the linear equation

$$u'_t + Au + B(u_0, u) = f(t), \quad u(0) = 0, \quad 0 < t < a.$$

Put $u(t) = e^{\lambda t}v(t)$ $(\lambda > 0)$. Then

$$v'_t + (A+\lambda)v + B(u_0, v) = e^{-\lambda t}f(t), \quad v(0) = 0.$$

From here and (2) and (3) we find that

$$\int_0^a |v'_t + (A+\lambda)v|^2\, dt \le 2\int_0^a |f|^2(t)\, dt + 2\int_0^a |B(u_0, v)|^2\, dt$$

$$\le C + C\int_0^a |(A+E)^{\tilde\gamma}u_0|^2(t)|(A+E)^{\tilde\gamma+1/2}v|^2 + |(A+E)^{\tilde\gamma}v|^2(t)|(A+E)^{\tilde\gamma+1/2}u_0|^2\, dt,$$

where $3/8 > \tilde{\gamma} > 1/4$, $\tilde{\gamma} < \gamma$. This choice $\tilde{\gamma}$ is possible in view of $\gamma > 1/4$. Therefore, using the condition of the lemma and choosing $\lambda \geq 1$, we find that

$$\int_0^a |g|^2 dt \equiv \int_0^a |v_t' + (A+\lambda)v|^2\, dt \leq C + C\int_0^a |(A+E)^{1-\varepsilon}v|^2\, dt + C \sup_{0<t<a} |(A+E)^{1/2-\varepsilon}v|^2$$

$$\leq C + \frac{C}{\lambda^\varepsilon}\int_0^a |(A+\lambda)^{1-\varepsilon/2}v|^2\, dt + \frac{C}{\lambda^\varepsilon} \sup_{0<t<a} |(A+\lambda)^{1/2-\varepsilon/2}v|^2$$

for a sufficiently small $\varepsilon > 0$, where g stands for the expression $v' + (A+\lambda)v$. Then $v' + (A+\lambda)v = g$ and $v(0) = 0$. Multiply this equation scalarly by $(A+\lambda)^{1-\varepsilon}v$ and integrate from 0 to t. We have

$$\frac{|(A+\lambda)^{\frac{1-\varepsilon}{2}}v|^2}{2} + \int_0^t |(A+\lambda)^{1-\frac{\varepsilon}{2}}v|^2\, dt = \int_0^t \langle g, (A+\lambda)^{1-\varepsilon}v\rangle\, dt$$

$$\leq \sqrt{\int_0^t |g|^2\, dt}\sqrt{\int_0^t |(A+\lambda)^{1-\frac{\varepsilon}{2}}v|^2 dt} \leq \frac{1}{2}\int_0^t |g|^2\, dt + \frac{1}{2}\int_0^t |(A+\lambda)^{1-\frac{\varepsilon}{2}}v|^2\, dt.$$

Choosing λ sufficiently large ($\lambda^{-\varepsilon}C < 1/2$), from this and the previous inequalities we obtain $v' + (A+\lambda)v \in H_2[0,a]$. Hence, passing from v to u, we obtain the claim of the lemma.

3. A condition for existence of a smooth solution and the properties of a separating function.

Definition 4. A function $f(t) \in H_2[0,a]$, $|f|_{H_2} \neq 0$, is called a *separating function* of (1) if

(a) problem (1) in $(0,a)$ has a solution $u(t)$ and $u' + Au \in H_2[0, a-\varepsilon]$ for every $\varepsilon \in (0,a)$ but $u' + Au \notin H_2[0,a]$;

(b) if $g(t) \in H_2[0,a]$ and $|g|_{H_2} < |f|_{H_2}$ then the problem

$$v'(t) + Av + B(v,v) = g(t), \quad v(0) = 0, \quad 0 < t < a,$$

has a solution and $v'(t) + Av \in H_2[0,a]$.

Theorem 1. *If there is $f(\cdot) \in H_2[0,a]$ such that* (1) *has no solution satisfying the condition $u' + Au \in H_2[0,a]$ then* (1) *possesses a separating function.*

Lemma 10. *There is a number $\varepsilon > 0$ such that if $|f|^2_{H_2[0,a]} \leq \varepsilon$ then* (1) *has a solution $u(t)$ for which $u' + Au \in H_2[0,a]$.*

Proof. This lemma for the Navier–Stokes equations is well known (see [3, p. 203, Theorems 8 and 9]). Nevertheless, we give its complete proof, since the proofs of the abstract lemmas like Lemma 10 available to the authors use various conditions (on $f(\cdot)$ and the nonlinearity).

Put $v = e^{-\lambda t}u$, $\lambda \geq 1$. For v we then obtain

$$v_t' + Av + \lambda v + e^{\lambda t}B(v,v) - e^{-\lambda t}f(t) = 0, \quad v(0) = 0.$$

Hence,

$$v = -T(\lambda)B(v,v)e^{\lambda t} + T(\lambda)e^{-\lambda t}f(t)$$

$$= \int_0^t e^{-(A+\lambda)(t-\xi)}(-B(v(\xi), v(\xi))e^{\lambda\xi} + e^{-\lambda\xi}f(\xi))\, d\xi = R(v).$$

Then

$$R(v_1) - R(v_2) = \int_0^t e^{-(A+\lambda)(t-\xi)+\lambda\xi}[-B(v_1, v_1) + B(v_2, v_2)]\, d\xi$$

$$= \int_0^t e^{-(A+\lambda)(t-\xi)+\lambda\xi}[B(v_1, v_2 - v_1) + B(v_2 - v_1, v_2)]\, d\xi.$$

Now, by Lemma 8,

$$|R(v_1) - R(v_2)|_{\theta-\frac{1}{2},1} \le C\left(|v_1|_{\frac{\theta}{2}-\frac{1}{8},1} + |v_2|_{\frac{\theta}{2}-\frac{1}{8},1}\right)\left(|v_1 - v_2|_{\frac{\theta}{2}-\frac{1}{8},1}\right).$$

Take $\theta = 1$. Then $1/2 = \theta - 1/2 > \theta/2 - 1/8 = 3/8$ and

$$|R(v_1) - R(v_2)|_{\frac{3}{8},1} \le |R(v_1) - R(v_2)|_{\frac{1}{2},1} \le C[|v_1|_{\frac{3}{8},1} + |v_2|_{\frac{3}{8},1}]|v_1 - v_2|_{\frac{3}{8},1}.$$

Hence, R is a contraction in the space with the norm $|\cdot|_{\frac{3}{8},1}$ if v_1 and v_2 are small in this norm.

By Lemma 5,

$$|T(\lambda)e^{-\lambda t}f(t)|^2_{\frac{3}{8},1} \le 2\int_0^a |(A+E)^{-1/8}f|^2\, dt \le 2\varepsilon.$$

Therefore, the equation $v = R(v)$ has a solution v of bounded norm $|v|_{\frac{3}{8},1}$. Returning from v to u, we find that $|u|_{\frac{3}{8},1}$ is bounded. But then Lemma 9 yields the claim of the lemma.

Proof of Theorem 1. Denote by r^* the least upper bound of all r such that if $|f|^2_{H_2} < r$ then (1) has a solution $u(t)$ such that $u' + Au \in H_2[0, a]$.

Suppose that there is $f \in H_2[0, a]$ such that (1) has no solution for which $u' + Au \in H_2[0, a]$; i.e.,

$$\int_0^{a-\delta} |u' + Au|^2\, dt = \infty$$

for some $0 \le \delta < a$. From here and Lemma 10 we conclude that r^* exists and $0 < r^* < \infty$.

It follows from the definition of r^* that there exist $f_n \in H_2[0, a]$ and $0 \le \delta_n < a$, $n = 1, 2, \dots$, such that

$$\int_0^a |f_n|^2(t)\, dt \ge r^*, \quad \int_0^a |f_n|^2(t)\, dt \to r^* \quad \text{as } n \to \infty,$$

$$\int_0^{a-\delta_n} |u_n' + Au_n|^2\, dt = \infty, \quad n = 1, 2, \dots,$$

$$\int_0^{a-\delta_n-\varepsilon} |u_n' + Au_n|^2\, dt < \infty \quad \text{for every } \varepsilon > 0,\ a - \delta_n - \varepsilon > 0,$$

where $u_n(t)$ is a solution to (1) on $(0, a - \delta_n)$.

Using translation of the time origin (for each n), we can assume that $\delta_n = 0$ $(n = 1, 2, \dots)$.

The sequence $\{f_n\}$ has a subsequence converging weakly to some $\tilde f$; denote it also by $\{f_n\}$. Thus,

$$u_n' + Au_n + B(u_n, u_n) = f_n \overset{H_2}{\rightharpoonup} \tilde f \in H_2[0,a] \quad \text{as } n \to \infty, \quad u_n|_{t=0} = 0, \tag{8}$$
$$\int\limits_0^{a-\varepsilon} |u_n' + Au_n|^2\,dt < \infty \quad \text{for every } 0 < \varepsilon < a,$$
$$\int\limits_0^{a} |u_n' + Au_n|^2\,dt = \infty, \quad |f_n|^2_{H_2[0,a]} \to r^*.$$

For $\tilde f$ in (8) the following cases are possible:

CASE 1. A solution to (1) with $f = \tilde f \in H_2[0,a]$ satisfies the condition $u' + Au \in H_2[0,a]$.

CASE 2. There is $\varepsilon \geq 0$ such that

$$\int\limits_0^{a-\varepsilon} |u' + Au|^2\,dt = \infty$$

for a solution to (1) with $f = \tilde f \in H_2[0,a]$.

Eliminate the first case.

Lemma 11. *If f_n converges weakly to $\tilde f$ ($f_n \rightharpoonup \tilde f$) in $H_2[0,a]$ and a solution $\tilde u(t)$ to (1) with $f = \tilde f$ satisfies the condition $\tilde u' + A\tilde u \in H_2[0,a]$ then there is n (sufficiently large) such that a solution u_n to (1) with $f = f_n$ satisfies also the condition $u_n' + Au_n \in H_2[0,a]$.*

PROOF. Let $\tilde u$ be a solution to (1) with $f = \tilde f$. By assumption, $\tilde u' + A\tilde u \in H_2[0,a]$. For the difference $\tilde u - u_n \equiv \omega_n$ we obtain

$$\begin{aligned} 0 &= \omega_n' + A\omega_n + B(\tilde u, \tilde u) - B(u_n, u_n) + f_n - \tilde f \\ &= \omega_n' + A\omega_n + B(\tilde u, \tilde u) - B(\tilde u - \omega_n, \tilde u - \omega_n) + f_n - \tilde f \\ &= \omega_n' + A\omega_n + B(\tilde u, \omega_n) + B(\omega_n, \tilde u) - B(\omega_n, \omega_n) + f_n - \tilde f, \quad \omega_n|_{t=0} = 0. \end{aligned}$$

Put $\omega_n = e^{\lambda t} y_n$. We have

$$y_n' + Ay_n + \lambda y_n E + B(\tilde u, y_n) + B(y_n, \tilde u) - e^{\lambda t} B(y_n, y_n) = (\tilde f - f_n)e^{-\lambda t} \equiv g_n, \ y_n|_{t=0} = 0.$$

Pass to the integral notation and use Lemma 1:

$$\begin{aligned} y_n(t) &= -T(\lambda)[B(\tilde u, y_n) + B(y_n, \tilde u) - e^{\lambda t} B(y_n, y_n)] + T(\lambda) g_n \\ &= -T(\lambda)[B(\tilde u, y_n) + B(y_n, \tilde u) - e^{\lambda t} B(y_n, y_n)] + T(\lambda, \alpha, b)[T(\lambda, 1-\alpha, -b) g_n], \end{aligned} \tag{9}$$

where $0 < \alpha < 1$. Take $-\alpha + b > -1$, $b \leq 0$. Then, by Lemma 6, $T(\lambda, 1-\alpha, -b)$ is compact in $H_2[0,a]$. Therefore, since $g_n = (\tilde f - f_n)e^{-\lambda t} \rightharpoonup 0$ in $H_2[0,a]$, from (9) we derive

$$\begin{aligned} y_n &= -T(\lambda)[B(\tilde u, y_n) + B(y_n, \tilde u) - e^{\lambda t} B(y_n, y_n)] + T(\lambda, \alpha, b) d_n, \\ &|d_n|_{H_2[0,a]} \to 0, \quad 0 < \alpha < 1, \ -\alpha + b > -1, \ b \leq 0, \end{aligned} \tag{10}$$

where $d_n = T(\lambda, 1-\alpha, -b) g_n$. By Corollary 1, we now have

$$|T(\lambda, \alpha, b) d_n|^2_{\gamma,\lambda} \leq C|d_n|^2_{H_2[0,a]} \to 0$$

as $n\to\infty$ for $\theta=0$, $\lambda\ge 1$, $\gamma+b\ge 0$, $\alpha-b-\gamma>1/2$, $-\alpha+b+1>0$, $b\le 0$, and $\alpha\in(0,1)$. (These conditions are satisfied, for example, if $\gamma=1/4+\delta$, $b=-\delta/2$, and $\alpha=3/4+\delta$, where $\delta>0$ is small.)

Consider (10) in the space with the norm $|\cdot|_{\gamma,\lambda}$:

$$y_n=-T(\lambda)[B(\tilde u,y_n)+B(y_n,\tilde u)-e^{\lambda t}B(y_n,y_n)]+r_n\equiv R_\lambda(y_n),$$
$$|r_n|_{\gamma,\lambda}\to 0\quad\text{as } n\to\infty,\ \text{where } r_n=T(\lambda,\alpha,b)d_n. \tag{11}$$

Now,

$$\begin{aligned}|R_\lambda(y)-R_\lambda(g)|^2_{\gamma,\lambda}&=|-T(\lambda)(B(\tilde u,y-g)+B(y-g,\tilde u))\\
&+T(\lambda)(e^{\lambda t}(B(y-g,y)+B(g,y-g)))|^2_{\gamma,\lambda}\le 2|T(\lambda)(B(\tilde u,y-g)\\
&+B(y-g,\tilde u))|^2_{\gamma,\lambda}+2|T(\lambda)(e^{\lambda t}B(y-g,y)+e^{\lambda t}B(g,y-g))|^2_{\gamma,\lambda}\\
&\le 2\lambda^{-2\varepsilon}|T(\lambda)(B(\tilde u,y-g)+B(y-g,\tilde u))|^2_{\gamma+\varepsilon,\lambda}\\
&+2e^{2a\lambda}|T(\lambda)(B(y-g,y)+B(g,y-g))|^2_{\gamma,\lambda}.\end{aligned}$$

Use Lemma 7 (taking $\alpha=1$, $b=0$, and $\theta=\gamma-1/2+2\varepsilon$, where $\varepsilon>0$ is small):

$$\begin{aligned}|R_\lambda(y)-R_\lambda(g)|^2_{\gamma,\lambda}&\le C\lambda^{-2\varepsilon}\big(|\tilde u|^2_{\frac{\gamma}{2}+\frac18+\frac{3\varepsilon}{2},1}|y-g|^2_{\frac{\gamma}{2}+\frac18+\frac{3\varepsilon}{2},1}\big)\\
&+Ce^{2\lambda a}\big(|y-g|^2_{\frac{\gamma}{2}+\frac18+\frac{3\varepsilon}{2},1}\big(|y|^2_{\frac{\gamma}{2}+\frac18+\frac{3\varepsilon}{2},1}+|g|^2_{\frac{\gamma}{2}+\frac18+\frac{3\varepsilon}{2},1}\big)\big).\end{aligned}$$

If $3/8>\gamma>1/4$ then we can take ε so small that $\gamma/2+1/8+3\varepsilon/2<\gamma$; therefore,

$$|R_\lambda(y)-R_\lambda(g)|^2_{\gamma,\lambda}\le C\lambda^{-2\varepsilon}\left(|\tilde u|^2_{\gamma,1}|y-g|^2_{\gamma,1}\right)+Ce^{2\lambda a}\big(|y-g|^2_{\gamma,1}\big(|y|^2_{\gamma,1}+|g|^2_{\gamma,1}\big)\big).$$

The value $|\tilde u|^2_{\gamma,1}$ is bounded; therefore, choosing λ large and the radius small of the ball in which we take y and g, we find that $R_\lambda(g)$ is a contraction. But then (11) for a large n has a solution with small norm $|\cdot|_{\gamma,1}$. Returning from y_n to $w_n=\tilde u-u_n$, we find that $|\tilde u-u_n|_{\gamma,1}<\infty$ for large n. Since $|\tilde u|_{\gamma,1}<\infty$, we conclude that $|u_n|_{\gamma,1}<\infty$. Therefore, u_n satisfies the conditions of Lemma 9. Consequently, $u_n'+Au_n\in H_2[0,a]$. The lemma is proven.

It follows from Lemma 11 that Case 1 is impossible. Thus, we have Case 2. It follows from (8) and the well-known properties of the weak limit that $|\tilde f(t)|_{H_2[0,a]}\le r^*$. Therefore, if $|g(t)|_{H_2[0,a]}<|\tilde f(t)|_{H_2[0,a]}$ then $|g(t)|_{H_2[0,a]}<r^*$. It follows from here and the definition of r^* that $v'+Av\in H_2[0,a]$, where $v(t)$ is a solution to the problem

$$v_t'+Av+B(v,v)=g(t),\quad v(0)=0,\quad 0<t<a.$$

Consequently, $\tilde f$ is a separating function. Theorem 1 is proven.

We now give some properties of a separating function of (1). Note first that the equality in the inequality $|\tilde f(t)|_{H_2[0,a]}\le r^*$ is attained; otherwise the definition of r^* would imply the condition $u'+Au\in H_2[0,a]$, where $u(t)$ is a solution to (1) with $f=\tilde f$. But this is impossible, since only Case 2 takes place; therefore, $|\tilde f(t)|_{H_2[0,a]}=r^*>0$.

Theorem 2. *Suppose that $s(t)$ is a separating function* ($|s|_{H_2[0,a]}>0$) *and $w(t)$ is a solution to* (1) *for $f=s(t)$. Then the pair $(w(t),s(t))$ satisfies* (4) (*see the introduction*).

Proof. Recall that the solution is understood in the sense of Definition 3. Thus, suppose that $w(t)$ is a solution to (1) in which $f=s(t)$ is a separating function, (w_1,w_2) is an arbitrary pair in D_a, and $B_ww_1\in H_2[0,a]$ (where D_a is taken from Definition 2). For $v=w(t)+\varepsilon w_1$ we obtain

$$\int_0^a|v'+Av+B(v,v)|^2\,dt=\int_0^a|s(t)+\varepsilon(w_1'+Aw_1+B_ww_1)+\varepsilon^2B(w_1,w_1)|^2\,dt$$

$$=\int_0^a(|s(t)|^2+2\varepsilon\langle s,w_1'+Aw_1+B_ww_1\rangle)\,dt+\varepsilon^2O(1).$$

Hence, the factor of the first power of ε must vanish; otherwise, choosing ε small, we can fulfill the inequality $|v' + Av + B(v,v)|_{H_2[0,a]} < |s(t)|_{H_2[0,a]}$; but then the definition of a separating function implies that $v' + Av \in H_2[0,a]$.

Hence, since w_1 is infinitely differentiable and satisfies condition (b) of Definition 2, we find that $w' + Aw \in H_2[0,a]$. This contradicts the fact that $s(t)$ is a separating function. Thus, we have the equality

$$\langle s(t), w_1' + Aw_1 + B_w w_1 \rangle_{H_2[0,a]} = 0.$$

Now, since $w' + Aw + B(w,w) = s(t)$, we obtain

$$\begin{aligned} 0 = \langle w' + Aw + B(w,w) - s, w_2 \rangle_{H_2} &= \langle -w_2' + Aw_2, w \rangle_{H_2} + \langle B(w,w) - s, w_2 \rangle_{H_2} \\ &= \langle -w_2' + Aw_2, w \rangle_{H_2} + \frac{1}{2}\langle B_w w, w_2 \rangle_{H_2} - \langle w_2, s \rangle_{H_2} \\ &= \langle -w_2' + Aw_2, w \rangle_{H_2} + \frac{1}{2}\langle w, B_w^* w_2 \rangle_{H_2} - \langle w_2, s \rangle_{H_2} \\ &= \left\langle -w_2' + Aw_2 + \frac{1}{2}B_w^* w_2, w \right\rangle_{H_2} - \langle w_2, s \rangle_{H_2}. \end{aligned}$$

Here $H_2 = H_2[0,a]$. Thus, $(w, s(t))$ is a solution to (4). The theorem is proven.

We can prove that a separating function satisfies the condition $s' - As \in H_2[0, a-\delta]$ for every $0 < \delta < a$.

Theorem 3. *Suppose that $f(t)$ is a separating function of* (1) *and $w(t)$ is a solution to* (1) *in* $(0,a)$. *Then*

$$(|w_t'|^2 - |Aw + B(w,w)|^2)_t' = 0$$

for $0 < t < a$.

Proof. Multiply the first equation in (4) scalarly by $w'(t)$ and integrate from 0 to t. After simple calculations we obtain the equality of the theorem.

Proof of Theorem A. Assume that (1) is not strongly solvable. Then there exist $a > 0$ and $f(t) \in H_2[0,a]$ such that a solution $u(t)$ to (1) satisfies $u' + Au \notin H_2[0,a]$. From Theorem 1 we find that there is a separating function $s(t)$. If $w(t)$ is a solution to (1) for $f(t) = s(t)$ then from Theorem 2 we find that the pair (w,s) is a nontrivial solution to (4).

Conversely, prove that if (4) has a nontrivial solution then (1) is not strongly solvable. Suppose the contrary. Assume that (4) on $[0,a]$, $a > 0$, has a nontrivial solution $(w(t), s(t))$, but (1) is strongly solvable. Then $w' + Aw \in H_2[0,a]$ by $s(t) \in H_2[0,a]$.

Take a pair (w_1, w_2) in the class D_a of test functions and $B_w w_1 \in H_2[0,a]$. Then from the definition of a solution we obtain

$$0 = \int_0^a \langle s, w_1' + Aw_1 + B_w w_1 \rangle\, dt = \int_0^a \langle s, w_1' + Aw_1 + B(w, w_1) + B(w_1, w) \rangle\, dt. \tag{12}$$

Consider the problem

$$\tilde{g}' + A\tilde{g} + B(w, \tilde{g}) + B(\tilde{g}, w) = s(t), \quad \tilde{g}(0) = 0, \quad 0 < t < a. \tag{13}$$

Put $\tilde{g} = e^{\lambda t} g$. For g we obtain

$$g' + (A + \lambda)g + B(w,g) + B(g,w) = e^{-\lambda t}s(t), \quad g(0) = 0, \quad 0 < t < a. \tag{14}$$

Let $g' + (A+\lambda)g \equiv v$. Then (14) is equivalent to the equation

$$v + D(\lambda)v = e^{-\lambda t}s(t), \quad \lambda \geq 1, \tag{15}$$

where $D(\lambda)$ is the operator acting by the formula

$$D(\lambda)v \equiv B(T(\lambda)v, w) + B(w, T(\lambda)v).$$

Take $\lambda \geq 1$ and $0 < \varepsilon < 1/2 - \delta_0$. Estimate the norm of $D(\lambda)$:

$$|D(\lambda)u|^2_{H_2[0,a]} = \int_0^a |D(\lambda)u|^2\,dt = \int_0^a |B(T(\lambda)u, w) + B(w, T(\lambda)u)|^2\,dt$$

$$\leq C\int_0^a |(A+E)^{\delta_0}T(\lambda)u|^2|(A+E)^{\delta_0+1/2}w|^2 + |(A+E)^{\delta_0}w|^2|(A+E)^{\delta_0+1/2}T(\lambda)u|^2\,dt$$

$$\leq C|(A+\lambda)^{-\varepsilon}|\int_0^a |(A+E)^{\delta_0+\varepsilon}T(\lambda)u|^2|(A+E)^{\delta_0+1/2}w|^2$$

$$+|(A+E)^{\delta_0}w|^2|(A+E)^{\delta_0+1/2+\varepsilon}T(\lambda)u|^2\,dt$$

$$\leq \frac{C}{\lambda^\varepsilon}\int_0^a |(A+E)^{1/2}T(\lambda)u|^2|(A+E)w|^2 + |(A+E)^{1/2}w|^2|(A+E)T(\lambda)u|^2\,dt.$$

Use the method of the proof of Lemma 9 to obtain

$$|D(\lambda)u|^2_{H_2[0,a]} \leq \frac{C}{\lambda^\varepsilon}|u|^2_{H_2[0,a]}|w' + Aw|^2_{H_2[0,a]} \leq \frac{C_1}{\lambda^\varepsilon}|u|^2_{H_2[0,a]}.$$

Hence, $\|D(\lambda)\| \leq 1/2$ for a sufficiently large λ. Therefore, (15) has a solution $v \in H_2[0,a]$. But $v = g' + (A+\lambda)g$; consequently, (14) has a solution g such that $g' + (A+\lambda)g \in H_2[0,a]$. Returning from g to $\tilde{g}$ we find that a solution $\tilde{g}$ to (13) satisfies the condition $\tilde{g}' + A\tilde{g} \in H_2[0,a]$.

Take $w_1 = w_{1n}$ in (12) $(n = 1, 2, \dots)$ so that

$$\frac{d}{dt}(w_{1n}) + Aw_{1n} \to \tilde{g}' + A\tilde{g} \tag{16}$$

in $H_2[0,a]$ as $n \to \infty$. This is possible, since $A \geq E$ is a constant selfadjoint operator and the set of smooth function is dense in the Sobolev space. Like in the estimation of the norm of $D(\lambda)$, we can show that

$$|B(w,v) + B(v,w)|_{H_2[0,a]} \leq C|v' + Av|_{H_2[0a]}$$

for every v if $v(0) = 0$ and $v' + Av \in H_2[0,a]$. Therefore, it follows from (16) that in $H_2[0,a]$

$$B(w_{1n}, w) + B(w, w_{1n}) \to B(\tilde{g}, w) + B(w, \tilde{g}).$$

From here and (12), passing to the limit as $n \to \infty$ we find that

$$0 = \int_0^a |s(t)|^2\,dt.$$

Consequently, $s(t) \equiv 0$. But then the second equation of (4) yields $w \equiv 0$; a contradiction. Theorem A is proven.

Remark 3.1. We can extend the main results of Sections 1–3 to more general equations, for example, as follows:

$$u'_t + Au + \theta(t)B(u) = f(t), \quad u|_{t=0} = 0, \tag{17}$$

where $A = A^*$ is a nonnegative selfadjoint operator with a compact resolvent, $\theta(t)$ is a continuous scalar function, and $B(\cdot)$ is a nonlinear operator satisfying the conditions

$$B(u) - B(u + \varepsilon w) = \varepsilon B_u w + D(u, w, \varepsilon)\varepsilon^2;$$

here B_u is a linear operator for each u, $D(u, w, \varepsilon)$ is a nonlinear operator such that

$$|B_u w| \leq \varphi(|A^{\gamma-1/2}u|)[|A^{\gamma}u||A^{\gamma-1/2}w| + |A^{\gamma-1/2}u||A^{\gamma}w|],$$

$$|D(u, w, \varepsilon)| \leq \psi(|A^{\gamma-1/2}u|, |A^{\gamma-1/2}w|, \varepsilon)[|A^{\gamma}u||A^{\gamma-1/2}w| + |A^{\gamma-1/2}u||A^{\gamma}w|],$$

where γ is a constant, $1/2 < \gamma < 1$, $\varphi(\cdot)$ is continuous on $[0, \infty)$, and $\psi(x, y, \varepsilon)$ is continuous for $0 \leq x < \infty$, $0 \leq y < \infty$, and $0 \leq \varepsilon \leq 1$. We can prove that, under the above conditions, the theorem on a separating function is valid for (17) and a theorem similar to Theorem A holds. Observe that the condition $A \geq E$ is inessential; it is sufficient that A be lower semibounded.

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