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DESCRIPTION OF INTERPOLATION SPACES FOR GENERAL LOCAL MORREY-TYPE SPACES

V.I. Burenkov, D.K. Darbayeva, E.D. Nursultanov

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Abstract. We consider the real interpolation method and prove that for general local Morrey-type spaces, in the case when they have the same integrability parameter, the interpolation spaces are again general local Morrey-type spaces with appropriately chosen parameters.

1 Introduction

Let $0 < p \leqslant \infty$ and $0 \leqslant \lambda \leqslant \frac{n}{p}$. The Morrey spaces M_p^λ were defined in [10] as the spaces of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x\in\mathbb{R}^n} \sup_{r>0} r^{-\lambda} \|f\|_{L_p(B(x,r))} < \infty ,$$

where $B(x,r)$ is the open ball of radius $r > 0$ with center at point $x \in \mathbb{R}^n$. If $\lambda = 0$, then $M_p^0 = L_p(\mathbb{R}^n)$, while if $\lambda = \frac{n}{p}$, then $M_p^{\frac{n}{p}} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > \frac{n}{p}$, then $M_p^\lambda = \Theta$, where Θ is the set of all functions that are equivalent to zero on $\mathbb{R}^n$.

The Morrey spaces and their generalizations have found wide applications in many problems of real analysis and partial differential equations. The boundedness of classical operators of real analysis in the Morrey spaces and in general Morrey-type spaces was studied by many authors. See survey papers [2, 3, 7, 8, 12, 14, 15, 11].

Interpolation of these spaces was considered in [16, 6, 11]. It follows by the results of [11] that

$$(M_p^{\lambda_0}, M_p^{\lambda_1})_{\theta,\infty} \subset M_p^\lambda ,$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$. See also [9], Theorem 3 (v). In [13, 1] it was established that this inclusion is strict, which raised the problem of giving a complete description of the interpolation spaces. This problem still remains open.

In [4] a similar problem was considered for a local variant of the Morrey spaces and for their generalizations involving an additional parameter.

The local Morrey-type spaces $LM^{\lambda}_{p,q}$ are defined for $\lambda \geqslant 0$, and $0 < p, q \leqslant \infty$ as the spaces of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ such that

$$\|f\|_{LM^{\lambda}_{p,q}} = \left(\int_0^{\infty} \left(r^{-\lambda} \|f\|_{L_p(B(0,r))} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} < \infty \,,$$

with the conventional modification for $q = \infty$.

Note that $LM^{\lambda}_{p,q} \neq \Theta$ if and only if $\lambda > 0$ for $q < \infty$ and $\lambda \geqslant 0$ for $q = \infty$. If $q = \infty$, then $LM^{0}_{p,\infty} = L_p(\mathbb{R}^n)$. Moreover, for $p = q$, we have

$$LM^{\lambda}_{p,p} = L_{p,\lambda}(\mathbb{R}^n)$$

and

$$\|f\|_{LM^{\lambda}_{p,p}} = (\lambda p)^{-\frac{1}{p}} \|f\|_{L_{p,\lambda}(\mathbb{R}^n)},$$

where $L_{p,\lambda}(\mathbb{R}^n)$ is the weighted Lebesgue space of all functions f Lebesgue measurable on $\mathbb{R}^n$ for which

$$\|f\|_{L_{p,\lambda}(\mathbb{R}^n)} = \|f(y)|y|^{-\lambda}\|_{L_p(\mathbb{R}^n)} < \infty.$$

It appeared that for $p_0 = p_1$, in contrast to the scale of the Morrey spaces M^{λ}_p, the scale of the local Morrey-type spaces $LM^{\lambda}_{p,q}$ is closed under the procedure of interpolation. Namely, the following statement was proved in [4].

Theorem 1.1. ([4]) *Let $0 < p, q_0, q_1, q \leqslant \infty$ and $0 < \theta < 1$. Suppose, in addition, that $\lambda_0 \neq \lambda_1$ and $0 < \lambda_0, \lambda_1 < \frac{n}{p}$ if $p < \infty$ and at least one of the parameters q_0, q_1 and q is finite, and $0 \leqslant \lambda_0, \lambda_1 \leqslant \frac{n}{p}$ if $q_0 = q_1 = q = \infty$. Then*

$$\left(LM^{\lambda_0}_{p,q_0}, \;\; LM^{\lambda_1}_{p,q_1} \right)_{\theta,q} = LM^{\lambda}_{p,q}, \tag{1.1}$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

Remark 1. Note that in Theorem 1.1 there are additional assumptions on λ_0 and λ_1: $0 < \lambda_0, \lambda_1 < \frac{n}{p}$ if $q < \infty$ and $0 \leqslant \lambda_0, \lambda_1 \leqslant \frac{n}{p}$ if $q = \infty$. They appeared because in [4] in the proof of Theorem 1.1 the equality $LM^{\lambda}_{p,q} = \widetilde{LM}^{\lambda}_{p,q}$ was used where $\widetilde{LM}^{\lambda}_{p,q}$ is the space of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{LM}^{\lambda}_{p,q}} = \left(\int_0^{\infty} \left(r^{-\lambda} \|f\|_{\widetilde{L}_p(B(0,r))} \right)^q \frac{dr}{r} \right)^{\frac{1}{q}} < \infty \,,$$

where

$$\|f\|_{\widetilde{L}_p(B(x,r))} = |B(x,r)|^{\frac{1}{p}} \sup_{\rho \geqslant r} \left(\frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |f(y)|^p dy \right)^{\frac{1}{p}},$$

and $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$, and this equality holds only under the additional assumptions on λ_0 and λ_1 mentioned above.

In present note we state a similar result for general local Morrey-type spaces $LM^{\lambda}_{p,q}(G,\mu)$.

2 Main result

Let (Ω, μ) be a space with a positive σ-finite Borel measure μ. By $G = \{G_t\}_{t>0}$ we denote a parametric family of μ-measurable subsets of Ω, for which

$$G_t \neq \Omega \text{ for some } t > 0, \quad G_{t_1} \subset G_{t_2} \text{ if } 0 < t_1 < t_2 < \infty \text{ and } \bigcup_{t>0} G_t = \Omega. \tag{2.1}$$

Definition 1. Let $0 < p, q \leqslant \infty$ and $0 < \lambda < \infty$ if $q < \infty$ and $0 \leqslant \lambda < \infty$ if $q = \infty$. We define the space $LM^{\lambda}_{p,q}(G, \mu)$ as the space of all functions f μ-measurable on Ω such that for $q < \infty$

$$\|f\|_{LM^{\lambda}_{p,q}(G,\mu)} = \left(\int_0^{\infty} \left(t^{-\lambda} \|f\|_{L_p(G_t,\mu)} \right)^q \frac{dt}{t} \right)^{1/q} < \infty,$$

and for $q = \infty$

$$\|f\|_{LM^{\lambda}_{p,\infty}(G,\mu)} = \sup_{t>0} t^{-\lambda} \|f\|_{L_p(G_t,\mu)} < \infty,$$

where

$$\|f\|_{L_p(G_t,\mu)} = \left(\int_{G_t} |f(x)|^p d\mu \right)^{\frac{1}{p}},$$

with the conventional modification for $p = \infty$.

Remark 2. Let $0 < p, q \leqslant \infty$ and $a \geqslant 0$. Let v be a function positive, locally absolutely continuous, strictly increasing on (a, ∞) and such that $v = 0$ on $(0, a]$, $\lim\limits_{t \to a^+} v(t) = \alpha$, $\lim\limits_{t \to +\infty} v(t) = \infty$. Moreover, let μ be the Lebesgue measure on $\mathbb{R}^n$. If in definition 1 we take $\lambda = 1$, $\Omega = \mathbb{R}^n$,

$$G_t = \begin{cases} \varnothing, & \text{if } 0 < t \leqslant a, \\ B(0, v^{(-1)}(t)), & \text{if } \alpha < t < \infty, \end{cases}$$

and

$$||f||_{L_p(G_t)} = \begin{cases} 0, & \text{if } 0 < t \leqslant a, \\ ||f||_{L_p(B(0,v^{(-1)}(t)))}, & \text{if } \alpha < t < \infty, \end{cases}$$

then

$$\|f\|_{LM^1_{p,q}(G,\mu)} = \left(\int_a^{\infty} \left(t^{-1} ||f||_{L_p(B(0,v^{(-1)}(t)))} \right)^q \frac{dt}{t} \right)^{1/q} = \left(v^{(-1)}(t) = r \right)$$

$$= ||f||_{LM^{v(\cdot)}_{pq}} \equiv \left(\int_{\alpha}^{\infty} \left(\frac{||f||_{L_p(B(0,r))}}{v(r)} \right)^q \frac{dv(r)}{v(r)} \right)^{1/q},$$

hence $LM^1_{p,q}(G, \mu) = LM^{v(\cdot)}_{pq}$.

A similar argument shows that for any $0 < \lambda < \infty$

$$||f||_{LM^{\lambda}_{p,q}(G,\mu)} = \lambda^{-\frac{1}{q}} ||f||_{LM^{v^{\lambda}(\cdot)}_{pq}}$$

and $LM^{\lambda}_{p,q}(G,\mu) = LM^{v^{\lambda}(\cdot)}_{pq}$.

Note also that if

$$G_t(\lambda) = \begin{cases} \varnothing, & \text{if } 0 < t \leqslant a, \\ B(0,(v^{\frac{1}{\lambda}})^{(-1)}(t)), & \text{if } \alpha < t < \infty, \end{cases}$$

then

$$||f||_{LM^{\frac{1}{\lambda}}_{p,q}(G(\lambda),\mu)} = \lambda^{-\frac{1}{q}} ||f||_{LM^{v(\cdot)}_{pq}}$$

and $LM^{\frac{1}{\lambda}}_{p,q}(G(\lambda),\mu) = LM^{v(\cdot)}_{pq}$.

Remark 3. Let $0 < p, q \leqslant \infty$, and let w be a positive measurable function on $(0,\infty)$ such that $||w||_{L_q(t,\infty)} < \infty$ for some $t > 0$. Set $a = \inf\{t > 0 : ||w||_{L_q(t,\infty)} < \infty\}$, and let in Remark 2

$$v(t) = \begin{cases} 0, & \text{if } 0 < t \leqslant a, \\ q^{-\frac{1}{q}} ||w||^{-1}_{L_q(t,\infty)}, & \text{if } a < t < \infty, \end{cases}$$

and $\alpha = \lim\limits_{t\to a^+} q^{-\frac{1}{q}} ||w||^{-1}_{L_q(t,\infty)}$. Clearly, if $q < \infty$, then $\lim\limits_{t\to+\infty} v(t) = \infty$. If $q = \infty$, then, in order that this equality also hold, we shall assume, in addition, that $\lim\limits_{t\to+\infty} ||w||_{L_\infty(t,\infty)} = 0$. In this case

$$LM^{1}_{p,q}(G,\mu) = LM_{pq,w(\cdot)}$$

– the general local Morrey-type space, studied in a number of papers by V.I. Burenkov and his coauthors [1]. See survey papers [2, 3, 5]. Moreover,

$$||f||_{LM^{1}_{p,q}(G,\mu)} = ||f||_{LM_{pq,w(\cdot)}} \equiv ||w(r)||f||_{L_p(B(0,r))}||_{L_q(0,\infty)},$$

because $v^{-q-1}(r)v'(r) = w^q(r)$ for almost all $r > a$.

A similar argument shows that for any $0 < \lambda < \infty$

$$||f||_{LM^{\lambda}_{p,q}(G,\mu)} = \lambda^{-\frac{1}{q}} ||f||_{LM_{pq,w^{\lambda}(\cdot)}}$$

and $LM^{\lambda}_{p,q}(G,\mu) = LM_{pq,w^{\lambda}(\cdot)}$.

Theorem 2.1. *Let $0 < p, q_0, q_1, q \leqslant \infty$, $0 < \lambda_0, \lambda_1 < \infty$, $\lambda_0 \neq \lambda_1$, and $0 < \theta < 1$, $\Omega \subset \mathbb{R}^n$, and let μ be a σ-finite Borel measure on Ω and $G = \{G_t\}_{t>0}$ be a family of μ-measurable sets G_t, satisfying (2.1). Then*

$$(LM^{\lambda_0}_{p,q_0}(G,\mu), LM^{\lambda_1}_{p,q_1}(G,\mu))_{\theta,q} = LM^{\lambda}_{p,q}(G,\mu),$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$. Moreover, there exist $c_1, c_2 > 0$, depending only on $p, q_0, q_1, q, \lambda_0, \lambda_1$ and θ, such that

$$c_1 ||f||_{LM^{\lambda}_{p,q}(G,\mu)} \leqslant ||f||_{(LM^{\lambda_0}_{p,q_0}(G,\mu), LM^{\lambda_1}_{p,q_1}(G,\mu))_{\theta,q}} \leqslant c_2 ||f||_{LM^{\lambda}_{p,q}(G,\mu)}$$

[1] Usually this space is considered under slightly weaker assumptions on w, namely it is assumed that $w \in \Omega_q \Leftrightarrow w$ is non-negative measurable on $(0,\infty)$, not equivalent to 0 on (t,∞) for all $t > 0$ and such that $||f||_{L_q(t,\infty)} < \infty$ for some $t > 0$. Recall that, given a function w non-negative measurable on $(0,\infty)$ and not equivalent to 0 (t,∞) for all $t > 0$, the space $LM_{pq,w(\cdot)}$ is non-trivial, i.e. consists not only of functions equivalent to 0 if and only if $w \in \Omega_q$. (See [2] for details.) So, $w \in \Omega_q$ is a minimal natural assumption on w when studying the spaces $LM_{pq,w(\cdot)}$.

for all $f \in LM^{\lambda}_{p,q}(G,\mu)$.

Corollary 2.1. *Equality* (3.2) *holds under the assumptions of Theorem* 2.1 *on the parameters.*

Corollary 2.2. *Let* $0 < p, q_0, q_1, q \leqslant \infty$, $0 < \lambda_0, \lambda_1 < \infty$, $\lambda_0 \neq \lambda_1$, $0 < \theta < 1$, *and function* v *be as in Remark* 2. *Then*

$$\left(LM^{v^{\lambda_0}(\cdot)}_{p,q_0}, LM^{v^{\lambda_1}(\cdot)}_{p,q_1}\right)_{\theta,q} = LM^{v^{\lambda}(\cdot)}_{p,q},$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$.

Remark 4. Note that the equality

$$\left(LM_{pq_0,w^{\lambda_0}(\cdot)}, LM_{pq_1,w^{\lambda_1}(\cdot)}\right)_{\theta,q} = LM_{pq,w^{\lambda}(\cdot)}, \tag{2.2}$$

where $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$, may not hold even for the case of the power function $w(r) = r^{-s}$, $s > 0$. In this case equality (2.2) holds if λ is replaced by

$$\nu = \lambda + \frac{1}{s}\left(\frac{1}{q} - \frac{1-\theta}{q_0} - \frac{\theta}{q_1}\right),$$

hence equality (2.2) holds only if

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{2.3}$$

Indeed, by Corollary 2.1

$$\left(LM_{pq_0,(r^{-s})^{\lambda_0}(\cdot)}, LM_{pq_1,(r^{-s})^{\lambda_1}(\cdot)}\right)_{\theta,q} = \left(LM^{s\lambda_0-\frac{1}{q_0}}_{p,q_0}, LM^{s\lambda_1-\frac{1}{q_1}}_{p,q_1}\right)_{\theta,q}$$

$$= LM^{\nu s-\frac{1}{q}}_{p,q} = LM_{pq,(r^{-s})^{\nu}}.$$

However, equality (2.2) holds for one special choice of λ_0 and λ_1.

Corollary 2.3. Let $0 < p, q_0, q_1, q < \infty$, $q_0 \neq q_1$, $0 < \theta < 1$, equality (2.3) be satisfied, and let w be a such positive measurable function on $(0,\infty)$, that $w \in L_1(t,\infty)$ for some $t > 0$. Then [2]

$$\left(LM_{p,q_0,w^{\frac{1}{q_0}}(\cdot)}, LM_{p,q_1,w^{\frac{1}{q_1}}(\cdot)}\right)_{\theta,q} = LM_{p,q,w^{\frac{1}{q}}(\cdot)}.$$

Let $M^{\uparrow}$ be the cone of all functions φ non-negative and non-decreasing on $(0,\infty)$. Moreover, for $0 < \lambda < \infty$ if $p < \infty$, and for $0 \leqslant \lambda < \infty$ if $p = \infty$, let $\Phi^{\uparrow}_{\lambda,p}$ denote the space of all functions $\varphi \in M^{\uparrow}$ for which

[2] In [17] this result is extended to all functions $w \in \Omega_1$.

$$||\varphi||_{\Phi_{\lambda,p}} = \left(\int_0^\infty \left(t^{-\lambda}|\varphi(t)|\right)^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty$$

if $p < \infty$, and

$$||\varphi||_{\Phi_{\lambda,\infty}} = \text{ess sup }_{x\in(0,\infty)} t^{-\lambda}|\varphi(t)| < \infty$$

if $p = \infty$.

Let, for $s > 0$ and a function $\varphi \in M^{\uparrow}$, the functions $A_s\varphi$ and $B_s\varphi$ be defined by

$$A_s\varphi(t) = \begin{cases} \varphi(t), & \text{if } 0 < t \leqslant s, \\ \varphi(s), & \text{if } t > s, \end{cases}$$

and

$$B_s\varphi(t) = \varphi(t)\chi_{(s,\infty)}(t), \; t > 0,$$

where $\chi_{(s,\infty)}$ is the characteristic function of the interval (s, ∞).

The proof of Theorem 2.1 is based on the following statement.

Theorem 2.2 *Let* $0 < p_0, p_1, q \leqslant \infty$, $0 < \theta < 1$, $0 < \lambda_0 < \lambda_1$, *and* $\lambda = (1-\theta)\lambda_0 + \theta\lambda_1$. *Then there exist* $c_1, c_2 > 0$ *depending only on* $p_0, p_1, q, \lambda_0, \lambda_1, \theta$ *such that*

$$c_1\,||\varphi||_{\Phi_{\lambda,q}} \leqslant \left\| \inf_{s>0} \left(||A_s\varphi||_{\Phi_{\lambda_0,p_0}} + t||B_s\varphi||_{\Phi_{\lambda_1,p_1}}\right) \right\|_{\Phi_{\theta,q}} \leqslant c_2\,||\varphi||_{\Phi_{\lambda,q}}$$

for all functions $\varphi \in \Phi^{\uparrow}_{\lambda,p}$.

Detailed proofs of Theorems 2.1 and 2.2 and of more general statements of such type are contained in [5].

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References

[1] O. Blasco, A. Ruiz, L. Vega, *Non interpolation in Morrey-Campanato and block spaces,* Ann. Scuola Norm. Super. Pisa 28 (1999), no. 1, 31-40.

[2] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces I,* Eurasian Mathematical Journal 3 (2012), no. 3, 11-32.

[3] V.I. Burenkov, *Recent progress in studying the boundedness of classical operators of real analysis in general Morrey-type spaces II,* Eurasian Mathematical Journal 4 (2013), no. 1, 21-45.

[4] V.I. Burenkov, E.D. Nursultanov, *Description of interpolation spaces for local Morrey-type spaces,* Trudy Math. Inst. Steklov 269 (2010), 46-56.

[5] V.I. Burenkov, D.K. Darbayeva, E.D. Nursultanov, *Description of interpolation spaces for the pair of local Morrey-type spaces and their generalizations,* Trudy Math. Inst. Steklov 2013 (accepted for publication).

[6] A. Campanato, M.K.V. Murthy, *Una generalizzazione del teorema di Riesz-Thorin,* Ann. Scuola Norm. Super. Pisa 19 (1965), 87-100.

[7] V.S. Guliyev, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators,* Eurasian Mathematical Journal 3 (2012), no. 3, 33-61.

[8] P.G. Lemarié-Rieusset, *The role of Morrey spaces in the study of Navier-Stokes and Euler equations,* Eurasian Mathematical Journal 3 (2012), no. 3, 62-93.

[9] P.G. Lemarié-Rieusset, *Multipliers and Morrey spaces.* Potential Analysis 38 (2013), 741-752.

[10] C.B. Morrey, *On the solution of quasi-linear elliptic partial differential equations,* Trans. Amer. Math. Soc. 43 (1938), 126-166.

[11] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces,* J. Func. Anal. 4 (1969), 71-87.

[12] M.A. Ragusa, *Operators in Morrey type spaces and applications,* Eurasian Mathematical Journal 3 (2012), no. 3, 94-109.

[13] A. Ruiz, L. Vega, *Corrigenda to "Unique continuation for Schrödinger operators" and a remark on interpolation on Morrey spaces,* Publ. Mat., Barc. 39 (1995), 405-411.

[14] W. Sickel, *Smoothness spaces related to Morrey spaces - a survey. I,* Eurasian Mathematical Journal 3 (2012), no. 3, 110-149.

[15] W. Sickel, *Smoothness spaces related to Morrey spaces - a survey. II,* Eurasian Mathematical Journal 4 (2013), no. 1, 82-124.

[16] G. Stampacchia, *$\mathcal{L}_{p,\lambda}$-spaces and interpolation,* Comm. Pure Appl. Math. 17 (1964), 293-306.

[17] T.V. Tararykova, *Comments on definitions of general local and global Morrey-type spaces,* Eurasian Mathematical Journal 4 (2013), no. 1, 125-134.

[18] M. Taylor, *Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations,* Comm. Partial Differential Equations 17 (1992), 1407-1456.

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