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SOME CHARACTERIZING CONDITIONS FOR THE HARDY INEQUALITY

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Abstract. In this paper, a new scale of necessary and sufficient conditions for the validity of the Hardy and *reverse* Hardy inequalities in the cases

$$0 < \tfrac{q}{p} < 1, \qquad p \in (-\infty, 0) \cup (1, \infty),$$

and

$$0 < \tfrac{p}{q} < 1, \qquad q \in (0, 1)$$

is found and estimates for the best constants are derived.

1 Introduction

The "classical" Hardy inequality

$$\left(\int_a^b \left(\int_a^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{1}$$

and the *reverse* Hardy inequality

$$\left(\int_a^b \left(\int_a^x f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \ge C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \tag{2}$$

for $f \ge 0$ with given weight functions u, v are completely characterized for all real values of p, q, $p \ne 0$, $q \ne 0$ (see, e.g., [5], [4], [2]).

Moreover, for the case

$$1 < p \le q < \infty$$

we have about 15 scales of (equivalent) conditions starting with the classical necessary and sufficient Muckenhoupt condition

$$A_M := \sup_{x\in(a,b)} A_M(x) < \infty$$

where

$$A_M(x) = U^{\frac{1}{q}}(x)V^{\frac{1}{p'}}(x), \quad p' = \frac{p}{p-1}$$

with

$$U(x) = \int_x^b u(t)dt, \quad V(x) = \int_a^x v^{1-p'}(t)dt. \tag{3}$$

For the case $q < p$ we have less conditions. The (again classical) Maz'ja–Rozin condition

$$B_{MR} := \left(\int_a^b U^{\frac{r}{q}}(x)V^{\frac{r}{q'}}(x)dV(x)\right)^{\frac{1}{r}} < \infty$$

for the case

$$0 < q < p < \infty, \quad p > 1, \quad q \neq 1, \quad \frac{1}{r} := \frac{1}{q} - \frac{1}{p},$$

was extended by the (Persson–Stepanov) condition

$$B_{PS} := \left(\int_a^b \left[\int_a^x u(t)V^q(t)dt\right]^{\frac{r}{q}} V^{-\frac{r}{q}}(x)dV(x)\right)^{\frac{1}{r}} < \infty$$

and both conditions have been extended to the scales (with $s \in (0,\infty)$)

$$\mathcal{B}_{MR}(s) := \left(\int_a^b \left[\int_t^b u(\tau)V^{q(1/p'-s)}(\tau)d\tau\right]^{\frac{r}{q}} V^{rs-1}(t)dV(t)\right)^{\frac{1}{r}} < \infty,$$

$$\mathcal{B}_{PS}(s) := \left(\int_a^b \left[\int_a^t u(\tau)V^{q(1/p'+s)}(\tau)d\tau\right]^{\frac{r}{q}} V^{-rs-1}(t)dV(t)\right)^{\frac{1}{r}} < \infty$$

which are mutually equivalent (see [6] for details; notice that B_{MR} and B_{PS} coincide with $\mathcal{B}_{MR}(\frac{1}{p'})$ and $\mathcal{B}_{PS}(\frac{1}{p})$, respectively).

The case of negative values of the parameters p, q and the *reverse* Hardy inequality is completely described in [7] and in [1]; for the case $\frac{q}{p} \in (0,1)$, i.e.

$$-\infty < p < q < 0$$

the corresponding (Prokhorov) condition reads

$$\mathcal{B}_P := \left(\int_a^b \tilde{U}^{\frac{r}{p}}(x)V^{\frac{r}{p'}}(x)d\tilde{U}(x)\right)^{-\frac{1}{r}} < \infty \tag{4}$$

with $\tilde{U}(x) := \int_a^x u(t)dt$.

In this paper, we extend these conditions for the region

$$0 < \tfrac{q}{p} < 1, \qquad p \in (-\infty, 0) \cup (1, \infty),$$

introducing the new scale

$$B_K(s) := \left(\int_a^b \left(\int_x^b u(z) V^{q(\frac{1}{p'} - \frac{s}{r})}(z) dz \right)^{\frac{r}{q}} V^{s-1}(x) dV(x) \right)^{\frac{q}{r}} \tag{5}$$

where $s \in (0, \infty)$, and for the region

$$0 < \tfrac{p}{q} < 1, \qquad q \in (0,1)$$

using the (dual) scale

$$B_K^*(s) := \left(\int_a^b \left(\int_a^x v^{1-p'}(z) U^{p'(\frac{1}{q} - \frac{s}{r})}(z) dz \right)^{\frac{r}{p'}} U^{s-1}(x) d(-U(x)) \right)^{\frac{p'}{r}}.$$

2 Main results

In the first main result we consider the case $p \in (-\infty, 0) \cup (1, \infty)$. Then we can rewrite the two inequalities (1) and (2) as one single inequality:

$$\int_a^b \left(\int_a^x f(t) dt \right)^q u(x) dx \le C^q \left(\int_a^b f^p(x) v(x) dx \right)^{\frac{q}{p}}, \tag{6}$$

and the main result reads:

Theorem 1. *Let* $p \in (-\infty, 0) \cup (1, \infty)$, $\frac{q}{p} \in (0,1)$ *and* $s \in (0, \infty)$. *Then inequality (6) holds if and only if*

$$B_K(s) < \infty,$$

and for the best constant we have the following estimates:

- *if* $p \in (-\infty, 0)$, *then*

$$C_1(s) B_K(s) \le C^q \le \left(1 - \frac{sp'}{r}\right)^{-\frac{q}{p'}} s^{\frac{q}{r}} B_K(s), \tag{7}$$

where

$$C_1(s) = \begin{cases} \frac{\frac{q}{r} s}{(\frac{s-1}{p} + 1)^q}, & s \in (0, 1-p), \\ \frac{q}{r}, & s \in [1-p, \infty). \end{cases}$$

- *if* $p \in (1, \infty)$, *then*

$$\frac{\frac{q}{r} s}{(\frac{s-1}{p} + 1)^q} B_K(s) \le C^q \le C_2(s) B_K(s), \tag{8}$$

where

$$C_2(s) = \begin{cases} \left(1 - \frac{sp'}{r}\right)^{-\frac{q}{p'}} s^{\frac{q}{r}}, & s \in (0, \frac{r}{p'}), \\ (p')^q (\frac{p}{r})^{\frac{q}{p}} s, & s \in [\frac{r}{p'}, \infty). \end{cases}$$

Remark. The result mentioned above can be reformulated for the case of the inequalities

$$\left(\int_a^b \left(\int_x^b f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \le C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \qquad (p\in(1,\infty))$$

and

$$\left(\int_a^b \left(\int_x^b f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \ge C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}} \qquad (p\in(-\infty,0)\cup(0,1))$$

by the usual way, replacing, roughly speaking, the functions U and V by the functions $\tilde{U}$ and $\tilde{V}$, respectively, where $\tilde{U}(x)=\int_a^x u(t)dt$, $\tilde{V}(x)=\int_x^b v^{1-p'}(t)dt$. The counterpart of $B_K(s)$ will have the form:

$$\tilde{B}_K(s) := \left(\int_a^b \left(\int_a^x u(z)\tilde{V}^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}} \tilde{V}^{s-1}(x)d(-\tilde{V}(x))\right)^{\frac{q}{r}} \tag{9}$$

and the theorem will be same.

As far as concerns the case $0<\frac{p}{q}<1$, $q\in(0,1)$, we can use the duality principle. By Theorem 3 in [7], inequality (2) is equivalent to the following inequality

$$\left(\int_a^b \left(\int_x^b f(t)dt\right)^{p'} v^{1-p'}(x)dx\right)^{\frac{1}{p'}} \ge C\left(\int_a^b f^{q'}(x)u^{1-q'}(x)dx\right)^{\frac{1}{q'}},$$

where $-\infty<q'<p'<0$. Then our second main result, Theorem 2, follows immediately from Remark.

Theorem 2. *Let $p,q\in(0,1)$, $\frac{p}{q}\in(0,1)$ and $s\in(0,\infty)$. Then inequality (2) holds if and only if*

$$B_K^*(s) := \left(\int_a^b \left(\int_a^x v^{1-p'}(z)U^{p'(\frac{1}{q}-\frac{s}{r})}(z)dz\right)^{\frac{r}{p'}} U^{s-1}(x)d(-U(x))\right)^{\frac{p'}{r}} < \infty.$$

Moreover, for the best possible constant we have

$$\bar{C}_1(s)B_K^*(s) \le C^{p'} \le \left(1-\frac{sq}{r}\right)^{-\frac{p'}{q}} s^{\frac{p'}{r}} B_K^*(s)$$

where

$$\bar{C}_1(s) = \begin{cases} \frac{\frac{p'}{r}s}{(\frac{s-1}{q'}+1)^{p'}}, & s\in(0,1-q'),\\ \frac{p'}{r}, & s\in[1-q',\infty).\end{cases}$$

If we suppose that $p > 1$ and $q/p \in (0,1)$ then $B_K(s) = \mathcal{B}_{MR}(s/r)$ for all $s > 0$. In this sense the condition $B_K(s) < \infty$ extends the Maz'ja–Rozin scale condition $\mathcal{B}_{MR}(s/r) < \infty$ to negative parameters p, q. By the same point of view, if we consider $\mathcal{B}_{PS}(s/r) < \infty$ as an extension of Persson–Stepanov scale condition, then these two extended conditions are in the following relations:

Proposition. *Let $p \in (-\infty, 0) \cup (1, \infty)$, $\frac{q}{p} \in (0,1)$ and $s \in (0, \infty)$. Then the following estimates hold:*

$$\frac{1}{2}\mathcal{B}^q_{PS}(\frac{s}{r}) \le \mathcal{B}^q_{MR}(\frac{s}{r}) \le J + 2\mathcal{B}^q_{PS}(\frac{s}{r}), \tag{10}$$

where $J = \left(\int_a^b \left(\int_t^b uV^{q(\frac{1}{p'}+\frac{s}{r})}\right)^{\frac{r}{q}} V^{s-1}(t)dV(t)\right)^{\frac{q}{r}} V^{-2s\frac{q}{r}}(b)$.

Consequently, if we additionally suppose that $V(b) = \infty$ then (10) takes the form:

$$\frac{1}{2}\mathcal{B}^q_{PS}(\frac{s}{r}) \le \mathcal{B}^q_{MR}(\frac{s}{r}) \le 2\mathcal{B}^q_{PS}(\frac{s}{r}).$$

3 Proofs

Proof of Theorem 1.
(**Necessity**) First, let us denote

$$I = \int_a^b \left(\int_a^x f(t)dt\right)^q u(x)dx,$$
$$J = \int_a^b f^p(x)v(x)dx,$$

then (6) takes the form:

$$I \le C^q J^{\frac{q}{p}}. \tag{11}$$

Let us choose the test function in the form:

$$f(x) = \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{pq}} V^{\frac{s-1}{p}}(x)v^{1-p'}(x).$$

Then we have

$$J = \int_a^b \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}} V^{s-1}(x)dV(x) \tag{12}$$

and

$$I = \int_a^b u(x)\left(\int_a^x \left(\int_t^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{pq}} V^{\frac{s-1}{p}}(t)dV(t)\right)^q dx.$$

Since $\frac{r}{p} > 0$, we have that for $t \in (a,\, x)$

$$\left(\int_t^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{p}} \ge \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{p}}$$

and supposing $\frac{s-1}{p}+1>0$ we get

$$
\begin{aligned}
I &\geq \int_a^b u(x)\left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{p}}\left(\int_a^x V^{\frac{s-1}{p}}(t)dV(t)\right)^q dx\\
&= \frac{1}{(\frac{s-1}{p}+1)^q}\int_a^b u(x)\left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{p}} V^{(\frac{s-1}{p}+1)q}(x)dx\\
&= \frac{1}{(\frac{s-1}{p}+1)^q}\int_a^b \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{p}} V^{s}(x)d\left(-\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)\\
&= \frac{s\frac{q}{r}}{(\frac{s-1}{p}+1)^q}\int_a^b \left[\int_a^x V^{s-1}(t)dV(t)\right] d\left(-\left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}}\right)\\
&= \frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}\int_a^b \left(\int_t^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}} V^{s-1}(t)dV(t),
\end{aligned}
$$

where we used Fubini's theorem in the last step. Consequently, we have

$$
I \;\geq\; \frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}\int_a^b \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}} V^{s-1}(x)dV(x) = \frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}J.
$$

This implies together with (11) and (12) that $\frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}J \leq C^q J^{\frac{q}{p}}$ or $\frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}J^{1-\frac{q}{p}} \leq C^q$, i.e.

$$
\frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}\left(\int_a^b \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}} V^{s-1}(x)dV(x)\right)^{\frac{q}{r}} \leq C^q,
$$

i.e.

$$
\frac{\frac{q}{r}s}{(\frac{s-1}{p}+1)^q}B_K(s) \leq C^q. \tag{13}
$$

To get the last estimate we supposed that $\frac{s-1}{p}+1>0$, which holds for all $p\in(1,\infty)$ and $s\in(0,\infty)$. This is true also for negative values of p if $s\in(0,1-p)$. To prove the necessity for parameters p negative and $s\in[1-p,\infty)$ we proceed as follows: First we use (13) for $s=1$, which implies the necessity of $B_K(1)$, and by using the monotonicity of the function $V(t)$ we estimate $B_K(1)$ by $B_K(s)$ from below, i.e.

$$
\begin{aligned}
B_K(1) &= \left(\int_a^b \left(\int_x^b u(z)V^{q-1}(z)dz\right)^{\frac{r}{q}} dV(x)\right)^{\frac{q}{r}}\\
&= \left(\int_a^b \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)V^{(s-1)\frac{q}{r}}(z)dz\right)^{\frac{r}{q}} dV(x)\right)^{\frac{q}{r}}\\
&\geq \left(\int_a^b \left(\int_x^b u(z)V^{q(\frac{1}{p'}-\frac{s}{r})}(z)dz\right)^{\frac{r}{q}} V^{s-1}(x)dV(x)\right)^{\frac{q}{r}} = B_K(s).
\end{aligned}
$$

This estimate and (13) imply

$$\frac{q}{r}B_K(s) \leq \frac{q}{r}B_K(1) \leq C^q,$$

where $s \in [1-p, \infty)$. The necessity part is completed.

(Sufficiency) Let be $p \in (-\infty, 0) \cup (1, \infty)$ and α a real positive parameter such that

$$1 - \frac{\alpha p'}{r} > 0. \tag{14}$$

Then the left hand side of inequality (11) can be estimated as follows:

$$\begin{aligned}
I &= \int_a^b \left(\int_a^x \left[V^{-\frac{\alpha}{r}}(t) v^{-\frac{1}{p}}(t) \right] \left[V^{\frac{\alpha}{r}}(t) v^{\frac{1}{p}}(t) f(t) \right] dt \right)^q u(x)dx \\
&\leq \int_a^b \left(\int_a^x V^{-\frac{\alpha p'}{r}}(t) v^{1-p'}(t)dt \right)^{\frac{q}{p'}} \left(\int_a^x V^{\frac{\alpha p}{r}}(t)v(t)f^p(t)dt \right)^{\frac{q}{p}} u(x)dx \\
&= \left(1 - \frac{\alpha p'}{r}\right)^{-\frac{q}{p'}} \int_a^b \left(\int_a^x V^{\frac{\alpha p}{r}}(t)v(t)f^p(t)dt \right)^{\frac{q}{p}} u(x) V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(x)dx \\
&= \left(1 - \frac{\alpha p'}{r}\right)^{-\frac{q}{p'}} \left[\left(\int_a^b \left(\int_a^x V^{\frac{\alpha p}{r}}(t)v(t)f^p(t)dt \right)^{\frac{q}{p}} u(x) V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(x)dx \right)^{\frac{p}{q}} \right]^{\frac{q}{p}} \\
&\leq \left(1 - \frac{\alpha p'}{r}\right)^{-\frac{q}{p'}} \alpha^{\frac{q}{r}} B_K(\alpha) \left(\int_a^b f^p(t)v(t)dt \right)^{\frac{q}{p}}.
\end{aligned} \tag{15}$$

Hence, it follows that the best constant satisfies:

$$C^q \leq \left(1 - \frac{\alpha p'}{r}\right)^{-\frac{q}{p'}} \alpha^{\frac{q}{r}} B_K(\alpha). \tag{16}$$

To get (15) we denote $g(t) = V^{\frac{\alpha p}{r}}(t)v(t)f^p(t)$ and use the following inequality:

$$\left(\int_a^b \left(\int_a^x g(t)dt \right)^{\frac{q}{p}} V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(x)u(x)dx \right)^{\frac{p}{q}} \leq \bar{C} \int_a^b g(t) V^{-\frac{\alpha p}{r}}(t)dt. \tag{17}$$

Let us show that this inequality holds, and investigate the constant $\bar{C}$.

Inequality (17) is a special case of the Hardy inequality and its validity is characterized by the finiteness of the following integral A_3 (see Theorem 5 in [3]):

$$A_3 = \left(\int_a^b \left(\int_x^b u(t)V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(t)dtV^{\frac{\alpha p}{r}}(x)\right)^{\frac{\frac{q}{p}}{1-\frac{q}{p}}} u(x)V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(x)dx\right)^{\frac{p}{q}-1}$$

$$= \left(\int_a^b \left(\int_x^b u(t)V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(t)dt\right)^{\frac{r}{p}} u(x)V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(x)V^{\alpha}(x)dx\right)^{\frac{p}{r}}$$

$$= \left(\frac{q\alpha}{r}\right)^{\frac{p}{r}}\left(\int_a^b \left(\int_a^x V^{\alpha-1}(t)dV(t)\right) d\left\{-\left(\int_x^b u(t)V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(t)dt\right)^{\frac{r}{q}}\right\}\right)^{\frac{p}{r}}$$

$$= \left(\frac{q\alpha}{r}\right)^{\frac{p}{r}}\left(\int_a^b \left(\int_x^b u(t)V^{(1-\frac{\alpha p'}{r})\frac{q}{p'}}(t)dt\right)^{\frac{r}{q}} V^{\alpha-1}(x)dV(x)\right)^{\frac{p}{r}}$$

$$= (\frac{q\alpha}{r})^{\frac{p}{r}} B_K^{\frac{p}{q}}(\alpha)$$

and for the constant $\bar{C}$ we have the estimate:

$$\bar{C} \le (1-\frac{q}{p})^{1-\frac{p}{q}} A_3 \le (\frac{q}{r})^{-\frac{p}{r}}(\frac{q\alpha}{r})^{\frac{p}{r}} B_K^{\frac{p}{q}}(\alpha) = \alpha^{\frac{p}{r}} B_K^{\frac{p}{q}}(\alpha).$$

So, we have the condition $B_K(\alpha) < \infty$ with α satisfying (14). If p is negative then (14) holds for all positive α's and in this case we have the sufficient condition if we replace α in (15) by s. In the case $p > 1$ the condition (14) holds if $\alpha \in (0, \frac{r}{p'})$. Again replacing α by s we get the sufficient condition for $(0, \frac{r}{p'})$. In the remaining case where $s \in [\frac{r}{p'}, \infty)$ we choose $\alpha = s - \varepsilon$, with arbitrary $\varepsilon \in (s - \frac{r}{p'}, s)$. By these choice of parameters we have condition (14) for α and

$$B_K(\alpha) = B(s-\varepsilon) = \left(\int_a^b \left(\int_x^b u(t)V^{(1-\frac{sp'}{r})\frac{q}{p'}+\frac{\varepsilon q}{r}}(t)dt\right)^{\frac{r}{q}} V^{s-\varepsilon-1}(x)dV(x)\right)^{\frac{q}{r}} \tag{18}$$

Now we show that

$$B(s-\varepsilon) \le \tilde{C}B(s),$$

where $\tilde{C}$ is a constant which depends on s and ε. To this aim we first estimate the inner integral in (18) in the following form:

$$\int_x^b u(t)V^{(1-\frac{sp'}{r})\frac{q}{p'}+\frac{\varepsilon q}{r}}(t)dt$$

$$= \int_x^b V^{\frac{\varepsilon q}{r}}(t)d\left(-\int_t^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau\right)$$

$$= \int_x^b V^{\frac{\varepsilon q}{r}}(t)d\left(-\int_t^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau\right)$$

$$= V^{\frac{\varepsilon q}{r}}(x)\int_x^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau + \frac{\varepsilon q}{r}\int_x^b \left(\int_t^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau\right) V^{\frac{\varepsilon q}{r}-1}(t)dV(t)$$

$$= I_1(x) + \frac{\varepsilon q}{r} I_2(x). \tag{19}$$

Let us denote $\gamma = ((s-\varepsilon)\frac{q}{r}+1)\frac{q}{p}$. Using Hölder's inequality with exponents $\frac{r}{q}$ and $\frac{p}{q}$ we estimate $I_2(x)$ in the form:

$$\begin{aligned} I_2(x) &= \int_x^b \left[\left(\int_t^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau\right) V^{\frac{\varepsilon q}{r}+\gamma-1}(t)\right]\left[V^{-\gamma}\right] dV(t) \\ &\le \left(\int_x^b \left[\left(\int_t^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau\right) V^{\frac{\varepsilon q}{r}+\gamma-1}(t)\right]^{\frac{r}{q}} dV(t)\right)^{\frac{q}{r}} \left(\int_x^b V^{-\frac{\gamma p}{q}} dV(t)\right)^{\frac{q}{p}} \\ &\le \frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}} \left(\int_x^b \left(\int_t^b u(\tau)V^{(1-\frac{sp'}{r})\frac{q}{p'}}(\tau)d\tau\right)^{\frac{r}{q}} V^{(\frac{\varepsilon q}{r}+\gamma-1)\frac{r}{q}}(t)dV(t)\right)^{\frac{q}{r}} V^{(1-\frac{\gamma p}{q})\frac{q}{p}}(x) \\ &= \frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}} I_3(x), \end{aligned}$$

which together with (19) implies

$$\int_x^b u(t)V^{(1-\frac{sp'}{r})\frac{q}{p'}+\frac{\varepsilon q}{r}}(t)dt \le I_1(x) + \frac{\varepsilon q}{r}\frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}} I_3(x).$$

Using this estimate, we obtain

$$\begin{aligned} B(s-\varepsilon) &= \left(\int_a^b \left(\int_x^b u(t)V^{(1-\frac{sp'}{r})\frac{q}{p'}+\frac{\varepsilon q}{r}}(t)dt\right)^{\frac{r}{q}} V^{s-\varepsilon-1}(x)dV(x)\right)^{\frac{q}{r}} \\ &\le \left(\int_a^b \left(I_1(x) + \frac{\varepsilon q}{r}\frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}} I_3(x)\right)^{\frac{r}{q}} V^{s-\varepsilon-1}(x)dV(x)\right)^{\frac{q}{r}} \\ &\le \left(\int_a^b I_1(x)^{\frac{r}{q}} V^{s-\varepsilon-1}(x)dV(x)\right)^{\frac{q}{r}} + \frac{\varepsilon q}{r}\frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}} \left(\int_a^b I_3(x)^{\frac{r}{q}} V^{s-\varepsilon-1}(x)dV(x)\right)^{\frac{q}{r}} \\ &\le B_K(s) + \frac{\varepsilon q}{r}\frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}(\frac{r}{p}-\gamma\frac{r}{q}+s-\varepsilon)^{\frac{q}{r}}} B_K(s) \\ &= (1 + \frac{\varepsilon q}{r}\frac{1}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}(\frac{r}{p}-\gamma\frac{r}{q}+s-\varepsilon)^{\frac{q}{r}}}) B_K(s), \end{aligned} \tag{20}$$

where we used the triangle inequality for the norm. From (20), (16) and from the definition of γ we get the estimate for the best constant:

$$\begin{aligned} C^q &\le \left(1 - \frac{(s-\varepsilon)p'}{r}\right)^{-\frac{q}{p'}} (s-\varepsilon)^{\frac{q}{r}} B_K(s-\varepsilon) \\ &\le \left(1 - \frac{(s-\varepsilon)p'}{r}\right)^{-\frac{q}{p'}} (s-\varepsilon)^{\frac{q}{r}} (1 + \frac{\frac{\varepsilon q}{r}}{(\frac{\gamma p}{q}-1)^{\frac{q}{p}}(\frac{r}{p}-\gamma\frac{r}{q}+s-\varepsilon)^{\frac{q}{r}}}) B_K(s) \\ &= \left(1 - \frac{\alpha p'}{r}\right)^{-\frac{q}{p'}} \alpha^{-\frac{q}{p}} s B_K(s). \end{aligned}$$

Consequently, we have that

$$C^q \le \left[\inf_{\alpha\in(0,\frac{r}{p'})}\left(1-\frac{\alpha p'}{r}\right)^{-\frac{q}{p'}}\alpha^{-\frac{q}{p}}\right] sB_K(s) = (p')^q(\frac{p}{r})^{\frac{q}{p}} sB_K(s),$$

where $s \in [\frac{r}{p'},\infty)$. The proof of sufficiency is completed. □

Proof of Proposition. Let $s \in (0,\infty)$ be a real parameter and $\varepsilon = \frac{q}{p}(1-\frac{q}{r}s)$. To prove the assertion we first show that

$$\mathcal{B}^q_{PS}(\frac{s}{r}) \le 2\mathcal{B}^q_{MR}(\frac{s}{r}). \tag{21}$$

For this aim, we estimate the inner integral of $\mathcal{B}^q_{PS}(\frac{s}{r})$ as follows, where we first use integration by parts and then the Hölder inequality with exponents $\frac{r}{q}$ and $\frac{p}{q}$:

$$\begin{aligned}
I_1(t) &= \int_a^t u(\tau)V^{q(\frac{1}{p'}+\frac{s}{r})}(\tau)d\tau \\
&= \int_a^t V^{2\frac{qs}{r}}(\tau)d\left(-\int_\tau^t uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right) \\
&= 2\frac{qs}{r}\int_a^t\left[\int_\tau^t uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]V^{2\frac{qs}{r}-1}(\tau)dV(\tau) \\
&= 2\frac{qs}{r}\int_a^t\left(\left[\int_\tau^t uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]V^{2\frac{qs}{r}-1+\varepsilon}(\tau)\right)V^{-\varepsilon}(\tau)dV(\tau) \\
&\le 2\frac{qs}{r}\left(\int_a^t\left[\int_\tau^t uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]^{\frac{r}{q}}V^{(2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)dV(\tau)\right)^{\frac{q}{r}}\left(\int_a^t V^{-\frac{\varepsilon p}{q}}(\tau)dV(\tau)\right)^{\frac{q}{p}} \\
&\le \frac{2\frac{qs}{r}}{(1-\varepsilon\frac{p}{q})^{\frac{q}{p}}}\left(\int_a^t\left[\int_\tau^b uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]^{\frac{r}{q}}V^{(2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)dV(\tau)\right)^{\frac{q}{r}}V^{(1-\frac{\varepsilon p}{q})\frac{q}{p}}(t) \\
&= \frac{2\frac{qs}{r}}{(1-\varepsilon\frac{p}{q})^{\frac{q}{p}}}I_2(t).
\end{aligned}$$

Using this result we can estimate $\mathcal{B}^q_{PS}(s)$ in the form:

$$\begin{aligned}
\mathcal{B}^q_{PS}(\frac{s}{r}) &= \left(\int_a^b I_1(t)^{\frac{r}{q}}V^{-s-1}(t)dV(t)\right)^{\frac{q}{r}} \le \\
&\le \frac{2\frac{qs}{r}}{(1-\varepsilon\frac{p}{q})^{\frac{q}{p}}}\left(\int_a^b I_2(t)^{\frac{r}{q}}V^{-s-1}(t)dV(t)\right)^{\frac{q}{r}} \le \frac{2\frac{qs}{r}}{(1-\varepsilon\frac{p}{q})^{\frac{q}{p}}}\times \\
&\times \left(\int_a^b\left(\int_a^t\left[\int_\tau^b uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]^{\frac{r}{q}}V^{(2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)dV(\tau)\right)V^{(1-\varepsilon\frac{p}{q})\frac{r}{p}-s-1}(t)dV(t)\right)^{\frac{q}{r}} =
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2\frac{qs}{r}}{(1-\varepsilon\frac{p}{q})^{\frac{q}{p}}}\left(\int_a^b\left[\int_\tau^b uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]^{\frac{r}{q}} V^{(2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)\times\right.\\
&\times \left.\left(\int_\tau^b V^{(1-\varepsilon\frac{p}{q})\frac{r}{p}-s-1}(t)dV(t)\right)dV(\tau)\right)^{\frac{q}{r}} \le\\
&\le \frac{2\frac{qs}{r}}{(1-\varepsilon\frac{p}{q})^{\frac{q}{p}}(\varepsilon\frac{r}{q}+s-\frac{r}{p})^{\frac{q}{r}}}\left(\int_a^b\left[\int_\tau^b uV^{q(\frac{1}{p'}-\frac{s}{r})}dz\right]^{\frac{r}{q}} V^{s-1}(\tau)dV(\tau)\right)^{\frac{q}{r}}.
\end{aligned}
$$

This estimate and the definition of ε imply (21).

Now we show the second part, i.e.

$$\mathcal{B}^q_{MR}(\frac{s}{r}) \le J + 2\mathcal{B}^q_{PS}(\frac{s}{r}).$$

Using integration by parts we easily estimate the inner integral of $\mathcal{B}^q_{MR}(\frac{s}{r})$ in the form:

$$
\begin{aligned}
&J_1(t) = \int_t^b u(\tau)V^{q(\frac{1}{p'}-\frac{s}{r})}(\tau)d\tau\\
&= \int_t^b V^{-2\frac{qs}{r}}(\tau)d\left(\int_t^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right)\\
&= V^{-2\frac{qs}{r}}(b)\left(\int_t^b uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right) + 2\frac{qs}{r}\int_t^b\left[\int_t^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right]V^{-2\frac{qs}{r}-1}(\tau)dV(\tau)\\
&\le V^{-2\frac{qs}{r}}(b)\left(\int_t^b uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right) + 2\frac{qs}{r}\int_t^b\left[\int_a^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right]V^{-2\frac{qs}{r}-1}(\tau)dV(\tau)\\
&= J_2(t) + 2\frac{qs}{r}J_3(t).
\end{aligned}
$$

Denoting $\varepsilon = \frac{q}{r}(\frac{q}{p}s+\frac{r}{p})$ and using Hölder's inequality we have:

$$
\begin{aligned}
&J_3(t) = \int_t^b\left[V^{-2\frac{qs}{r}-1+\varepsilon}(\tau)\int_a^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right]V^{-\varepsilon}(\tau)dV(\tau)\\
&\le \left(\int_t^b\left[\int_a^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right]^{\frac{r}{q}}V^{(-2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)dV(\tau)\right)^{\frac{q}{r}}\left(\int_t^b V^{-\varepsilon\frac{p}{q}}(\tau)dV(\tau)\right)^{\frac{q}{p}}\\
&\le \frac{1}{(\varepsilon\frac{p}{q}-1)^{\frac{q}{p}}}\left(\int_t^b\left[\int_a^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\right]^{\frac{r}{q}}V^{(-2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)dV(\tau)\right)^{\frac{q}{r}}V^{(1-\varepsilon\frac{p}{q})\frac{q}{p}}(t)\\
&= \frac{1}{(\varepsilon\frac{p}{q}-1)^{\frac{q}{p}}}J_4(t).
\end{aligned}
$$

Consequently, we have

$$J_1(t) \le J_2(t) + \frac{2\frac{qs}{r}}{(\varepsilon\frac{p}{q}-1)^{\frac{q}{p}}}J_4(t).$$

From this estimate we obtain:

$$\begin{aligned}
\mathcal{B}^q_{MR}(\frac{s}{r}) &= \Big(\int_a^b J_1(t)^{\frac{r}{q}} V^{s-1}(t)dV(t)\Big)^{\frac{q}{r}} \\
&= \left(\int_a^b \left[J(t) + \frac{2^{\frac{qs}{r}}}{(\varepsilon^{\frac{p}{q}}-1)^{\frac{q}{p}}} J_4(t)\right]^{\frac{r}{q}} V^{s-1}(t)dV(t)\right)^{\frac{q}{r}} \\
&\le \Big(\int_a^b J_2(t)^{\frac{r}{q}} V^{s-1}(t)dV(t)\Big)^{\frac{q}{r}} + \frac{2^{\frac{qs}{r}}}{(\varepsilon^{\frac{p}{q}}-1)^{\frac{q}{p}}} \Big(\int_a^b J_4(t)^{\frac{r}{q}} V^{s-1}(t)dV(t)\Big)^{\frac{q}{r}} \\
&= J + \frac{2^{\frac{qs}{r}}}{(\varepsilon^{\frac{p}{q}}-1)^{\frac{q}{p}}} J_4,
\end{aligned}$$

where

$$J = \left(\int_a^b \Big(\int_t^b uV^{q(\frac{1}{p'}+\frac{s}{r})}\Big)^{\frac{r}{q}} V^{s-1}(t)dV(t)\right)^{\frac{q}{r}} V^{-2s\frac{q}{r}}(b)$$

and by Fubini's theorem we rewrite J_4 in the form:

$$\begin{aligned}
J_4 &= \left(\int_a^b \left(\int_t^b \Big[\int_a^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\Big]^{\frac{r}{q}} V^{(-2\frac{qs}{r}-1+\varepsilon)\frac{r}{q}}(\tau)dV(\tau)\right) V^{(1-\frac{\varepsilon p}{q})\frac{r}{p}+s-1}(t)dV(t)\right)^{\frac{q}{r}} \\
&= \frac{1}{((1-\varepsilon^{\frac{p}{q}})\frac{r}{p}+s)^{\frac{q}{r}}} \left(\int_a^b \Big[\int_a^\tau uV^{q(\frac{1}{p'}+\frac{s}{r})}dz\Big]^{\frac{r}{q}} V^{-s-1}(\tau)dV(\tau)\right)^{\frac{q}{r}} \\
&= \frac{\mathcal{B}^q_{PS}(\frac{s}{r})}{((1-\varepsilon^{\frac{p}{q}})\frac{r}{p}+s)^{\frac{q}{r}}}.
\end{aligned}$$

From these and from the definition of ε we have:

$$\mathcal{B}^q_{MR}(\frac{s}{r}) \le J + 2\mathcal{B}^q_{PS}(\frac{s}{r}).$$

□

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