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EQUICONVERGENCE THEOREMS FOR STURM–LIOVILLE OPERATORS WITH SINGULAR POTENTIALS (RATE OF EQUICONVERGENCE IN W_2^θ–NORM)

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Abstract. We study the Sturm–Liouville operator $Ly = l(y) = -\dfrac{d^2y}{dx^2} + q(x)y$ with Dirichlet boundary conditions $y(0) = y(\pi) = 0$ in the space $L_2[0, \pi]$. We assume that the potential has the form $q(x) = u'(x)$, where $u \in W_2^\theta[0, \pi]$ with $0 < \theta < 1/2$. Here $W_2^\theta[0, \pi] = [L_2, W_2^1]_\theta$ is the Sobolev space. We consider the problem of equiconvergence in $W_2^\theta[0, \pi]$–norm of two expansions of a function $f \in L_2[0, \pi]$. The first one is constructed using the system of the eigenfunctions and associated functions of the operator L. The second one is the Fourier expansion in the series of sines. We show that the equiconvergence holds for any function f in the space $L_2[0, \pi]$.

1 Introduction

In this paper we deal with the Sturm–Liouville operator

$$Ly = l(y) = -\frac{d^2y}{dx^2} + q(x)y, \tag{1}$$

with Dirichlet boundary conditions $y(0) = y(\pi) = 0$ in the space $L_2[0, \pi]$. We assume that the potential q is complex-valued and has the form $q(x) = u'(x)$, where $u \in W_2^\theta[0, \pi]$ with $0 < \theta < 1/2$. Here the derivative is treated in the distributional sense, and $W_2^\theta[0, \pi] = [L_2, W_2^1]_\theta$ is the Sobolev space with fractional order of smoothness defined by interpolation. This class of operators was defined in the papers of A.M. Savchuk and A.A. Shkalikov [7]–[9]. In particular, it was shown there that L is bounded from below and has purely discrete spectrum.

We consider the problem of equiconvergence in $W_2^\theta[0, \pi]$–norm of two expansion of a function $f \in L_2[0, \pi]$. The first one is constructed using the system of the eigenfunctions and associated functions of the operator L, while the second one is the Fourier expansion in the series of sines.

The problem of uniform equiconvergence (i.e. in the norm of the space $C[0,\pi]$) is well studied in the classical theory of Sturm–Liouville operators for regular potentials $q \in L_1(0,\pi)$. In the monograph of V.A. Marchenko [4] the uniform equiconvergence was proved for any function $f \in L_2[0,\pi]$. In 1991 V.A. Il'in [2] proved this result for any $f \in L_1[0,\pi]$. V.A. Vinokurov and V.A. Sadovnichii [10] proved a theorem on the equiconvergence for Sturm–Liouville operators whose potential is the derivative of a function of bounded variation (and also for any $f \in L_1[0,\pi]$).

The problem of the rate of equiconvergence (for regular potentials) was studied in the paper of A.M. Gomilko and G.V. Radzievskii [1]. In the author's paper [5] results were obtained in the case in which $q = u'$, $u \in W_2^\theta[0,\pi]$ with $0 < \theta < 1/2$. It was shown that for any $f \in L_2[0,\pi]$ one can estimate the rate of equiconvergence uniformly over the ball $u \in B_{\theta,R} = \{v \in W_2^\theta[0,\pi] : \|v\|_{W_2^\theta} \leqslant R\}$. This result is new even in the classical case $q \in L_2[0,\pi]$.

In this paper we prove equiconvergence and obtain a similar estimate of the rate of equiconvergence in the norm of the space $W_2^\theta[0,\pi]$ with $0 < \theta < 1/2$.

2 Preliminary results

We begin with the following results about operator (1).

Let us recall that the operator L is bounded from below and has purely discrete spectrum. Let $\{\lambda_n\}_1^\infty$ be the sequence of all eigenvalues of the operator L. We shall enumerate the eigenvalues in such a way that $|\lambda_1| \leqslant |\lambda_2| \leqslant \dots$, and we assume that each eigenvalue is repeated as many times as its algebraic multiplicity. By $\{y_n\}_1^\infty$ we denote the system of the eigenfunctions and associated functions. We assume that the function y_n corresponds to the eigenvalue λ_n and $\|y_n\|_{L_2} = 1$ if y_n is an eigenfunction.

Statement 1 (A.M. Savchuk, A.A. Shkalikov)**.** *Let $u \in L_2[0,\pi]$. Then the system $\{y_n\}_1^\infty$ of the eigenfunctions and associated functions of the operator L forms a Riesz basis in the space $L_2[0,\pi]$.*

By the above there exists the biorthogonal system $\{w_n\}_1^\infty$, i.e. $(y_n, w_m) = \delta_{nm}$ where $(f,g) = \int\limits_0^\pi f(x)\overline{g(x)}dx$. For more details see [5].

Let

$$l_2^\theta = \left\{ x = \{x_n\}_{n=1}^\infty \colon \ \left(\sum_{n=1}^\infty |x_n|^2 n^{2\theta} \right)^{1/2} = \|\{x_n\}\|_{l_2^\theta} < \infty \right\}.$$

We recall that for any $u \in W_2^\theta[0,\pi]$ with $0 < \theta < 1/2$

$$C_1 \|\{u_n\}\|_{l_2^\theta} \leqslant \|u\|_{W_2^\theta} \leqslant C_2 \|\{u_n\}\|_{l_2^\theta}$$

where $u_n = \sqrt{\dfrac{2}{\pi}}(u(x), \sin nx)$ and C_1, $C_2 > 0$ are independent of u.

Statement 2 (A.M. Savchuk, A.A. Shkalikov)**.** *Let $R > 0$, $0 < \theta < 1/2$, and $u \in B_{\theta,R}$. Then there exists a natural number* [2] *$N = N_{\theta,R}$ such that for all $n \geqslant N$ the eigenvalues λ_n of the operator L are simple,*

$$\begin{aligned} y_n(x) &= \sqrt{\tfrac{2}{\pi}}\sin nx + \varphi_n(x), \quad w_n(x) = \sqrt{\tfrac{2}{\pi}}\sin nx + \psi_n(x), \\ y_n'(x) &= n\left(\sqrt{\tfrac{2}{\pi}}\cos nx + \eta_n(x)\right) + u(x)\left(\sqrt{\tfrac{2}{\pi}}\sin nx + \varphi_n(x)\right), \end{aligned} \tag{2}$$

where the functions φ_n, ψ_n and η_n are such that the sequence $\{\gamma_n\}_{n=N}^{\infty} = \{\|\varphi_n(x)\|_C + \|\psi_n(x)\|_C + \|\eta_n(x)\|_C\}_{n=N}^{\infty} \in l_2^{\theta}$ and its norm in this space is bounded by a quantity depending only on θ and R. Moreover for $n \geqslant N$

$$\psi_n(x) = \psi_{n,0}(x) + \psi_{n,1}(x),$$

where

$$\psi_{n,0}(x) = \alpha_n \sin nx + \beta_n x \cos nx - \sqrt{\frac{2}{\pi}}\int_0^x \overline{u(t)}\sin n(x-2t)dt, \tag{3}$$

and the numbers α_n, β_n and the functions $\psi_{n,1}$ are such that the norm of the sequence $\{|\alpha_n| + |\beta_n|\}_{n=N}^{\infty}$ in the space l_2^{θ} is bounded by a quantity depending only on θ and R, and the sequence $\{\|\psi_{n,1}(x)\|_C\}_{n=N}^{\infty}$ belongs to the space l_1^{τ} for any $\tau < 2\theta$.

Statement 3. *Let $R > 0$, $0 < \theta < 1/2$ and $u \in B_{\theta,R}$. Then there exists a natural number $N = N_{\theta,R}$ such that for all $n \geqslant N$ the operator*

$$P_n(u):\, L_2[0,\pi] \to W_2^1[0,\pi], \qquad P_n(u):\, f \mapsto \sum_{k=1}^{n}(f, w_k)y_k,$$

is continuous on $B_{\theta,R}$, i.e. for any $u_0 \in B_{\theta,R}$

$$\|P_n(u) - P_n(u_0)\|_{L_2 \to W_2^1} \to 0 \text{ as } \|u - u_0\|_{W_2^{\theta}} \to 0, \quad u \in B_{\theta,R}.$$

This statement follows from the results of the paper [8] (Theorem 1.9) and classical theorems (see [3, Theorems IV.2.23 and IV.3.16]). For detailed proof see [6].

3 Main theorem

Theorem. *Let $R > 0$, $0 < \theta < 1/2$. Consider operator (1) acting in the space$L_2[0,\pi]$ with the homogeneous Dirichlet boundary conditions. Suppose that the complex-valued potential $q(x) = u'(x)$, where $u(x) \in B_{\theta,R}$.*

Let $\{y_n(x)\}_{n=1}^{\infty}$ be the system of the eigenfunctions and associated functions of the operator L and $\{w_n(x)\}_{n=1}^{\infty}$ be the biorthogonal system.

[2]Here and in the sequel when we write $N_{\theta,R}$, $C_{\theta,R}$ etc we mean that these quantities depend only on θ and R.

For an arbitrary function $f \in L_2[0,\pi]$ denote

$$c_n := (f(x), w_n(x)), \quad c_{n,0} := \sqrt{2/\pi}(f(x), \sin nx).$$

Then there exist a natural number $M = M_{\theta,R}$ and a positive number $C = C_{\theta,R}$ such that for all $m \geqslant M$ and for all $f \in L_2[0,\pi]$

$$\left\| \sum_{n=1}^{m} c_n y_n(x) - \sum_{n=1}^{m} \sqrt{\frac{2}{\pi}} c_{n,0} \sin nx \right\|_{W_2^\theta} \leqslant C \left(\sqrt{\sum_{n \geqslant m^{1-\theta}} |c_{n,0}|^2} + \frac{\|f\|_{L_2}}{m^{\theta(1-\theta)}} \right). \qquad (4)$$

Proof.

Step 1 (operators $B_{m,N}$ and B_m).

For given integers m, $m \geqslant N$, we introduce the operator $B_{m,N} : L_2[0,\pi] \to W_2^\theta[0,\pi]$ defined by

$$B_{m,N} f(x) := \sum_{n=N}^{m} c_n y_n(x) - \sum_{n=N}^{m} \sqrt{\frac{2}{\pi}} c_{n,0} \sin nx$$

and denote $B_{m,1} =: B_m$. Then for any function $f \in L_2[0,\pi]$ the following equality holds:

$$B_{m,N} f(x) = \sum_{n=N}^{m} \sqrt{\frac{2}{\pi}} (f(t), \psi_n(t)) \sin nx +$$

$$+ \sum_{n=N}^{m} \sqrt{\frac{2}{\pi}} (f(t), \sin nt) \varphi_n(x) + \sum_{n=N}^{m} (f(t), \psi_n(t)) \varphi_n(x). \qquad (5)$$

Let $N = N_{\theta,R}$ be the maximal of the positive integers defined in Statements 2 and 3.

Step 2 (estimation of the norm of the operator $B_{m,N}$).

There exists a positive number $C_{\theta,R}$ such that for all natural m satisfying $m \geqslant N$

$$\|B_{m,N}(u)\|_{L_2 \to W_2^\theta} \leqslant C_{\theta,R}. \qquad (6)$$

Let us estimate each term of the right-hand side in (5) separately. By asymptotic formulae (2) we have:

$$\left\| \sum_{n=N}^{m} (f(t), \psi_n(t)) \sin nx \right\|_{W_2^\theta} \leqslant C_1 \left(\sum_{n=N}^{m} |(f(t), \psi_n(t))|^2 n^{2\theta} \right)^{1/2} \leqslant$$

$$\leqslant C_1 \left(\sum_{n=N}^{m} \|f\|_{L_2}^2 \|\psi_n\|_{L_2}^2 n^{2\theta} \right)^{1/2} \leqslant C_2 \|f\|_{L_2},$$

where C_1, $C_2 > 0$ are independent of f. Consider the second and the third terms. We will show that $\{\|\varphi_n\|_{W_2^\theta}\}_{n=N}^{\infty} \in l_2$ provided that $0 < \theta < 1/2$. Asymptotics (2) imply the estimate

$$\|\varphi_n\|_{W_2^\theta} \leqslant C_3 \|\varphi_n\|_{L_2}^{1-\theta} \|\varphi_n\|_{W_2^1}^{\theta} \leqslant C_4 \|\varphi_n\|_{L_2}^{1-\theta} \|\varphi_n'\|_{L_2}^{\theta} \leqslant$$

$$\leqslant C_4\|\varphi_n\|_{L_2}^{1-\theta}\left(\|\eta_n\|_{L_2}^{\theta}n^{\theta}+\left(\sqrt{\frac{2}{\pi}}\right)^{\theta}\|u\|_{L_2}^{\theta}+\|u\varphi_n\|_{L_2}^{\theta}\right)$$

$$\leqslant C_5\|\varphi_n\|_{L_2}^{1-\theta}\left(\|\eta_n\|_{L_2}^{\theta}n^{\theta}+1\right)=$$

$$=C_5\left(\|\varphi_n\|_{L_2}n^{\theta}\right)^{1-\theta}\left((\|\eta_n\|_{L_2}n^{\theta})^{\theta}+(n^{\theta-1})^{\theta}\right), \tag{7}$$

where C_3, C_4, $C_5>0$ are independent of n. Here we used the inequality $\|f\|_{W_2^1}\leqslant\sqrt{\pi^2+1}\|f'\|_{L_2}$ fulfilled for all $f\in W_2^1[0,\pi]$ satisfying $f(0)=0$. Next we use Hölder's inequality

$$\sum_{n=N}^{m}(a_n^{1-\theta}b_n^{\theta})^2=\sum_{n=N}^{m}(a_n^2)^{1-\theta}(b_n^2)^{\theta}\leqslant\left(\sum_{n=N}^{m}a_n^2\right)^{1-\theta}\left(\sum_{n=N}^{m}b_n^2\right)^{\theta}$$

and note that the sequences $\{\|\varphi_n\|_{L_2}n^{\theta}\}_{n=N}^{\infty}$, $\{\|\eta_n\|_{L_2}n^{\theta}\}_{n=N}^{\infty}$, and $\{n^{\theta-1}\}_{n=1}^{\infty}$ belong to l_2 space for any $0<\theta<1/2$. Therefore (7) implies the inequality

$$\sum_{n=N}^{m}\|\varphi_n\|_{W_2^{\theta}}^2\leqslant C_6, \tag{8}$$

where $C_6>0$ is independent of m and N. Then

$$\left\|\sum_{n=N}^{m}(f(t),\sin nt)\varphi_n(x)\right\|_{W_2^{\theta}}\leqslant\sum_{n=N}^{m}|f_n|\|\varphi_n\|_{W_2^{\theta}}\leqslant C_7\|f\|_{L_2}, \tag{9}$$

where $f_n=(f(t),\sin nt)$ and $C_7=\sqrt{C_6}$.

For the last term of the right-hand side in (5) we have:

$$\left\|\sum_{n=N}^{m}(f(t),\psi_n(t))\varphi_n(x)\right\|_{W_2^{\theta}}\leqslant\sum_{n=N}^{m}\|f\|_{L_2}\|\psi_n\|_{L_2}\|\varphi_n\|_{W_2^{\theta}}\leqslant$$

$$\leqslant\|f\|_{L_2}\left(\sum_{n=N}^{m}\|\psi_n\|_{L_2}^2\right)^{1/2}\left(\sum_{n=N}^{m}\|\varphi_n\|_{L_2}^2\right)^{1/2}\leqslant C_8\|f\|_{L_2}, \tag{10}$$

where $C_8>0$ is independent of m and N, because the conditions of our theorem imply that $\{\|\psi_n\|_{L_2}\}_{n=N}^{\infty}\in l_2$.

We proved inequality (6). Step 2 is completed.

Step 3 (estimation of the norm of the operator B_m).

There exists a positive number $C_{\theta,R}$ such that for all natural m satisfying $m\geqslant N$

$$\|B_m(u)\|_{L_2\to W_2^{\theta}}\leqslant C_{\theta,R}. \tag{11}$$

Obviously the operator $B_m(u)$ can be represented as the sum:

$$B_m(u)=B_{N-1}(u)+B_{m,N}(u)=P_{N-1}(u)-P_{N-1}(0)+B_{m,N}(u),$$

where $P_n(u)$ is the operator defined in statement 3. Hence

$$\|B_m(u)\|_{L_2\to W_2^\theta} \leqslant 2 \sup_{v\in B_{\theta,R}} \|P_{N-1}(v)\|_{L_2\to W_2^\theta} + \|B_{m,N}(u)\|_{L_2\to W_2^\theta} \leqslant$$

$$\leqslant 2 \sup_{v\in B_{\theta,R}} \|P_{N-1}(v)\|_{L_2\to W_2^1} + \|B_{m,N}(u)\|_{L_2\to W_2^\theta}.$$

Let $0<\theta_1<\theta$, say $\theta_1=\frac{\theta}{2}$. By statement 3 the function $\|P_{N-1}(v)\|_{L_2\to W_2^1}: B_{\theta_1,R}\to \mathbb{R}$ is continuous on $B_{\theta_1,R}$. It is known that the ball $B_{\theta,R}$ is a compact subset of $B_{\theta_1,R}$. Therefore the function $\|P_{N-1}(v)\|_{L_2\to W_2^1}$ is bounded on $B_{\theta,R}$ by a constant depending only on θ and R. This, together with inequality (6), imply (11). Step 3 is completed.

Step 4 (proof of the equiconvergence).

Let $f\in L_2[0,\pi]$. Then

$$\lim_{m\to\infty} \|B_m f\|_{W_2^\theta} = 0. \tag{12}$$

First let us check that the system $\{y_k\}_1^\infty$ of the eigenfunctions and associated functions is minimal in the space $W_2^\theta[0,\pi]$. Assume the converse. Then $y_n \in \overline{span}\{y_k\}_{k\neq n}$ (here the closure is taken subject to $W_2^\theta[0,\pi]$ norm) for some natural number n. Therefore $y_n \in \overline{span}\{y_k\}_{k\neq n}$, where the closure is taken subject to $L_2[0,\pi]$ norm. It means that the system $\{y_k\}_1^\infty$ is not minimal in the space $L_2[0,\pi]$ — contradiction with statement 1. Next, we proved (see (8)) that the system $\{y_k\}_1^\infty$ is close to the orthogonal basis $\{\sqrt{2/\pi}\sin kx\}_1^\infty$ in the space $W_2^\theta[0,\pi]$. Let us now consider the orthonormal basis $\left\{\sqrt{\frac{2}{\pi}}(1+k^2)^{-\theta/2}\sin kx\right\}_1^\infty$ and the system $\{(1+k^2)^{-\theta/2}y_k\}_1^\infty$ in $W_2^\theta[0,\pi]$. We see that these two systems are also close in the space $W_2^\theta[0,\pi]$. It can be easily proved that the system $\{(1+k^2)^{-\theta/2}y_k\}_1^\infty$ is minimal in the space $W_2^\theta[0,\pi]$. Hence, this system forms the Bari basis in the space $W_2^\theta[0,\pi]$ and consequently the system $\{y_k\}_1^\infty$ is total in this space.

Let us consider the image of the function y_k under the map B_m:

$$(B_m y_k)(x) = \sum_{n=1}^{m}(y_k(x), w_n(x))y_n(x) - \frac{2}{\pi}\sum_{n=1}^{m}(y_k(x),\sin nx)\sin nx.$$

The first term of the right-hand side in the last equality is equal to 0 for $m<k$ and is equal to $y_k(x)$ for $m\geqslant k$. The second one is the partial sum of the Fourier series for the function y_k. Recall that the function $y_k\in W_2^1$. This implies that its Fourier series converges to y_k in W_2^1–norm. Consequently this series converges to y_k in the space W_2^θ. This yields that (12) holds for any function $f\in span\{y_k\}$. To conclude the proof of the Step 4, it remains to apply completeness of the system $\{y_k\}$ and inequality (11).

Now let us prove inequality (4). Let $g_k(x)=\sum\limits_{n=1}^{k} c_n y_n(x)$, where $k\geqslant N$. It is clear that for any function $f\in L_2[0,\pi]$ and any natural number m

$$\|B_m f\|_{W_2^\theta} \leqslant \|B_m(f-g_k)\|_{W_2^\theta} + \|B_m g_k\|_{W_2^\theta}. \tag{13}$$

Step 5 (estimation of the norm of $B_m(f-g_k)$ in the space $W_2^\theta[0,\pi]$). *There exists a positive number $C_{\theta,R}$ such that for all natural k, m satisfying $k,m\geqslant N$ and for all $f\in L_2[0,\pi]$*

$$\|B_m(f-g_k)\|_{W_2^\theta}\leqslant C_{\theta,R}\left(\left(\sum_{n=k+1}^{\infty}|c_{n,0}|^2\right)^{1/2}+\frac{\|f\|_{L_2}}{k^\theta}\right). \tag{14}$$

By asymptotic formula (2) we have:

$$\|f(x)-g_k(x)\|_{L_2}=\left\|\sum_{n=k+1}^{\infty}c_ny_n(x)\right\|\leqslant$$

$$\leqslant\left\|\sum_{n=k+1}^{\infty}\frac{2}{\pi}(f(x),\sin nx)\sin nx\right\|_{L_2}+\left\|\sum_{n=k+1}^{\infty}\sqrt{\frac{2}{\pi}}(f(x),\psi_n(x))\sin nx\right\|_{L_2}+$$

$$+\left\|\sum_{n=k+1}^{\infty}\sqrt{\frac{2}{\pi}}(f(x),\sin nx)\varphi_n(x)\right\|_{L_2}+\left\|\sum_{n=k+1}^{\infty}(f(x),\psi_n(x))\varphi_n(x)\right\|_{L_2}\leqslant$$

$$\leqslant\left(\left(\sum_{n=k+1}^{\infty}|c_{n,0}|^2\right)^{1/2}+\left(\sum_{n=k+1}^{\infty}|(f(x),\psi_n(x))|^2\right)^{1/2}\right)\times$$

$$\times\left(1+\left(\sum_{n=k+1}^{\infty}\|\varphi_n\|_{L_2}^2\right)^{1/2}\right)\leqslant$$

$$\leqslant\left(\left(\sum_{n=k+1}^{\infty}|c_{n,0}|^2\right)^{1/2}+\frac{\|f\|_{L_2}}{k^\theta}\right)\left(1+\frac{C_{\theta,R}}{k^\theta}\right). \tag{15}$$

Now inequality (14) follows from (15) and (11). Step 5 is completed.

Next we estimate the second term in (13). For a given natural m, $m>k$, we introduce the operator $S_m:\,W_2^1[0,\pi]\to W_2^\theta[0,\pi]$ defined by

$$S_mh(x)=2/\pi\sum_{n=m+1}^{\infty}(h(t),\sin nt)\sin nx\,.$$

Note that

$$B_mg_k(x)=g_k(x)-\frac{2}{\pi}\sum_{n=1}^{m}(g_k(t),\sin nt)\sin nx=S_mg_k(x)\,.$$

(The expression $S_mg_k(x)$ is well defined, since all eigenfunctions and associated functions of the operator L belong to the space $W_2^1[0,\pi]$.) Therefore

$$\|B_mg_k\|_{W_2^\theta}\leqslant\|S_m(g_k-g_N)\|_{W_2^\theta}+\|S_mg_N\|_{W_2^\theta}. \tag{16}$$

Step 6 (estimation of the norm of $S_m(g_k - g_N)$ in the space $W_2^\theta[0,\pi]$).
There exists a positive number $C_{\theta,R}$ such that for all natural k, m satisfying $m > k \geqslant N$ and for all $f \in L_2[0,\pi]$

$$\|S_m(g_k - g_N)\|_{W_2^\theta} \leqslant C_{\theta,R}\|f\|_{L_2} m^{\theta-1}k^{1-\theta}. \tag{17}$$

First let us recall that $y_n(x) = \sqrt{2/\pi}\sin nx + \varphi_n(x)$ (see (2)) and represent the left-hand side of inequality (17) as follows:

$$\|S_m(g_k(x) - g_N(x))\|_{W_2^\theta} = \left\|\sum_{n=N+1}^{k} c_n S_m y_n(x)\right\|_{W_2^\theta} = \left\|\sum_{n=N+1}^{k} c_n S_m \varphi_n(x)\right\|_{W_2^\theta}.$$

Note that [3]

$$\sum_{n=N+1}^{k} |c_n|^2 \leqslant \sum_{n=N+1}^{\infty} |c_n|^2 \leqslant$$

$$\leqslant 2\left(\frac{2}{\pi}\sum_{n=N+1}^{\infty} |(f(x), \sin nx)|^2 + \sum_{n=N+1}^{\infty} |(f(x), \psi_n(x))|^2\right) \leqslant C_{\theta,R}^{(1)}\|f\|_{L_2}^2,$$

Therefore

$$\left\|\sum_{n=N+1}^{k} c_n S_m \varphi_n(x)\right\|_{W_2^\theta} \leqslant \left(\sum_{n=N+1}^{k} |c_n|^2\right)^{1/2}\left(\sum_{n=N+1}^{k} \|S_m\varphi_n(x)\|_{W_2^\theta}^2\right)^{1/2} \leqslant$$

$$\leqslant \sqrt{C_{\theta,R}^{(1)}}\,\|f\|_{L_2}\left(\sum_{n=N+1}^{k} \|S_m\varphi_n(x)\|_{W_2^\theta}^2\right)^{1/2}.$$

Furthermore

$$\|S_m\varphi_n(x)\|_{W_2^\theta} \leqslant C_{\theta,R}^{(2)}\left(\sum_{j=m+1}^{\infty} (j^{2\theta}+1)|(\varphi_n(x), \sin jx)|^2\right)^{1/2} =$$

$$= C_{\theta,R}^{(2)}\left(\sum_{j=m+1}^{\infty} \frac{j^{2\theta}+1}{j^2}|(\varphi_n'(x), \cos jx)|^2\right)^{1/2} \leqslant C_{\theta,R}^{(2)} m^{\theta-1}\|\varphi_n(x)\|_{W_2^1}. \tag{18}$$

Here we applied integration by parts and boundary conditions $\varphi_n(0) = \varphi_n(\pi) = 0$. By asymptotic formulae (2) we obtain $\varphi_n'(x) = n\eta_n(x) + u(x)y_n(x)$, where $\{\|\eta_n(x)\|_{L_2} + n^{\theta-1}\} \in l_2^\theta$. Consequently

$$\|\varphi_n'(x)\|_{L_2} \leqslant (n\|\eta_n(x)\|_{L_2} + \|u(x)y_n(x)\|_{L_2}) = n^{1-\theta}\tau_n,$$

[3] Here $C_{\theta,R}^{(1)}$ and in the sequel $C_{\theta,R}^{(j)}$ with $j = 2, 3...$ are some positive quantities depending only on θ and R.

where $\|\{\tau_n\}\|_{l_2} \leqslant C^{(3)}_{\theta,R}$. Therefore,

$$\|S_m\varphi_n(x)\|_{W_2^\theta} \leqslant C^{(4)}_{\theta,R}\, m^{\theta-1}n^{1-\theta}\tau_n.$$

This implies that the first term in (16) satisfies the following inequality:

$$\|S_m(g_k - g_N)\|_{W_2^\theta} \leqslant C^{(5)}_{\theta,R}\ \|f\|_{L_2}\left(\sum_{n=1}^{k} m^{2\theta-2}n^{2-2\theta}\tau_n^2\right)^{1/2} \leqslant C^{(6)}_{\theta,R}\ \|f\|_{L_2}m^{\theta-1}k^{1-\theta}.$$

Step 6 is completed.

Step 7 (estimation of the norm of $S_m g_N$ in the space $W_2^\theta[0,\pi]$).

Finally we consider the second term in (16). We shall estimate the norm $\|S_m g_N\|_C$ in the same way as in (18):

$$\|S_m g_N(x)\|_C \leqslant C^{(7)}_{\theta,R}\, m^{\theta-1}\|g_N'(x)\|_{L_2} \leqslant C^{(7)}_{\theta,R}\, m^{\theta-1}\|g_N\|_{W_2^1}.$$

Since $\|P_N(u)\|_{L_2\to W_2^1} \leqslant C_{\theta,R}$ (see Step 3) and $g_N = P_N f$ we have $\|g_N\|_{W_2^1} \leqslant C_{\theta,R}\|f\|_{L_2}$. Therefore,

$$\|S_m g_N\|_C \leqslant C^{(8)}_{\theta,R}\ \|f\|_{L_2}\, m^{\theta-1}. \tag{19}$$

Hence inequalities (16), (17) and (19) imply that

$$\|B_m g_k\|_C \leqslant C^{(9)}_{\theta,R}\ \|f\|_{L_2}\, m^{\theta-1}k^{1-\theta}. \tag{20}$$

Let us put $k = [m^{1-\theta}] + 1$ for any natural number $m \geqslant N^2$. Inequalities (13), (14) and (20) imply inequality (4). $\square$

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