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NECESSARY AND SUFFICIENT CONDITIONS FOR THE BOUNDEDNESS OF GENUINE SINGULAR INTEGRAL OPERATORS IN LOCAL MORREY-TYPE SPACES [1]

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Abstract. The problem of the boundedness of a Calderon-Zygmund singular integral operator T in local Morrey-type spaces is reduced to the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions. This allows obtaining sufficient conditions for the boundedness of T in local Morrey-type spaces for all admissible values of the parameters. Moreover, for a certain range of the parameters, for a genuine Calderon-Zygmund singular integral operator these sufficient conditions coincide with the necessary ones.

1 Introduction

Let T be a singular integral Calderon-Zygmund operator, briefly a Calderon-Zygmund operator, i. e., a linear operator bounded from $L_2(\mathbb{R}^n)$ in $L_2(\mathbb{R}^n)$ taking all infinitely continuously differentiable functions f with compact support to the functions $Tf \in L_1^{\rm loc}(\mathbb{R}^n)$ represented by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,dy \qquad \text{a. e.} \quad \text{on} \ \ \mathbb{R}^n \setminus \operatorname{supp} f. \tag{1}$$

Here $K(x,y)$ is a continuous function away from the diagonal which satisfies the standard estimates: there exist $c_1 > 0$ and $0 < \varepsilon \le 1$ such that

$$|K(x,y)| \le c_1 |x-y|^{-n} \tag{2}$$

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for all $x, y \in \mathbb{R}^n$, $x \neq y$, and

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq c_1 \left(\frac{|x-x'|}{|x-y|} \right)^{\varepsilon} |x-y|^{-n} \tag{3}$$

whenever $2|x-x'| \leq |x-y|$. Such operators were introduced in [9].

In the theory of partial differential equations, together with weighted $L_{p,w}$-spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 [17] and defined as follows. *For* $0 \leq \lambda \leq n$, $1 \leq p \leq \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if* $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball centered at x of radius r. Note that $\mathcal{M}_{p,0} = L_p(\mathbb{R}^n)$ and $\mathcal{M}_{p,n} = L_\infty(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\mathcal{M}_{p,\lambda} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

These spaces appeared to be quite useful in the study of the local behaviour of solutions to elliptic partial differential equations, apriori estimates and other topics in the theory of partial differential equations.

We also denote by $W\mathcal{M}_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p-space.

The classical result for Calderon-Zygmund operators states that if $1 < p < \infty$ then T is bounded from $L_p(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$, and if $p = 1$ then T is bounded from $L_1(\mathbb{R}^n)$ to $WL_1(\mathbb{R}^n)$ (see, for example, [22]).

J. Peetre [20] studied the boundedness of singular integral operators in Morrey spaces, and his results imply the following statement for Calderon-Zygmund operators T.

Theorem 1. *Let* $1 < p < \infty$, $0 < \lambda < n$. *Then* T *is bounded from* $\mathcal{M}_{p,\lambda}$ *to* $\mathcal{M}_{p,\lambda}$.

If $\lambda = 0$, the statement of Theorem 1 reduces to the aforementioned result for $L_p(\mathbb{R}^n)$.

If in place of the power function $r^{-\frac{\lambda}{p}}$ in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive measurable weight function w, then it becomes generalised Morrey space $\mathcal{M}_{p,w}$. T. Mizuhara [16], E. Nakai [18] and V. S. Guliyev [11] (see also [14]) generalised Theorem 1 and obtained sufficient conditions on weights w_1 and w_2 ensuring the boundedness of T from $\mathcal{M}_{p,w_1}$ to $\mathcal{M}_{p,w_2}$. The following statement, containing the results in [16], [18] was proved in [11] (see also [14]).

Theorem 2. *Let* $1 < p < \infty$. *Moreover, let* w_1, w_2 *be positive measurable functions satisfying the following condition: there exits* $c_2 > 0$ *such that for all* $t > 0$

$$\left\| w_1^{-1}(r)\, r^{-\frac{n}{p}-1} \right\|_{L_1(t,\infty)} \leq c_2 w_2^{-1}(t)\, t^{-\frac{n}{p}}. \tag{4}$$

Then T *is bounded from* $\mathcal{M}_{p,w_1}$ *to* $\mathcal{M}_{p,w_2}$.

Earlier, in the [16], [18] Theorem 2 was proved for the case $w_1 = w_2 = w$, where w satisfies the pointwise doubling condition, namely for some $c_3 > 0$

$$c_3^{-1} w(t) \leq w(r) \leq c_3 w(t)$$

for all t, $r > 0$ such that $0 < r \leq t \leq 2r$.

We say that T is a *genuine* Calderon-Zygmund operator if it is a Calderon-Zygmund operator and for $n \geq 2$ there exist $c_4, c_5 > 0$ and a rotation $\mathcal{R}$ such that

$$K(x,y) \geq \frac{c_4}{|x-y|^n} \tag{5}$$

for all $x \in \mathbb{R}^n$ and for all $y \in C_x = x + \mathcal{R}(C)$, where

$$C = \{y = (y_1, ..., y_n) \equiv (\overline{y}, y_n) \in \mathbb{R}^n : y_n > c_5 |\overline{y}|\}.$$

If $n = 1$ then we assume that there exists $c_4 > 0$ such that

$$K(x,y) \geq \frac{c_4}{|x-y|}$$

for all $x \in \mathbb{R}$ and for all $y > x$ or for all $x \in \mathbb{R}$ and for all $y < x$.

Clearly the Hilbert transform $\mathcal{H}$, in which case $K(x,y) = \frac{1}{x-y}$, is a genuine Calderon-Zygmund operator because $K(x,y) \geq \frac{1}{|x-y|}$ for all $x \in \mathbb{R}$ and for all $y < x$. Let T be a Calderon-Zygmund operator of the form

$$K(x,y) = \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n},$$

where Ω is a continuous function on the unit sphere S^{n-1} in $\mathbb{R}^n$, $\Omega \not\equiv 0$, which is homogeneous of order zero and such that $\int_{S^{n-1}} \Omega(\eta) d\eta = 0$. The properties of Ω imply that there exist $c_6 > 0$, $\eta_0 \in S^{n-1}$ and $\delta > 0$ such that $\Omega(\eta) \geq c_6$ for all $\eta \in S^{n-1} \cap B(\eta_0, \delta)$. Hence condition (5) is satisfied.

In this paper we consider local and global Morrey-type spaces $LM_{p\theta,w}$ and $GM_{p\theta,w}$ as in [2]–[7], [11]–[15], where the boundedness of the maximal operator, fractional maximal operator and Riesz potentials was studied. We study the boundedness of Calderon-Zygmund operators from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$. We obtain sufficient conditions on weight functions w_1 and w_2 ensuring the boundedness of Calderon-Zygmund operators T from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$ for all admissible values of the parameters p, θ_1, θ_2.

Moreover, for a certain range of the parameters p, θ_1, θ_2 we obtain necessary and sufficient conditions on weight functions w_1 and w_2 for a genuine Calderon-Zygmund operator T to be bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$. In particular, in Theorem 8 we prove that if $1 < p < \infty$, $0 < \theta_1 \leq \theta_2 \leq \infty$ and $\theta_1 \leq 1$, then the condition

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p} \right\|_{L_{\theta_2}(0,\infty)} \leq c_7 \|w_1\|_{L_{\theta_1}(t,\infty)} \tag{6}$$

for all $t > 0$, where $c_7 > 0$ is independent of t, is necessary and sufficient for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$.

We note that condition (6) is also necessary and sufficient for the boundedness of the maximal operator from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ for a wider range of the parameter $\theta_1 : \theta_1 \le p$ [3,5].

In Theorem 9 it is proved that, under additional assumption of regularity on w_2, condition (6) is necessary and sufficient for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ also for $\theta_1 \le p$.

We also consider separately the case in which $LM_{p\theta_1,w_1}$ is replaced [2] by L_p.

Most of the results of this paper were formulated without proofs in [8].

2 Definitions and basic properties of Morrey-type spaces

Definition 1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p\theta,w}$ and $GM_{p\theta,w}$ the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms*

$$\|f\|_{LM_{p\theta,w}} \equiv \|f\|_{LM_{p\theta,w}(\mathbb{R}^n)} = \left\| w(r)\|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p\theta,w}}$$

respectively.

Note that

$$\|f\|_{LM_{p\infty,1}} = \|f\|_{GM_{p\infty,1}} = \|f\|_{L_p}.$$

Furthemore, $GM_{p\infty,r^{-\frac{\lambda}{p}}} \equiv \mathcal{M}_{p,\lambda}$, $0 \le \lambda \le n$ and $GM_{p\infty,w} \equiv \mathcal{M}_{p,w}$.

In [2, 3] the following statement was proved.

Lemma 1. *Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$.*

1. If for all $t > 0$

$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{7}$$

then $LM_{p\theta,w} = GM_{p\theta,w} = \Theta$.

2. If for all $t > 0$

$$\|w(r)r^{n/p}\|_{L_\theta(0,t)} = \infty, \tag{8}$$

then for $0 < p < \infty$ $GM_{p\theta,w} = \Theta$.

Definition 2. *Let $0 < p, \theta \le \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0,\infty)$, not equivalent to 0 and such that for some $t > 0$*

$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{9}$$

[2]Here and in the sequel we write just L_p for $L_p(\mathbb{R}^n)$, $0 < p \le \infty$. If $G \ne \mathbb{R}^n$, then we preserve the full notation $L_p(G)$. The same refers to the case of weighted Lebesgue spaces $L_{p,v}$.

Moreover, we denote by $\Omega_{p\theta}$ *the set of all functions* w *which are non-negative, measurable on* $(0,\infty)$, *not equivalent to* 0 *and such that for some* $t_1, t_2 > 0$

$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r) r^{\frac{n}{p}}\|_{L_\theta(0,t_2)} < \infty. \tag{10}$$

In the sequel, keeping in mind Lemma 1, we always assume that, for the spaces $LM_{p\theta,w}$, $w \in \Omega_\theta$ and, for the spaces $GM_{p\theta,w}$, $w \in \Omega_{p\theta}$.

Let $w \in \Omega_\theta$ and $f \in LM_{p\theta,w}$, then $f \in L_p(B(0,r))$ for all $r > 0$. If $f \in L_p$, then $\big\|w(r)\|f\|_{L_p(B(0,r))}\big\|_{L_\theta(t,\infty)} < \infty$ for any $t > 0$, and the fact that $f \in LM_{p\theta,w}$ completely depends on the behaviour of $f(x)$ for small $|x|$. However, if $f \notin L_p$, then the fact that $f \in LM_{p\theta,w}$ depends both on the behaviour of $f(x)$ for small and large $|x|$.

3 Corollaries of weighted $L_{p,w}$-estimates

For a measurable set $G \subset \mathbb{R}^n$ and a function v non-negative and measurable on G, let $L_{p,v}(G)$ be the weighted L_p-space of all functions f measurable on G for which [3]

$$\|f\|_{L_{p,v}(G)} = \|vf\|_{L_p(G)} < \infty.$$

If $0 < p \le \theta \le \infty$, then

$$\|f\|_{LM_{p\theta,w}} \le \|f\|_{L_{p,W}}, \tag{11}$$

and if $0 < \theta \le p \le \infty$, then

$$\|f\|_{L_{p,W}} \le \|f\|_{LM_{p\theta,w}}, \tag{12}$$

where for all $x \in \mathbb{R}^n$

$$W(x) = \|w\|_{L_\theta(|x|,\infty)}.$$

These inequalities are particular cases of general inequalities of such type for the Lebesgue spaces with mixed quasinorms (see, for example,[19, section 3.37]). In particular, for $0 < p \le \infty$

$$\|f\|_{LM_{pp,w}} = \|f\|_{L_{p,V}},$$

where for all $x \in \mathbb{R}^n$

$$V(x) = \|w\|_{L_p(|x|,\infty)}.$$

Hence the results for $LM_{pp,w}$ follow from the known results for weighted $L_{p,V}$-spaces.

We start by quoting Theorem 3.4.2 in [10] stating necessary and sufficient conditions on v_1 and v_2 ensuring the validity of the inequality

$$\|Tf\|_{L_{p,v_2}} \le c_8 \|f\|_{L_{p,v_1}}, \tag{13}$$

where v_1 and v_2 are functions non-negative and measurable on $\mathbb{R}^n$ and $c_8 > 0$ is independent of $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

[3]See footnote in Section 1.

Theorem 3. *Let $1 < p < \infty$ and let v_1, v_2 be functions non-negative and measurable on $\mathbb{R}^n$. Moreover, let for some $c_9 > 0$*

$$\sup_{\frac{1}{4}|x| \le |y| \le 4|x|} v_2(y) \le c_9 v_1(x) \quad \text{for a.e. } x \in \mathbb{R}^n \tag{14}$$

or for some $c_{10} > 0$

$$v_2(x) \sup_{\frac{1}{4}|x| \le |y| \le 4|x|} \frac{1}{v_1(y)} \le c_{10} \quad \text{for a.e. } x \in \mathbb{R}^n. \tag{15}$$

Then inequality (13) holds if, and for the Hilbert transform $\mathcal{H}$ only if, the following two conditions are satisfied:

$$\mathcal{J}_1 = \sup_{r>0} \|v_2\|_{L_p(B(0,r))} \, \| \, |y|^{-n} v_1(y)^{-1} \|_{L_{p'}({}^{\complement}B(0,2r))} < \infty \tag{16}$$

and

$$\mathcal{J}_2 = \sup_{r>0} \|v_1^{-1}\|_{L_{p'}(B(0,r))} \, \| \, |y|^{-n} v_2(y) \|_{L_p({}^{\complement}B(0,2r))} < \infty, \tag{17}$$

where ${}^{\complement}B(0,t)$ is the complement of $B(0,t)$.

Moreover, the sharp constant c_8^* in (13) satisfies the inequality

$$c_8^* \le c_{11}(\mathcal{J}_1 + \mathcal{J}_2),$$

and for the Hilbert transform $\mathcal{H}$ also

$$c_{12}(\mathcal{J}_1 + \mathcal{J}_2) \le c_8^*,$$

where c_{11}, $c_{12} > 0$ are independent of v_1 and v_2.

The application of Theorem 3 immediately implies the following result for the case of local Morrey-type spaces.

Theorem 4. *Let $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. Moreover, let for some $c_{13} > 0$*

$$\sup_{\frac{1}{4}|x| \le |y| \le 4|x|} W_2(y) \le c_{13} W_1(x) \quad \text{for a. e. } x \in \mathbb{R}^n \tag{18}$$

or for some $c_{14} > 0$

$$W_2(x) \sup_{\frac{1}{4}|x| \le |y| \le 4|x|} \frac{1}{W_1(y)} \le c_{13} \quad \text{for a. e. } x \in \mathbb{R}^n, \tag{19}$$

where for all $x \in \mathbb{R}^n$

$$W_1(x) = \|w_1\|_{L_{\theta_1}(|x|,\infty)}, \qquad W_2(x) = \|w_2\|_{L_{\theta_2}(|x|,\infty)}.$$

If $\theta_1 \le p \le \theta_2$,

$$\sup_{r>0} \|W_2\|_{L_p(B(0,r))} \, \| \, |y|^{-n} W_1(y)^{-1} \|_{L_{p'}({}^{\complement}B(0,2r))} < \infty \tag{20}$$

and

$$\sup_{r>0} \|W_1^{-1}\|_{L_{p'}(B(0,r))} \; \| \, |y|^{-n} W_2(y)\|_{L_p(\complement_{B(0,2r)})} < \infty, \tag{21}$$

then the operator T is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$. (In the latter case we assume that $w_1 \in \Omega_{p\theta_1}$, $w_2 \in \Omega_{p\theta_2}$.)

If $\theta_2 \le p \le \theta_1$, then conditions (20) *and* (21) *are necessary for the boundedness of the Hilbert transform $\mathcal{H}$ from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$.*

In particular, if $\theta_1 = \theta_2 = p$, then conditions (20) *and* (21) *are necessary and sufficient for the boundedness of $\mathcal{H}$ from LM_{pp,w_1} to LM_{pp,w_2}.*

Proof. Let $\theta_1 \le p \le \theta_2$. By applying (11), the sufficiency of (20) and (21) for the boundedness of T and (12) we get

$$\|Tf\|_{LM_{p\theta_2,w_2}} \le \|Tf\|_{L_{p,W_2}} \le c_{15}\|f\|_{L_{p,W_1}} \le c_{15}\|f\|_{LM_{p\theta_1,w_1}}, \tag{22}$$

where $c_{15} > 0$ is independent of f.

Conversely if $\theta_2 \le p \le \theta_1$ and

$$\|\mathcal{H}f\|_{LM_{p\theta_2,w_2}} \le c_{16}\|f\|_{LM_{p\theta_1,w_1}},$$

where $c_{16} > 0$ is independent of f, then by (11) and (12)

$$\|\mathcal{H}f\|_{L_{p,W_2}} \le c_{16}\|f\|_{L_{p,W_1}} \tag{23}$$

and one may apply the necessity of (20) and (21) for the validity of (23).

Also (22) implies that

$$\|Tf\|_{GM_{p\theta_2,w_2}} \le c_{15}\|f\|_{GM_{p\theta_1,w_1}}.$$

□

4 Singular integrals and Hardy operator

In order to obtain necessary and sufficient conditions on w_1 and w_2 ensuring the boundedness of T for other values of the parameters we shall reduce the problem of the boundedness of T in the local Morrey-type spaces to the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions.

We start with quoting the following result proved in [11] (see also [14]).

Lemma 2. *Let $1 < p < \infty$ and $\gamma \ge 1$, then there exists $c_{17} > 0$ such that*

$$\|Tf\|_{L_p(B(0,r))} \le c_{17} r^{\frac{n}{p}} \int_{\gamma r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(0,t))} dt \tag{24}$$

for all $r > 0$ and for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$.

Corollary 1. *Let $1 < p < \infty$, $0 < \delta < \frac{n}{p}$ and $\gamma \geq 1$. Then there exists $c_{18} > 0$ such that*

$$\|Tf\|_{L_p(B(0,r))} \leq c_{18} r^{n/p-\delta} \left(\int_{\gamma r}^{\infty} \left(\int_{B(0,t)} |f(x)|^p dx \right) \frac{dt}{t^{n-\delta p+1}} \right)^{1/p} \tag{25}$$

for all $r > 0$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Proof. It suffices to apply Hölder's inequality to the integral in the right-hand side of (24). □

Let H be the Hardy operator

$$(Hg)(r) = \int_0^r g(t)dt, \quad 0 < r < \infty.$$

Lemma 3. *Let $1 < p < \infty$, $0 < \theta \leq \infty$ and $w \in \Omega_\theta$. Then there exists $c_{19} > 0$ such that*

$$\|Tf\|_{LM_{p\theta,w}} \leq c_{19} \, \|H\hat{g}\|_{L_{\theta,\hat{v}}(0,\infty)} \tag{26}$$

for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, where

$$\hat{g}(t) = \|f\|_{L_p\left(B\left(0,t^{-\frac{p}{n}}\right)\right)} \tag{27}$$

and

$$\hat{v}(r) = w\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}\left(\frac{n}{p}+\frac{1}{\theta}\right)-\frac{1}{\theta}}. \tag{28}$$

Proof. By Lemma 2

$$\begin{aligned}
\|Tf\|_{LM_{p\theta,w}} &= \big\|w(r)\|Tf\|_{L_p(B(0,r))}\big\|_{L_\theta(0,\infty)} \\
&\leq c_{17} \left\| w(r) r^{\frac{n}{p}} \int_r^\infty t^{-\frac{n}{p}-1} \|f\|_{L_p(B(0,t))} dt \right\|_{L_\theta(0,\infty)} \\
&= \frac{c_{17} p}{n} \left\| w(r) r^{\frac{n}{p}} \int_0^{r^{-\frac{n}{p}}} \|f\|_{L_p(B(0,t^{-\frac{p}{n}}))} dt \right\|_{L_\theta(0,\infty)}
\end{aligned}$$

$$\begin{aligned}
&= c_{20} \left\| w(r^{-\frac{p}{n}}) r^{-1-(\frac{p}{n}+1)\frac{1}{\theta}} \int_0^r \|f\|_{L_p(B(0,t^{-\frac{p}{n}}))} dt \right\|_{L_\theta(0,\infty)} \\
&= c_{20} \|H\hat{g}\|_{L_{\theta,\hat{v}}(0,\infty)},
\end{aligned}$$

where $c_{20} > 0$ depends only on n, p and θ. □

Lemma 4. *Let $1 < p < \infty$, $0 < \delta < \frac{n}{p}$, $0 < \theta \leq \infty$ and $w \in \Omega_\theta$. Then there exists $c_{21} > 0$ such that*

$$\|Tf\|_{LM_{p\theta,w}} \leq c_{21} \|Hg_\delta\|^{\frac{1}{p}}_{L_{\frac{\theta}{p},v_\delta}(0,\infty)}$$

for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$, *where*

$$g_\delta(t) = \int_{B\left(0,t^{\frac{1}{\delta p - n}}\right)} |f(y)|^p dy \tag{29}$$

and

$$v_\delta(r) = \left[w\left(r^{\frac{1}{\delta p-n}}\right) r^{\frac{1}{\delta p-n}\left(\frac{n}{p}-\delta+\frac{1}{\theta}\right)-\frac{1}{\theta}} \right]^p . \tag{30}$$

Proof. By Corollary 1

$$\begin{aligned}
\|Tf\|_{LM_{p\theta,w}} &= \big\| w(r) \|Tf\|_{L_p(B(0,r))} \big\|_{L_\theta(0,\infty)} \\
&\le c_{18} \left\| w(r) r^{\frac{n}{p}-\delta} \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^p dx \right) \frac{dt}{t^{n-\delta p+1}} \right)^{\frac{1}{p}} \right\|_{L_\theta(0,\infty)} \\
&= c_{18} (n-\delta p)^{-\frac{1}{p}} \left\| w(r) r^{\frac{n}{p}-\delta} \left(\int_0^{r^{\delta p - n}} \left(\int_{B(0,\tau^{\delta p-n})} |f(x)|^p dx \right) d\tau \right)^{\frac{1}{p}} \right\|_{L_\theta(0,\infty)}
\end{aligned}$$

$$\begin{aligned}
&= c_{18}(n-\delta p)^{-\frac{1}{p}} \left(\int_0^\infty \left(w(r) r^{\frac{n}{p}-\delta} \right)^\theta \left(\int_0^{r^{\delta p-n}} g(\tau) d\tau \right)^{\frac{\theta}{p}} dr \right)^{\frac{1}{\theta}} \\
&= c_{18}(n-\delta p)^{-\frac{1}{p}-\frac{1}{\theta}} \left(\int_0^\infty \left(w\left(\rho^{\frac{1}{\delta p-n}}\right) \rho^{\frac{n}{p(\delta p-n)}} \right)^\theta \rho^{\frac{1}{\delta p-n}-1} \left(Hg(\rho)\right)^{\frac{\theta}{p}} d\rho \right)^{\frac{1}{\theta}} \\
&= c_{21} \|Hg\|^{\frac{1}{p}}_{L_{\frac{\theta}{p},v_\delta}(0,\infty)},
\end{aligned}$$

where $c_{21} > 0$ depends only on n, p, θ and δ. □

Remark 1. *Recall that for the maximal operator* M, *as proved in [5], for* $1 < p < \infty$, $0 < \theta \le \infty$ *and* $w \in \Omega_\theta$, *there exists* $c_{22} > 0$ *such that*

$$\|Mf\|_{LM_{p\theta,w}} \le c_{22} \|Hg_0\|^{1/p}_{L_{\frac{\theta}{p},v_0}(0,\infty)}$$

for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 5. *Let* $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $0 < \delta < \frac{n}{p}$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$. *Moreover, let*

$$v_{1,\delta}(r) = \left[w_1\left(r^{\frac{1}{\delta p-n}}\right) r^{\frac{1}{(\delta p-n)\theta_1}-\frac{1}{\theta_1}} \right]^p , \tag{31}$$

$$v_{2,\delta}(r) = \left[w_2\left(r^{\frac{1}{\delta p-n}}\right) r^{\frac{1}{\delta p-n}\left(\frac{n}{p}-\delta+\frac{1}{\theta_2}\right)-\frac{1}{\theta_2}} \right]^p \tag{32}$$

and

$$\hat{v}_1(r) = w_1\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n\theta_1}-\frac{1}{\theta_1}}, \tag{33}$$

$$\hat{v}_2(r) = w_2\left(r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}\left(\frac{n}{p}+\frac{1}{\theta}2\right)-\frac{1}{\theta_2}}. \tag{34}$$

Assume that the operator H is bounded from $L_{\frac{\theta_1}{p},v_{1,\delta}}(0,\infty)$ to $L_{\frac{\theta_2}{p},v_{2,\delta}}(0,\infty)$ or from $L_{\theta_1,\hat{v}_1}(0,\infty)$ to $L_{\theta_2,\hat{v}_2}(0,\infty)$ on the cone of all non-negative functions φ non-increasing on $(0,\infty)$ and satisfying $\lim\limits_{t\to\infty}\varphi(t)=0$.

Then T is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$. (In the latter case we assume that $w_1\in\Omega_{p\theta_1}$, $w_2\in\Omega_{p\theta_2}$.)

Proof. Let H be bounded from $L_{\frac{\theta_1}{p},v_{1,\delta}}(0,\infty)$ to $L_{\frac{\theta_2}{p},v_{2,\delta}}(0,\infty)$. Then by Lemma 4 applied to $LM_{p\theta_2,w_2}$

$$\|Tf\|_{LM_{p\theta_2,w_2}} \le c_{23}\|Hg_\delta\|^{1/p}_{L_{\frac{\theta_2}{p},v_{2,\delta}}(0,\infty)},$$

where $c_{23}>0$ is independent of f.

Since g is non-negative, non-increasing on $(0,\infty)$ and $\lim\limits_{t\to+\infty} g_\delta(t)=0$ and H is bounded from $L_{\frac{\theta_1}{p},v_{1,\delta}}(0,\infty)$ to $L_{\frac{\theta_2}{p},v_{2,\delta}}(0,\infty)$ on the cone of functions containing g_δ, we have

$$\|Tf\|_{LM_{p\theta_2,w_2}} \le c_{24}\|g_\delta\|^{1/p}_{L_{\frac{\theta_1}{p},v_{1,\delta}}(0,\infty)},$$

where $c_{24}>0$ is independent of f.

Hence

$$\begin{aligned}
\|Tf\|_{LM_{p\theta_2,w_2}} &\le c_{24}\left(\int_0^\infty v_{1,\delta}(t)^{\frac{\theta_1}{p}}\|f\|^{\theta_1}_{L_p\left(B\left(0,t^{\frac{1}{\delta p-n}}\right)\right)}dt\right)^{1/\theta_1}\\
&= c_{24}(n-\delta p)^{\frac{1}{\theta_1}}\left(\int_0^\infty v_{1,\delta}(r^{\delta p-n})^{\frac{\theta_1}{p}} r^{\delta p-n-1}\|f\|^{\theta_1}_{L_p(B(0,r))}dr\right)^{1/\theta_1}\\
&= c_{25}\left(\int_0^\infty \left(w_1(r)\|f\|_{L_p(B(0,r))}\right)^{\theta_1}dr\right)^{1/\theta_1}\\
&= c_{25}\|f\|_{LM_{p\theta_1,w_1}},
\end{aligned} \tag{35}$$

where $c_{25}>0$ is independent of f.

If H is bounded from $L_{\theta_1,\hat{v}_1}(0,\infty)$ to $L_{\theta_2,\hat{v}_2}(0,\infty)$ then the argument is similar. (Lemma 4 should be replaced by Lemma 3.) □

5 Sufficient conditions

In order to obtain explicit sufficient conditions on the weight functions ensuring the boundedness of T, first we shall apply the following well-known simple sufficient conditions ensuring the boundedness of the Hardy operator H from one weighted Lebesgue space to another one (see, for example, [3]).

Lemma 5. *Let $0 < \theta_1, \theta_2 \le \infty$ and let v_1, v_2 be functions positive and measurable on $(0,\infty)$. Then the condition*

$$\Big\| v_2(r) \Big\| t^{-\frac{(1-\theta_1)_+}{\theta_1}} v_1^{-1}(t) \Big\|_{L_{\frac{\theta_1}{(\theta_1-1)_+}}(0,r)} \Big\|_{L_{\theta_2}(0,\infty)} < \infty \tag{36}$$

is sufficient for the boundedness of H from $L_{\theta_1,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$ in the case $1 \le \theta_1 \le \infty$ and for the boundedness of H from $L_{\theta_1,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$ on the cone of all non-negative non-increasing functions on $(0,\infty)$ in the case $0 < \theta_1 < 1$.

If $\theta_1 = \infty$, then condition (36) is also necessary for the boundedness of H from $L_{\infty,v_1}(0,\infty)$ to $L_{\theta_2,v_2}(0,\infty)$.

The statements of Lemma 5 follow by applying Hölder's inequality if $1 \le \theta_1 \le \infty$ and the inequality

$$\left(\int_a^b \varphi(t)dt\right)^{\theta_1} \le \theta_1 \int_a^b (t-a)^{\theta_1-1} \varphi(t)^{\theta_1} dt$$

for all $-\infty < a < b \le \infty$ and for all functions φ non-negative and non-increasing on $(0,\infty)$ if $0 < \theta_1 < 1$. (See, for example, [1].)

Theorem 5 and Lemma 5 imply the following sufficient conditions for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$.

Theorem 6. *Let $1 < p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$. If, for some $0 < \delta < \frac{n}{p}$,*

$$\Big\| w_2(r) r^{\frac{n}{p}-\delta} \Big\| w_1^{-1}(t) t^{\delta-\frac{n}{p}-\frac{1}{\min\{p,\theta_1\}}} \Big\|_{L_s(r,\infty)} \Big\|_{L_{\theta_2}(0,\infty)} < \infty, \tag{37}$$

where $s = \frac{p\theta_1}{(\theta_1-p)_+}$ (if $\theta_1 \le p$, then $s = \infty$) or if

$$\Big\| w_2(r) r^{\frac{n}{p}} \Big\| w_1^{-1}(t) t^{-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}} \Big\|_{L_\sigma(r,\infty)} \Big\|_{L_{\theta_2}(0,\infty)} < \infty, \tag{38}$$

where $\sigma = \frac{\theta_1}{(\theta_1-1)_+}$ (if $\theta_1 \le 1$, then $\sigma = \infty$), then T is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$. (In the latter case we assume that $w_1 \in \Omega_{p\theta_1}$, $w_2 \in \Omega_{p\theta_2}$.)

Remark 2. *Recall that, as proved in [5], if condition (37) is satisfied and, moreover, if it is satisfied with $\delta = 0$, then the maximal operator M is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$.*

Under some assumptions on the regularity of the weight w_1 conditions (37) and (38) can be weakened. To prove this we shall need the following lemmas proved in [7].

Lemma 6. *Let φ be a non-negative measurable function on $(0,\infty)$, $c_{26}>0$ and*

$$\int_r^\infty \varphi(t)dt \le c_{26} r\varphi(r) \tag{39}$$

for all $r>0$. Then for all $\delta\in(0,\frac{1}{c_{26}})$

$$r^{-\delta}\int_r^\infty \varphi(t)t^\delta dt \le (1-\delta c_{26})^{-1}\int_r^\infty \varphi(t)dt \tag{40}$$

for all $r>0$.

Corollary 2. *Let $0<p,q\le\infty$ and $c_{27}>0$. Moreover, let φ and ψ be non-negative measurable functions on $(0,\infty)$ and*

$$\|\varphi\|_{L_p(r,\infty)} \le c_{27} r^{\frac{1}{p}}\varphi(r) \tag{41}$$

for all $r>0$. Then for all $\delta\in(0,\frac{1}{pc_{27}})$

$$\|\psi(r)\, r^{-\delta}\|\varphi(t)\, t^\delta\|_{L_p(r,\infty)}\|_{L_q(0,\infty)} \le (1-\delta p c_{27})^{-\frac{1}{p}}\ \|\psi(r)\ \|\varphi(t)\|_{L_p(r,\infty)}\|_{L_q(0,\infty)}. \tag{42}$$

Lemma 7. *Let φ be a non-negative measurable function on $(0,\infty)$, $c_{28}>0$ and*

$$\int_0^r \varphi(t)dt \le c_{28} r\varphi(r) \tag{43}$$

for all $r>0$. Then for all $\delta\in(0,\frac{1}{c_{28}})$

$$r^{\delta}\int_0^r \varphi(t)t^{-\delta} dt \le (1-\delta c_{28})^{-1}\int_0^r \varphi(t)dt \tag{44}$$

for all $r>0$.

Theorem 7. *Let $1<p<\infty$, $0<\theta_1,\theta_2\le\infty$, $w_1\in\Omega_{\theta_1}$ and $w_2\in\Omega_{\theta_2}$. If, for some $c_{29}>0$, for all $r>0$*

$$\|w_1^{-1}(t)t^{-\frac{n}{p}-\frac{1}{\min\{p,\theta_1\}}}\|_{L_s(r,\infty)} \le c_{29} w_1^{-1}(r) r^{-\frac{n}{p}-\frac{1}{\min\{p,\theta_1\}}+\frac{1}{s}} \tag{45}$$

and

$$\left\|w_2(r)r^{\frac{n}{p}}\left\|w_1^{-1}(t)t^{-\frac{n}{p}-\frac{1}{\min\{p,\theta_1\}}}\right\|_{L_s(r,\infty)}\right\|_{L_{\theta_2}(0,\infty)}<\infty, \tag{46}$$

or if, for some $c_{30}>0$, for all $r>0$

$$\|w_1^{-1}(t)t^{-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}\|_{L_\sigma(r,\infty)} \le c_{30} w_1^{-1}(r) r^{-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}+\frac{1}{\sigma}} \tag{47}$$

and for all $\delta>0$

$$\left\|w_2(r)r^{\frac{n}{p}+\delta}\left\|w_1^{-1}(t)t^{-\delta-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}\right\|_{L_\sigma(r,\infty)}\right\|_{L_{\theta_2}(0,\infty)}<\infty, \tag{48}$$

then T is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$. (In the latter case we assume that $w_1\in\Omega_{p\theta_1}$, $w_2\in\Omega_{p\theta_2}$.)

Proof. 1. Note that regularity condition (45) coincides with condition (41) of Corollary 2, where $p = s$ and $\varphi(t) = w_1^{-p}(t)t^{-n-\frac{p}{\min\{p,\theta_1\}}}$. By Corollary 2 condition (44) is satisfied for some $\delta > 0$ with $q = \theta_2$ and $\psi(t) = w_2^{-p}(r)r^{-\frac{1}{p}}$, which implies condition (37). Therefore Theorem 6 implies that T is bounded from $LM_{p\infty,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\infty,w_1}$ to $GM_{p\theta_2,w_2}$.

2. Let condition (48) be satisfied. Then, in particular, it is satisfied for $0 < \delta < (\sigma c_{30})^{-1}$. Next condition (47) implies that

$$\left\|w_1^{-1}(t)t^{-\delta-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}\right\|_{L_\sigma(r,\infty)} \le r^{-\delta}\left\|w_1^{-1}(t)t^{-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}\right\|_{L_\sigma(r,\infty)} \le c_{30}w_1^{-1}(r)r^{-\delta-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}+\frac{1}{\sigma}}.$$

Hence condition (41) is satisfied for $p = \sigma$, $\varphi(t) = w_1^{-1}(t)t^{-\delta-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}$ and $c_{27} = c_{30}$. Therefore Corollary 2 with $\psi(t) = w_2(t)t^{\frac{n}{p}+\delta}$, $q = \theta_2$ implies condition (38) because

$$\left\|w_2(r)r^{\frac{n}{p}}\left\|w_1^{-1}(t)t^{-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}\right\|_{L_\sigma(r,\infty)}\right\|_{L_{\theta_2}(0,\infty)} \le (1-\delta\sigma c_{30})^{-\frac{1}{\sigma}}\left\|w_2(r)r^{\frac{n}{p}+\delta}\left\|w_1^{-1}(t)t^{-\delta-\frac{n}{p}-\frac{1}{\min\{1,\theta_1\}}}\right\|_{L_\sigma(r,\infty)}\right\|_{L_{\theta_2}(0,\infty)} < \infty.$$

Therefore Theorem 6 implies that T is bounded from $LM_{p\infty,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\infty,w_1}$ to $GM_{p\theta_2,w_2}$. □

Corollary 3. *Let $1 < p < \infty$, $0 < \theta_2 \le \infty$, $w_1 \in \Omega_\infty$ and $w_2 \in \Omega_{\theta_2}$. If, for some $c_{31} > 0$, for all $r > 0$*

$$\|w_1^{-1}(t)t^{-\frac{n}{p}-1}\|_{L_1(r,\infty)} \le c_{31}w_1^{-1}(r)r^{-\frac{n}{p}} \tag{49}$$

and for all $\delta > 0$

$$\|w_2(r)r^{\delta+\frac{n}{p}}\|w_1^{-1}(t)t^{-\delta-\frac{n+1}{p}}\|_{L_p(r,\infty)}\|_{L_{\theta_2}(0,\infty)} < \infty, \tag{50}$$

then T is bounded from $LM_{p\infty,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\infty,w_1}$ to $GM_{p\theta_2,w_2}$. (In the latter case we assume that $w_1 \in \Omega_{p\infty}$, $w_2 \in \Omega_{p\theta_2}$.)

Proof. Note that condition (50) and Hölder's inequality imply condition (48) for all $\delta > 0$ in the case $\theta_1 = \infty$ and therefore Corollary 2 implies Corollary 3. □

Corollary 4. *Let $1 < p < \infty$ and $w_1, w_2 \in \Omega_\infty$. If, for some c_{32}, $c_{33} > 0$, for all $r > 0$*

$$\|w_1^{-1}(t)t^{-\frac{n}{p}-1}\|_{L_1(r,\infty)} \le c_{32}w_1^{-1}(r)r^{-\frac{n}{p}} \tag{51}$$

and for some $\delta > 0$

$$\|w_1^{-1}(t)t^{-\delta-\frac{n+1}{p}}\|_{L_p(r,\infty)} \le c_{33}w_2^{-1}(r)r^{-\delta-\frac{n}{p}}, \tag{52}$$

then T is bounded from $LM_{p\infty,w_1}$ to $LM_{p\infty,w_2}$ and from $GM_{p\infty,w_1}$ to $GM_{p\infty,w_2}$. (In the latter case we assume that $w_2 \in \Omega_{p\infty}$.)

Remark 3. *We note that condition* (52) *is weaker than condition* (4) *in Theorem 2. However in Theorem 2 there is no additional regularity assumptions on* w_1. *If* $w_1 = w_2$ *then the statements of Theorem 2 and Corollary 4 coincide, because in this case by Lemma 6 condition* (51), *coinciding in this case with condition* (4), *implies condition* (52).

Remark 4. *When proving the results of this section we applied inequalities* (24) *and* (25) *with* $\gamma = 1$. *One may trace that inequalities* (24) *and* (25) *in fact imply the statements of this section in a slightly stronger version. Namely the norms in* $L_q(r,\infty)$ *with* q *equal to* s, σ, 1 *or* p *of the expressions containing* $w_1^{-1}(t)$ *in formulas* (45)–(52) *can be replaced by the norms of the same expressions but in* $L_q(\gamma r,\infty)$ *with arbitrary* $\gamma \geq 1$.

6 Necessary and sufficient conditions

For the majority of cases the necessary and sufficient conditions for the validity of the inequality

$$\|H\varphi\|_{L_{\theta_2,v_2}(0,\infty)} \leq c_{34} \|\varphi\|_{L_{\theta_1,v_1}(0,\infty)}, \tag{53}$$

where $c_{34} > 0$ is independent of φ, for all non-negative non-increasing functions φ are known; for detailed information see [23], [24]. Application of Theorem 5 and of any of those conditions gives sufficient conditions for the boundedness of Calderon-Zygmund operators from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GM_{p\theta_2,w_2}$.

However, there is no guarantee that the application of the necessary and sufficient conditions on v_1 and v_2 ensuring the validity of (53) will imply the necessary and sufficient conditions for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$.

Fortunately for certain values of the parameters this is the case, namely for $1 < p < \infty$ and $0 < \theta_1 \leq \theta_2 \leq \infty$, $\theta_1 \leq 1$.

Theorem 8. *Let* $1 < p < \infty$, $0 < \theta_1, \theta_2 \leq \infty$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

1. If T *is a genuine Calderon-Zygmund operator, then the condition*

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{n/p} \right\|_{L_{\theta_2}(0,\infty)} \leq c_{35} \|w_1\|_{L_{\theta_1}(t,\infty)}, \tag{54}$$

where $c_{35} > 0$ *is independent of* $t > 0$, *is necessary for the boundedness of* T *from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$.

2. If T *is a Calderon-Zygmund operator,* $\theta_1 \leq \theta_2$ *and* $\theta_1 \leq 1$, *then condition* (54) *is sufficient for the boundedness of* T *from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$. *(In the latter case we assume that* $w_1 \in \Omega_{p\theta_1}$ *and* $w_2 \in \Omega_{p\theta_2}$.*)*

3. In particular, if T *is a genuine Calderon-Zygmund operator,* $\theta_1 \leq \theta_2$ *and* $\theta_1 \leq 1$, *then condition* (54) *is necessary and sufficient for the boundedness of* T *from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$.

Proof. 1. Let T be an operator (1) where $K(x,y)$ is a continuous function away from the diagonal which satisfies condition (5). Assume that, for some $c_{36} > 0$ and for all $f \in LM_{p\theta_1,\omega_1}$

$$\|Tf\|_{LM_{p\theta_2,\omega_2}} \leq c_{36} \|f\|_{LM_{p\theta_1,\omega_1}}.$$

Consider 'piece-of-cake' test functions, i. e. take here

$$f(x)=\chi_{(B(0,2t)\backslash B(0,t))\cap\mathcal{R}(\beta C)}(x),\quad x\in\mathbb{R}^n,$$

where sufficiently small $\beta>0$ will be chosen later.

Then as in [5] for some $c_{37}>0$

$$\|f\|_{LM_{p\theta_1,\omega_1}}\le c_{37}t^{\frac{n}{p}}\,\|\omega_1\|_{L_{\theta_1}(t,\infty)}\,.$$

Moreover

$$\|Tf\|_{LM_{p\theta_2,\omega_2}}=\left\|\omega_2(r)\left\|\int_{(B(0,2t)\backslash B(0,t))\cap\mathcal{R}(\beta C)}K(x,y)dy\right\|_{L_p(B(0,r))}\right\|_{L_{\theta_2}(0,\infty)}\ge$$

$$\ge\left\|\omega_2(r)\left\|\int_{(B(0,2t)\backslash B(0,t))\cap\mathcal{R}(\beta C)}K(x,y)dy\right\|_{L_p(B(0,\rho)\cap\mathcal{R}(\beta C))}\right\|_{L_{\theta_2}(0,\infty)},$$

where $\rho=\frac{1}{2}\min\{r,t\}$. Assume that $\beta>0$ is such that for all $t>0$ and for all $x\in B\,(0,t/2)\cap\mathcal{R}(\beta C)$ we have

$$B\,(0,t/2)\cap\mathcal{R}(\beta C)\subset x+\mathcal{R}(C)=C_x.$$

Then by condition (5) for all $x\in B(0,\rho)\cap\mathcal{R}(\beta C)$ and for all $y\in(B(0,2t)\backslash B(0,t))\cap\mathcal{R}(\beta C)$ we have $K(x,y)\ge\frac{c_4}{|x-y|^n}\ge\frac{2^nc_4}{t^n}$, hence

$$\left|\int_{(B(0,2t)\backslash B(0,t))\cap\mathcal{R}(\beta C)}K(x,y)dy\right|\ge\frac{2^nc_4}{t^n}\,|(B(0,2t)\backslash B(0,t))\cap\mathcal{R}(\beta C)|\ge c_{38},$$

where $c_{38}>0$ is independent of t.

Therefore

$$\|Tf\|_{LM_{p\theta_2,\omega_2}}\ge c_{38}\left\|\omega_2(r)\,|B(0,\rho)\cap\mathcal{R}(\beta C)|^{\frac{1}{p}}\right\|_{L_{\theta_2}(0,\infty)}\ge c_{39}\left\|\omega_2(r)\min\{r,t\}^{\frac{n}{p}}\right\|_{L_{\theta_2}(0,\infty)}.$$

Thus

$$\left\|\omega_2(r)\min\{r,t\}^{\frac{n}{p}}\right\|_{L_{\theta_2}(0,\infty)}\le\frac{c_{37}}{c_{39}}t^{\frac{n}{p}}\,\|\omega_1\|_{L_{\theta_1}(t,\infty)}\,.$$

Since

$$\min\left\{\frac{r}{t},1\right\}\ge\frac{r}{t+r},\quad 0<t,r<\infty,$$

this implies condition (54).

2. It is known [24] that the necessary and sufficient condition for the boundedness of the operator H from $L_{\theta_1,\hat{v}_1}(0,\infty)$ to $L_{\theta_2,\hat{v}_2}(0,\infty)$ on the cone of all non-negative non-increasing on $(0,\infty)$ functions φ, where $\theta_1\le1$, has the form: for some $c_{40}>0$, for all $t>0$

$$\|\hat{v}_2(r)\min\{t,r\}\|_{L_{\theta_2}(0,\infty)}\le c_{40}\,\|\hat{v}_1(r)\|_{L_{\theta_1}(0,t)}\,,\qquad(55)$$

where $\hat{v}_1$ and $\hat{v}_2$ are defined by (33) and (34). This condition is equivalent to condition (54). Indeed, replacing $r^{-\frac{p}{n}}$ by ρ and $t^{-\frac{p}{n}}$ by τ, we get an equivalent form of (55) for some $c_{41} > 0$ for all $t > 0$

$$\left\| w_2(\rho)\rho^{n/p} \min\{\tau^{-\frac{n}{p}}, \rho^{-\frac{n}{p}}\} \right\|_{L_{\theta_2}(0,\infty)} \le c_{41} \left\| w_1 \right\|_{L_{\theta_1}(\tau,\infty)} .$$

In its turn this condition is equivalent to (54) since there exist c_{42}, $c_{43} > 0$ such that

$$c_{42} \left(\frac{r}{r+\tau} \right)^{\frac{n}{p}} \le r^{n/p} \min\{\tau^{-\frac{n}{p}}, r^{-\frac{n}{p}}\} \le c_{43} \left(\frac{r}{r+\tau} \right)^{\frac{n}{p}}$$

for all r, $\tau > 0$. Hence by Theorem 5 the second statement of the theorem follows. □

Theorem 9. *Let* $1 < p < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\theta_1 \le p$, $w_1 \in \Omega_{\theta_1}$ *and* $w_2 \in \Omega_{\theta_2}$.

1. If T is a Calderon-Zygmund operator, then the condition: for some $0 < \delta < \frac{n}{p}$, $c_{44} > 0$, *for all* $t > 0$

$$\left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p}-\delta} \right\|_{L_{\theta_2}(0,\infty)} \le c_{44} \left\| w_1 \right\|_{L_{\theta_1}(t,\infty)} \tag{56}$$

is sufficient for the boundedness of T from $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$ *and from* $GM_{p\theta_1,w_1}$ *to* $GM_{p\theta_2,w_2}$. *(In the latter case we assume that* $w_1 \in \Omega_{p\theta_1}$ *and* $w_2 \in \Omega_{p\theta_2}$.*)*

2. If T is a genuine Calderon-Zygmund operator and for some $c_{45} > 0$ *for all* $t > 0$

$$\left\| w_2(r) r^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,t)} \le c_{45} w_2(t) t^{\frac{n}{p}+\frac{1}{\theta_2}}, \tag{57}$$

then condition (54) *is necessary and sufficient for the boundedness of T from* $LM_{p\theta_1,w_1}$ *to* $LM_{p\theta_2,w_2}$.

Proof. 1. Since $\frac{\theta_1}{p} \le \frac{\theta_2}{p}$ and $\frac{\theta_1}{p} \le 1$, the operator H is bounded from $L_{\frac{\theta_1}{p},v_{1,\delta}}(0,\infty)$ to $L_{\frac{\theta_2}{p},v_{2,\delta}}(0,\infty)$ on the cone of non-negative non-increasing functions if and only if

$$\left\| v_{2,\delta}(r) \min\{t, r\} \right\|_{L_{\frac{\theta_2}{p}}(0,\infty)} \le c_{40} \left\| v_{1,\delta}(r) \right\|_{L_{\frac{\theta_1}{p}}(0,t)} . \tag{58}$$

An argument similar to the one used in the proof of statement 2 of Theorem 8 shows that this condition is equivalent to condition (56).

2. Assume that conditions (54) and (57) are satisfied. By (57) and Lemma 7 there exist $0 < \delta < \frac{n}{p}$ and $c_{46} > 0$ such that for all $t > 0$

$$t^{\delta} \left\| w_2(r) r^{\frac{n}{p}-\delta} \right\|_{L_{\theta_2}(0,t)} \le c_{46} \left\| w_2(r) r^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,t)} \le c_{46} (2t)^{\frac{n}{p}} \left\| w_2(r) \left(\frac{r}{t+r} \right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,t)} . \tag{59}$$

Hence by (54) and (59)

$$\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}-\delta} \right\|_{L_{\theta_2}(0,\infty)}$$
$$\leq 2^{(\frac{1}{\theta_1}-1)_+} \left(\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}-\delta} \right\|_{L_{\theta_2}(0,t)} + \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}-\delta} \right\|_{L_{\theta_2}(t,\infty)} \right)$$
$$\leq 2^{(\frac{1}{\theta_1}-1)_+} \left(t^{-\frac{n}{p}+\delta} \left\| w_2(r) r^{\frac{n}{p}-\delta} \right\|_{L_{\theta_2}(0,t)} + 2^{\delta} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(t,\infty)} \right)$$
$$\leq c_{47} \left(\left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,t)} + \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(t,\infty)} \right)$$
$$\leq 2c_{47}c_{35} \left\| w_1 \right\|_{L_{\theta_1}(t,\infty)},$$

where a_+ denotes the positive part of $a \in \mathbb{R}$ and $c_{47} > 0$ is independent of t, which implies condition (56). Therefore by the first statement of the theorem T is bounded from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$. Thus condition (54), under the assumption (57), is sufficient for the boundedness of T. It is also necessary by the first statement of Theorem 8. □

Theorem 10. *Let T be a Calderon-Zygmund operator, $1 < p < \infty$, $0 < \theta_2 \leq \infty$ and $w_2 \in \Omega_{\theta_2}$. Then the condition*

$$w_2 \in L_{\theta_2}(0,\infty) \tag{60}$$

is sufficient, and for a a genuine Calderon-Zygmund operator is also necessary, for the boundedness of T from L_p to $LM_{p\theta_2,w_2}$ and from L_p to $GM_{p\theta_2,w_2}$. (In the latter case we assume that $w_2 \in \Omega_{p\theta_2}$.)

Proof. First consider the case of spaces $LM_{p\theta_2,w_2}$.

By the first statement of Theorem 9 with $w_1 \equiv 1$ and $\theta_1 = \infty$ the condition

$$I = \sup_{t>0} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,\infty)} < \infty$$

is necessary for the boundedness T from $LM_{p\theta_1,w_1}$ to $LM_{p\theta_2,w_2}$. Hence it suffices to notice that by the Monotone Convergence Theorem

$$\|w_2\|_{L_{\theta_2}(0,\infty)} = \lim_{t\to 0+} \left\| w_2(r) \left(\frac{r}{t+r}\right)^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,\infty)} = I < \infty.$$

The sufficiency of (60) follows by the boundedness of T from L_p to L_p:

$$\|Tf\|_{LM_{p\theta_2,w_2}} = \left\| w_2(r) \, \|Tf\|_{L_p(B(0,r))} \right\|_{L_{\theta_2}(0,\infty)} \leq c_{48} \, \|w_2\|_{L_{\theta_2}(0,\infty)} \, \|f\|_{L_p},$$

where $c_{48} > 0$ is independent of f.

In the case of the spaces $GM_{p\theta_2,w_2}$ the necessity of condition (60) follows since $\|Tf\|_{GM_{p\theta_2,w_2}} \geq \|Tf\|_{LM_{p\theta_2,w_2}}$, and the sufficiency of this condition follows since

$$\|Tf\|_{GM_{p\theta_2,w_2}} = \sup_{x\in\mathbb{R}^n} \|(Tf)(x+\cdot)\|_{LM_{p\theta_2,w_2}} \leq c_{48}\|w_2\|_{L_{\theta_2}(0,\infty)} \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{L_p} = c_{48}\|w_2\|_{L_{\theta_2}(0,\infty)}\|f\|_{L_p}.$$

□

7 The case of weak Morrey-type spaces

Next we consider the local and global weak Morrey-type spaces and study the boundedness of Calderon-Zygmund operators in these spaces.

Definition 3. *Let $0 < p, \theta \leq \infty$ and let w be a non-negative measurable function on $(0, \infty)$. Denote by $LWM_{p\theta,w}$ and $GWM_{p\theta,w}$, the local weak Morrey-type spaces, the global weak Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorms*

$$\|f\|_{LWM_{p\theta,w}} \equiv \|f\|_{LWM_{p\theta,w}} = \left\|w(r)\|f\|_{WL_p(B(0,r))}\right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GWM_{p\theta,w}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LWM_{p\theta,w}},$$

respectively, where

$$\|f\|_{WL_p(B(0,r))} = \sup_{t>0} t\,(meas\ \{x \in B(0,r) : |f(x)| > t\})^{1/p},$$

if $p < \infty$ and $\|f\|_{WL_\infty(B(0,r))} = \|f\|_{L_\infty(B(0,r))}$.

The spaces $LWM_{p\theta,w}$, $GWM_{p\theta,w}$ are aimed at describing the behaviour of $\|f\|_{WL_p(B(0,r))}$, $\|f\|_{WL_p(B(x,r))}$ respectively, for small $r > 0$.

Note that for any $0 < p, \theta \leq \infty$

$$\|f\|_{LWM_{p\theta,w}} \leq \|f\|_{LM_{p\theta,w}}, \quad \|f\|_{GWM_{p\theta,w}} \leq \|f\|_{GM_{p\theta,w}}$$

for all functions $f \in LM_{p\theta,w}$, $f \in GM_{p\theta,w}$ respectively.

We shall use the following theorem stating necessary and sufficient conditions for the validity of the following inequality

$$\|Tf\|_{WL_{p,v_2}} \leq c_{49}\|f\|_{L_{p,v_1}}, \tag{61}$$

where v_1 and v_2 are functions non-negative and measurable on $\mathbb{R}^n$ and $c_{49} > 0$ is independent of $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ (see [10] and [21]).

Theorem 11. *Let $1 \le p < \infty$. Then inequality* (61) *holds if, and only if, inequality* (16) *holds. Moreover, the sharp (minimal possible) constant c_{49}^* in* (61), *satisfies the inequality*

$$c_{50}\mathcal{J}_1 \le c_{49}^* \le c_{51}\mathcal{J}_1,$$

where $c_{50}, c_{51} > 0$ are independent of v_1 and v_2.

Lemma 8. [11], [14] *Let $1 \le p < \infty$ and $\gamma \ge 1$, then there exists $c_{52} > 0$ such that*

$$\|Tf\|_{WL_p(B(0,r))} \le c_{52} r^{\frac{n}{p}} \int_{\gamma r}^{\infty} t^{-\frac{n}{p}-1} \|f\|_{L_p(B(0,t))} dt \tag{62}$$

for all $r > 0$ and for all $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Consequently, Corollary 1 holds if $L_p(B(0,r))$ is replaced by $WL_p(B(0,r))$ and the condition $1 < p < \infty$ is replaced by $1 \le p < \infty$, and Lemmas 3 – 4 and Theorem 5 hold if $LM_{p\theta,w}$ and $GM_{p\theta,w}$ are replaced by $LWM_{p\theta,w}$, $GWM_{p\theta,w}$ respectively, and the condition $1 < p < \infty$ is replaced by $1 \le p < \infty$.

Theorem 12. *Let $1 \le p < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $w_1 \in \Omega_{\theta_1}$ and $w_2 \in \Omega_{\theta_2}$.*

1. If T is a genuine Calderon-Zygmund operator, then the condition (54) *is necessary for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LWM_{p\theta_2,w_2}$.*

2. If T is a Calderon-Zygmund operator, $\theta_1 \le \theta_2$ and $\theta_1 \le 1$, then condition (54) *is sufficient for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LWM_{p\theta_2,w_2}$ and from $GM_{p\theta_1,w_1}$ to $GWM_{p\theta_2,w_2}$. (In the latter case we assume that $w_1 \in \Omega_{p\theta_1}$ and $w_2 \in \Omega_{p\theta_2}$.)*

3. In particular, if T is a genuine Calderon-Zygmund operator, $\theta_1 \le \theta_2$ and $\theta_1 \le 1$, then condition (54) *is necessary and sufficient for the boundedness of T from $LM_{p\theta_1,w_1}$ to $LWM_{p\theta_2,w_2}$.*

Proof. Sufficiency follows by Theorem 5 for the weak case as in the proof of Theorem 8. The proof of necessity is also essentially the same as in the proof of Theorem 8, because (with the same notation as in the proof of Theorem 8)

$$\|Tf\|_{LWM_{p\theta_2,\omega_2}} \ge \left\| \omega_2(r) \left\| \int_{(B(0,2t)\setminus B(0,t))\cap \mathcal{R}(\beta C)} K(x,y)dy \right\|_{WL_p(B(0,r)\cap\mathcal{R}(\beta C))} \right\|_{L_{\theta_2}(0,\infty)}$$

$$\ge c_{38} \left\| \omega_2(r) \left| B(0,\rho) \cap \mathcal{R}(\beta C) \right|^{\frac{1}{p}} \right\|_{L_{\theta_2}(0,\infty)} \ge c_{39} \left\| \omega_2(r) \min\{r,t\}^{\frac{n}{p}} \right\|_{L_{\theta_2}(0,\infty)}.$$

□

Theorem 13. *Let T be a Calderon-Zygmund operator, $1 \le p < \infty$, $0 < \theta_2 \le \infty$ and $w_2 \in \Omega_{\theta_2}$. Then condition* (60) *is sufficient, and for a genuine Calderon-Zygmund operator is also necessary, for the boundedness of T from L_p to $LWM_{p\theta_2,w_2}$ and from L_p to $GWM_{p\theta_2,w_2}$. (In the latter case we assume that $w_2 \in \Omega_{p\theta_2}$.)*

Proof. The proof is similar to the proof of Theorem 10. □

8 Concluding remark

When defining the global Morrey-type spaces, it may make sense to consider a weight function w depending not only on $r > 0$, but also on $x \in \mathbb{R}^n$, and consider the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\Big\| \big\| w(x,r) \|f\|_{L_p(B(x,r))} \big\|_{L_\theta(0,\infty)} \Big\|_{L_\infty} < \infty.$$

For the case $\theta = \infty$ such quasinorms were considered in [18]. Moreover, it is also reasonable to replace L_∞ by L_η, where $0 < \eta \le \infty$, thus assuming that

$$\|f\|_{GM_{p\theta\eta,w}} = \Big\| \big\| w(x,r) \|f\|_{L_p(B(x,r))} \big\|_{L_\theta(0,\infty)} \Big\|_{L_\eta} < \infty.$$

Let in Theorem 5 formulas (31) and (32) be replaced by

$$v_{1,\delta}(x,r) = \left[w_1\left(x, r^{\frac{1}{\delta p-n}}\right) r^{\frac{1}{(\delta p-n)\theta_1} - \frac{1}{\theta_1}} \right]^p,$$
$$v_{2,\delta}(x,r) = \left[w_2\left(x, r^{\frac{1}{\delta p-n}}\right) r^{\frac{1}{\delta p-n}\left(\frac{n}{p} - \delta + \frac{1}{\theta_2}\right) - \frac{1}{\theta_2}} \right]^p$$

and formulas (33) and (34) be replaced by

$$\hat{v}_1(x,r) = w_1\left(x, r^{-\frac{p}{n}}\right) r^{-\frac{p}{n\theta_1} - \frac{1}{\theta_1}},$$

$$\hat{v}_2(x,r) = w_2\left(x, r^{-\frac{p}{n}}\right) r^{-\frac{p}{n}\left(\frac{n}{p} + \frac{1}{\theta}_2\right) - \frac{1}{\theta_2}}.$$

An argument similar to the one of the proof of Theorem 5 shows that if

$$\Big\| \, \|H\|_{C \cap L_{\frac{\theta_1}{p}, v_{1,\delta}(x,r)}(0,\infty) \to C \cap L_{\frac{\theta_2}{p}, v_{2,\delta}(x,r)}(0,\infty)} \Big\|_{L_\eta} < \infty,$$

$$\Big\| \, \|H\|_{C \cap L_{\theta_1, \hat{v}_1(x,r)}(0,\infty) \to C \cap L_{\theta_2, \hat{v}_2(x,r)}(0,\infty)} \Big\|_{L_\eta} < \infty$$

respectively, where C is the cone of all non-negative functions φ non-increasing on $(0,\infty)$ satisfying $\lim\limits_{t \to +\infty} \varphi(t) = 0$, then T is bounded from $GM_{p\theta_1\eta,w_1}$ to $GM_{p\theta_2\eta,w_2}$.

Similar remarks refer to all other inequalities of the paper involving global Morrey-type spaces or global weak Morrey-type spaces.

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