

# Thick brane solutions

Vladimir Dzhunushaliev,<sup>1,2</sup> \* Vladimir Folomeev,<sup>2</sup> † and Masato Minamitsuji<sup>3</sup><sup>1</sup>

<sup>1</sup>*Department of Physics and Microelectronic Engineering,*

*Kyrgyz-Russian Slavic University, Bishkek, Kievskaya Str. 44, 720021, Kyrgyz Republic*

<sup>2</sup>*Institute of Physicotechnical Problems and Material Science of the NAS of the Kyrgyz Republic,*  
*265 a, Chui Street, Bishkek, 720071, Kyrgyz Republic*

<sup>3</sup>*Center for Quantum Spacetime, Sogang University,*  
*Shinsu-dong 1, Mapo-gu, 121-742 Seoul, South Korea*

This article gives a comprehensive review on thick brane solutions and related topics. Such models have attracted much attention from many aspects since the birth of the brane world scenario. In many works, it has been usually assumed that a brane is an infinitely thin object; however in more general situations, one can no longer assume this. It is also widely considered that more fundamental theories such as string theory would have a minimal length scale. Many multidimensional field theories coupled to gravitation have exact solutions of gravitating topological defects, which can represent our brane world. The inclusion of brane thickness can realize a variety of possible brane world models. Given our understanding, the known solutions can be classified into topologically non-trivial solutions and trivial ones. The former class contains solutions of a single scalar (domain walls), multi-scalar, gauge-Higgs (vortices), Weyl gravity and so on. As an example of the latter class, we consider solutions of two interacting scalar fields. Approaches to obtain cosmological equations in the thick brane world are reviewed. Solutions with spatially extended branes (S-branes) and those with an extra time-like direction are also discussed.

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\* Email: vdzhunus@krsu.edu.kg

† Email: vfolomeev@mail.ru

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## I. INTRODUCTION

At present, the presence of extra dimensions is an integral part of almost all physical theories playing a fundamental role in trying to explain physical interactions based on common principles. Also, any realistic candidate for a grand unified theory, such as superstring/M-theory, should be multidimensional by necessity, otherwise, it will contain undesirable physical consequences. Traditionally, it has been considered that in such theories the observable 4-dimensional spacetime appears as a result of compactification of extra dimensions, where the characteristic size of extra dimensions becomes much less than that of 4-dimensional spacetime. But, recently, a new idea, called *brane world*, has come up, where we live on a thin leaf (brane) embedded into some multidimensional space (bulk). Because of the lack of experimental and observational data, there is no reason to prefer a particular class of multidimensional models of gravity. Actually, various kinds of multidimensional theory have been studied in modern theoretical physics.

In this article, we will review brane world models where our brane universe has a finite thickness<sup>1</sup>. The reason is mainly two-fold: Firstly, such solutions appear in various multidimensional field theories coupled to gravity, leading to a variety of possibilities of brane world, which are interesting by itself. Secondly, brane thickness should be an essential ingredient to extend the idea of brane world to multidimensional spacetimes.

Before coming to our main subject, it is important to keep the main history of multidimensional gravity in mind. Thus, we briefly discuss the history of multidimensional gravity and especially why the idea of brane world arose.

### A. Prebrane epoch of multidimensional theory

#### 1. Why is the visible universe 4-dimensional?

After the birth of Einstein's general relativity, the natural question about the dimensionality of the world which we live in appeared. Within the framework of Einstein's gravitational theory, space and time are unified and it allows us

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<sup>1</sup> In this review, we will basically follow the notations, sign conventions and definitions of fields which were adopted in the original papers.

to realize that the surrounding world is a 4-dimensional one. This statement is a consequence of a combination of the principle of Occam's razor and Einstein's idea about the geometrical nature of the gravitational field. Hereupon, the gravitational field is not an external field on the background of a specified spacetime, but a geometrical characteristic, i.e., metric, self-controlled on any sufficiently smooth manifold.

One of the questions inevitably arising in such a case is about the dimensionality of our world. Naturally, it is necessary to consider two questions concerning the dimensionality of the physical world:

- Why the physically observed dimensions of our Universe = 3 + 1 (space + time)?
- If the real dimensions are still more than 4, why extra dimensions are not observable, and what about any consequences of multidimensionality for a 4-dimensional observer?

Strictly speaking, the answer to the second question could serve as an answer to the first question. Indeed, if extra dimensions are hidden from direct observation then our world looks 4-dimensional effectively.

Although at the present time, it is not possible to give an exact answer to the first question about reasons of the 4-dimensionality of our world, investigations in this direction show that the 4-dimensional spacetime is marked out somehow among spacetimes with a different number of dimensions. Let us enumerate these properties:

- Einstein's equations in Lorentzian manifolds with dimensions  $d < 4$  lead to a flat metric.
- Maxwell's equations are conformally invariant only in a four-dimensional spacetime.
- In Newtonian gravity, circular orbits of point-like bodies in central-symmetric fields are stable only on manifolds with dimensions  $d < 4$ .
- In spacetime with dimensions  $d > 4$  the Schrödinger equation either does not have any energy level, or an energy spectrum is unbounded below.
- Huygens principle is valid only for spaces with odd dimensions.
- Quantum electrodynamics is renormalizable only in a spacetime with dimensions  $d < 4$ .

## 2. Kaluza-Klein theory

The present stage of the development of multidimensional theories of gravity began with Kaluza's paper [1] where the foundations of modern multidimensional gravitational theories have been laid. The essence of Kaluza's idea consists in the proposal that the 5-dimensional Kaluza-Klein gravitation is equivalent to 4-dimensional Einstein gravitation coupled to Maxwell's electromagnetism. In the primordial paper there was the proposal that the  $G_{55}$  metric component is a constant, and it is not varied when obtaining the gravitational equations. Later on, this proposal was excluded. Let us show the basic advantages and disadvantages of the standard 5-dimensional Kaluza-Klein theory. The advantages are:

- The 5-dimensional Einstein equations could be reduced to a form identical to the equations of 4-dimensional gravitation interacting with electromagnetic and scalar fields.
- The geodesics equations in a 5-dimensional space lead to the equations of motion for charged particles in the electromagnetic and gravitational fields.
- If one supposes that all quantities in the Kaluza-Klein theory do not depend on the 5-th coordinate  $x^5$  then it transforms as a transformation of coordinates in the following way:

$$x'^5 = x^5 + f(x^\mu), \quad \mu = 0, 1, 2, 3 \quad (1)$$

This coordinate transformation leads to the usual gradient transformations in Maxwell's electromagnetism.

The disadvantages are the following:

- Why is the 5-th coordinate unobservable in our world?
- Einstein noted [2] that it is not possible to give a natural physical meaning to the metric component  $G_{55}$ .
- The equation for a scalar field  $\mathcal{R}_{55} - G_{55}\mathcal{R} = 0$  gives the stiff and unusual connection between the 4-dimensional scalar curvature  $R$  and the invariant  $F_{\mu\nu}F^{\mu\nu}$  of the electromagnetic field.

### 3. Multidimensional theories

Next, multidimensional Kaluza-Klein theories were considered. One of the most important questions is the question about the unobservability and independence of 4-dimensional quantities on extra dimensions. The most general approach for a solution of this problem consists in obtaining the solutions of the multidimensional Einstein equations, which yield characteristic linear 4-dimensional spacetime sizes of many orders of magnitude bigger than linear sizes in extra coordinates. It is also necessary that the 4-dimensional metric of the obtained equations does not depend on extra coordinates. This could be done by refusing vacuum equations, and introducing torsion [3–6] or quadratic terms of curvature [7], or by adding multidimensional matter. The last variant called “spontaneous compactification” of the extra dimensions was suggested by Cremmer & Scherk [8, 9]. Unfortunately, this approach gave up Einstein’s original idea about the geometrization of physics. Following [10], multidimensional Kaluza-Klein theories can be classified into compactified theories [3–9, 11, 12] and projective theories [13]–[15].

According to the definition, compactified theories should satisfy the following properties:

- some periodicity condition with respect to extra coordinates is fulfilled;
- extra dimensions form some compact manifold  $M$ ;
- the linear sizes of this manifold  $M$  are small in comparison with the sizes of spacetime [11, 12].

It is necessary to note that the metric which satisfies the above properties is a solution of the vacuum (4+d)-dimensional Einstein equations only if  $M$  is a flat manifold [16]. Manifold  $M$  with constant curvature can be a solution of the Einstein equations only in the existence of multidimensional matter [8, 9], torsion [3–6] or quadratic terms of curvature [7]. In the projective geometry, a point is a set of projective points for which all relations of  $(n + 1)$ -homogeneous coordinates are the same. In such a case, each projective line is a point in a real space. Such an approach leads to the conclusion that the metric does not depend on an extra coordinate. Projective field theories do not agree with experiment. At the present time, they are not considered as realistic candidates for an unified field theory.

### 4. Problem of unobservability of extra dimensions

In the last few decades, the idea that our space, indeed, has a number of dimensions more than four became quite popular again. It is supposed that many problems of elementary particles physics could be solved by introducing strings, supersymmetry, and by increasing the number of spacetime dimensions. In increasing the number of dimensions, a legitimate question arises: why physical properties of extra dimensions differ strongly from those of the observed 4-dimensional spacetime. Two solutions to this question were suggested, namely a mechanism of spontaneous compactification and brane world models.

In the case of compactified extra dimensions, it is supposed that these dimensions are rather small and unobservable. In the papers [17]–[23] the modern approach to the problem of the unobservability of compactified extra dimensions was introduced. In a  $d$ -dimensional spacetime, it is possible to consider gravity coupled to matter. For the purpose of compactification, a special solution of the block form  $M^d = M^4 \times M^{d-4}$  is searched for, where  $M^4$  is the 4-dimensional spacetime and  $M^{d-4}$  is a compact space of extra dimensions.  $M^4$  and  $M^{d-4}$  are spaces of constant curvature. The block representation of the metric is consistent with the gravitational equations only if one supposes that the energy momentum tensor takes the block form  $T_{\mu\nu} = -\gamma_1 g_{\mu\nu}$  in  $M^4$ , and  $T_{mn} = \gamma_2 g_{mn}$  in  $M^{d-4}$ , where  $g_{\mu\nu}$  and  $g_{mn}$  are metrics of  $M^4$  and  $M^{d-4}$ . There are various mechanisms of spontaneous compactification: Freund-Rubin compactification [24] with a special ansatz for antisymmetric tensors; Englert compactification [25] setting a gauge field in an internal space equal to the spin connection; suitable embedding in a gauge group [26]; monopole or instanton mechanism [27]; compactification using scalar chiral fields [28], and compactification using radiative corrections [29].

### B. Brane world scenario

In this subsection, the new approach to the problem of unobservability of extra dimensions, called brane world scenario, is explained. This approach is quite different from the traditional compactification approach and allows even noncompact extra dimensions. According to the new approach, particles corresponding to electromagnetic, weak and strong interactions are confined on some hypersurface (called a brane) which, in turn, is embedded in some multidimensional space (called a bulk). Only gravitation and some exotic matter (e.g., the dilaton field) could propagate in the bulk. It is supposed that our Universe is such a brane-like object. This idea for the first time was formulated phenomenologically in [30]–[34], and got confirmation within the framework of string theory. Within the

brane world scenario, restriction on sizes of extra dimensions becomes weaker. It happens because particles belonging to the Standard Model propagate only in three space dimensions. But Newton's law of gravitation is sensible to existence of extra dimensions.

The prototype idea of brane world appeared a rather long time ago. Now the term "brane model" means different ways for solving some fundamental problems in high-energy physics. The pioneering works in this direction were done in the papers [30] and [31]. These prototype models have been constructed phenomenologically and considered as topological defects, i.e., thick branes in the modern terminology.

In [30], an idea that extra dimensions could be noncompactified was suggested. It was supposed that we live in a vortex enveloping multidimensional space. Akama showed that "... The Einstein equation is induced just as in Sakharov's pregeometry ...". The main idea of this paper consists in using the Higgs lagrangian in a 6-dimensional flat spacetime

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^M + D_M\phi^\dagger D^M\phi + a|\phi|^2 - b|\phi|^4 + c, \quad (2)$$

where  $F_{MN} = \partial_M A_N - \partial_N A_M$  and  $D_M\phi = \partial_M + ieA_M$ . It allows one to obtain the induced Einstein gravitation on a vortex. The corresponding field equations have the vortex solution

$$A_M = \epsilon_{0123MN}A(r)\frac{X^N}{r}, \phi = \varphi(r)e^{in\theta}, r^2 = (x^5)^2 + (x^6)^2, \quad (3)$$

where  $A(r)$  and  $\varphi(r)$  are the solutions of the Nielsen-Olsen differential equations describing the vortex. Our Universe (the vortex) is localized inside the region  $\mathcal{O}(\epsilon)$  ( $\epsilon = 1/\sqrt{a}$ ). Further, introducing curvilinear coordinates

$$X_M = Y^M(x^\mu) + n_m^M x^m, \quad M = 0, 1, 2, 3, 5, 6, \quad \mu = 0, 1, 2, 3, \quad m = 5, 6, \quad (4)$$

where  $X^M$  are the Cartesian coordinates and  $n_m^M$  are the normal vectors of the vortex, one can show that the Einstein action is induced through vacuum polarizations.

In the paper [31] the idea that we live on a domain wall (brane in modern language) and "... ordinary particles are confined inside a potential wall ...", was suggested. The quantum toy model with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_A\varphi\partial^A\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{4}\varphi^4, \quad A = 0, 1, \dots, 4 \quad (5)$$

is under consideration, which describes a scalar field in the (4+1)-dimensional spacetime  $M^{(4,1)}$  with metric  $g_{AB} = \text{diag}(1, -1, -1, -1, -1)$ . The classical field equations for the scalar field  $\varphi$  admit a domain wall solution (which is a (1+1)-dimensional kink)

$$\varphi^{cl}(x) = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{mx^4}{\sqrt{2}}\right). \quad (6)$$

This kink solution can be considered as the domain wall. In the paper the possibility of trapping particles with spins 0 and 1/2 on the domain wall was also considered.

For particles with spin 0, the linearized equation of motion for the field  $\varphi' = \varphi - \varphi^{cl}$  is under consideration

$$-\partial_A\partial^A\varphi' - m^2\varphi' - 3\lambda(\varphi^{cl})^2\varphi' = 0. \quad (7)$$

It was shown that there exist three types of perturbations:

1. the first one is of particles confined inside the wall and with the energy  $E = \vec{k}^2$ ;
2. the second one is of particles confined inside the wall, but with the energy  $E = \vec{k}^2 + \frac{3}{2}m^2$ ;
3. the third one is not confined inside the wall and at large  $x^4$  the energy is  $E = \vec{k}^2 + (k^4)^2 + 2m^2$ .

The possibility of interaction of the first type of particles with creation of the third type of particles and with subsequent escape of them beyond the domain wall was discussed.

Furthermore, the question about trapping fermions on the domain wall was considered. The Lagrangian of massless fermions is

$$\mathcal{L}_\psi = i\bar{\Psi}\Gamma^A\partial_A\Psi + \hbar\bar{\Psi}\Psi\varphi. \quad (8)$$

It was shown that for such an interaction between a fermion  $\Psi$  and a domain scalar field  $\varphi$ , the Dirac equation has a solution

$$\Psi^{(0)}(x^0, \vec{x}, x^4) = \psi(x^0, \vec{x}) \exp\left(-\hbar \int_0^{x^4} \varphi^{cl}(x'^4) dx'^4\right), \quad (9)$$

where  $\psi(x^0, \vec{x})$  is a left-handed massless (3+1)-dimensional spinor. It was also shown that there exist two types of fermion perturbations:

1. the first one is of fermions (9) confined inside the wall;
2. the second one is not confined inside the wall and the energy exceeds  $\hbar m/\sqrt{\lambda}$ . It can be created only in high energy collisions.

Note that in both papers [30] and [31] the idea about noncompactified extra dimensions was suggested. The difference consists in that in [30] our 4-dimensional world is the interior of a vortex, while in [31] our 4-dimensional world is a domain wall.

We should also mention interesting earlier attempts in the context of supergravity and superstring theory. In [35], models of domain walls in  $N = 1$  supergravity theories were considered. Using this model, it was shown that there exists a class of domain wall solutions which need not to be  $\mathbb{Z}_2$  symmetric. Such a solution describes a stable domain wall that divides two isolated but non-degenerate supersymmetric vacua, and at least one of them is an anti-de Sitter one. Moreover, it was shown that in supergravity theories gravity plays a non-trivial and crucial role for topological defects (in particular, for domain walls).

Ref. [36] examined the possibility that some of the internal dimensions, at relatively low energy scales, may be of the order of a few TeV, within the framework of a perturbative string theory which could relate the supersymmetry breaking scale to the size of the internal dimensions.

Although several ideas have been presented during the 80s and the early 90s, one of the most striking facts which activated the studies on brane models was the development in superstring theory and M-theory since the mid of 90s, especially the discovery of D-brane solutions [37, 38]. A valuable contribution to development of this model was made in the papers [39] and [40], following an earlier idea from [36]. The authors considered a flat bulk geometry of  $(4 + d)$  dimensions, when  $d$  extra dimensions are compact with radius  $R$ . All the Standard Model particles are assumed to be localized on the branes at the boundary of the  $d$ -dimensional compact space. But the gravitational interaction feels the presence of extra dimensions on the distance scales less than  $R$  and differs from Newton's law there. By integrating over the internal space, it turns out that the 4-dimensional Planck mass  $M_{Pl}$  and the  $(4 + d)$ -dimensional Planck mass  $M_{\text{fund}}$  are connected by the following relation:

$$M_{Pl}^2 = M_{\text{fund}}^{2+d} R^d. \quad (10)$$

Since Newton's law has been checked up to the distances of the order of 0.1 mm,  $R$  can be of order 0.1 mm or less.

In the papers [41–43] the model of the Universe expanding as a 3-shell in a 5-dimensional spacetime was considered. It was shown that the model can solve the hierarchy problem. Also the relation between the size of the Universe and its thickness was found.

In the papers [44] and [45], Randall and Sundrum considered a non-flat bulk space. The metric has the form

$$ds^2 = e^{-2kr_c|y|} \eta_{\mu\nu} dx^\mu dx^\nu - r_c^2 dy^2, \quad (11)$$

where  $k$  is a parameter of order of the fundamental Planck mass;  $r_c$  is the radius of a 5-th coordinate. This metric is a solution of the 5-dimensional gravitational equations following from the Lagrangian

$$S = \frac{M_*^3}{16\pi} \int d^4x dy \sqrt{-g} (-R - \lambda) + \sum_{i=1,2} \int d^4x \sqrt{-h^{(i)}} (\mathcal{L}^{(i)} - V^{(i)}), \quad (12)$$

where  $h_{\mu\nu}^{(i)}$ ,  $\mathcal{L}^{(i)}$  and  $V^{(i)}$  are the 4-dimensional metric, the Lagrangian and the vacuum energy on  $(i)$ th-brane, respectively. One can show that the relation between the 4-dimensional  $M_{Pl}$  and the fundamental  $M_*$  Planck masses is:

$$M_{Pl}^2 = \frac{M_*^3}{k} [1 - e^{-2kr_c\pi}] \sim \frac{M_*^3}{k}. \quad (13)$$

Namely, the scale of physical phenomena on the brane is fixed by the value of the warp factor. On the brane near  $y = \pi$  (observed brane), the conformal factor for the 4-dimensional metric is  $\rho^2 = e^{-2kr_c\pi}$ , and physical masses should be calibrated by taking this factor into account. For example, if  $kr_c \sim 12$  then masses of order TeV could be generated from the fundamental Planck mass  $\sim 10^{16}$  TeV.

Various physical problems have been considered in five-dimensional (and even multidimensional) brane models, especially in the Randall-Sundrum models and their extensions. See e.g., [46–50] for reviews. In many works, it was assumed that the brane is infinitely thin. Although in the thin brane approximation many interesting results have been obtained, in some situations the effects of the brane thickness cannot be neglected.

### C. Thick branes

From a realistic point of view, a brane should have a thickness. It is also widely considered that the most fundamental theory would have a minimal length scale. In some cases the effects of brane thickness can be important. The inclusion of brane thickness gives us new possibilities and new problems. In many multidimensional field theories coupled to gravity there are solutions of topological defects. They have lead to a richer variety of brane worlds (see sections II and III). At the same time, brane thickness also brings ambiguity in the definition of the effective four-dimensional quantities (see section V).

We now give our precise definition of thick branes, to avoid possible problems related to possible differences in terminology. Our definition is based on the following form of a multidimensional metric: For five-dimensional problems, the solutions whose metric are

$$ds^2 = a^2(y)g_{\mu\nu}dx^\mu dx^\nu - dy^2, \quad (14)$$

where  $-\infty < y < \infty$  is the coordinate of the extra dimension, are considered. The four-dimensional  $g_{\mu\nu}$  is Minkowski or de Sitter (or anti-de Sitter) spacetime.  $a(y)$  is the warp function, which is regular, has a peak at the brane and falls off rapidly away from the brane. A typical behavior of the warp function in thin and thick brane solutions is shown in Fig. 1. When the  $Z_2$ -symmetry is assumed,  $a(y) = a(-y)$ . The normalizability of the graviton zero mode gives the condition that  $\int_{-\infty}^{\infty} dy a(y)^4$  is nonvanishing and finite. In five-dimensional problems, the thin brane approximation is

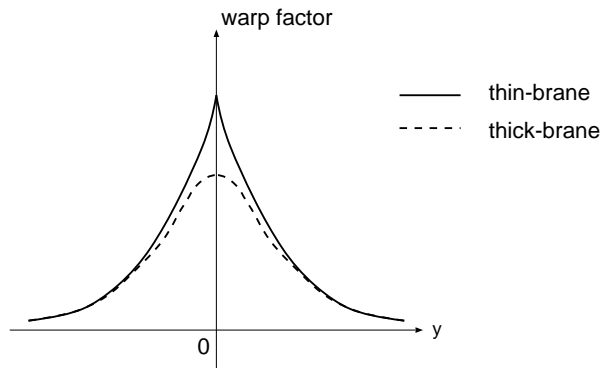


FIG. 1: The typical behavior of the warp factor  $a(y)$  in the thin and the thick brane solutions are shown. The dashed line denotes the warp factor of the thick brane, while the solid one does that of the thin brane.

valid as long as the brane thickness cannot be resolved, in other words, if the energy scale of the brane thickness is much higher than those in the bulk and on the brane. In contrast, when thickness becomes as large as the scale of interest, its effect is no longer negligible.

With more than six-dimensional spacetimes,  $n > 1$ , it is assumed that the metric function has the form

$$ds^2 = a^2(r)g_{\mu\nu}dx^\mu dx^\nu - b^2(r)(dr^2 + r^2 d\Omega_{n-1}^2). \quad (15)$$

The four-dimensional metric  $g_{\mu\nu}$  also represents Minkowski or de Sitter spacetime.  $d\Omega_{n-1}^2$  is the line element of an  $(n-1)$ -dimensional compact manifold (the unit  $(n-1)$ -sphere in most cases). The extra coordinates are composed of the radial one  $r$  ( $0 < r < \infty$ ), and those of the  $(n-1)$ -dimensional ones. It is possible to put the brane at the center

$r = 0$  by an appropriate coordinate choice. Here  $a(r)$  is the warp factor, which has a peak at the brane. The functions  $a(r)$  and  $b(r)$  are regular everywhere and the integral  $\int_0^\infty dr a^4 b^n r^{n-1}$  is finite, in order to obtain a localized graviton zero mode. In such higher-codimensional cases, the situation is quite different from that in the five-dimensional one. In the context of the thin brane approximation, in approaching the brane, usually self-interactions of bulk matter, e.g., gravitation, are divergent. Then, it is impossible to describe the matter interactions localized on the brane. To study such models, brane thickness becomes an essential ingredient which plays the role of an effective UV cut-off.

Note that it is obvious that such a definition is rather ambiguous: for example, one can redefine the coordinate  $r$  in the following way

$$\sqrt{b(r)}dr = dz, \quad (16)$$

and then the brane metric can be rewritten as follows

$$ds^2 = a^2(z)g_{\mu\nu}dx^\mu dx^\nu - [dz^2 + r^2(z)d\Omega_{n-1}^2]. \quad (17)$$

In sections II-IV, exact solutions of static thick brane solutions in various classes of field theories will be considered. In sections V-VI, time-dependent situations are considered, and then the bulk metric is no longer the form of Eq. (14) or Eq. (15).

#### D. Purpose and construction

In this paper, we will give a comprehensive review of works devoted to thick brane solutions, and related topics. To our knowledge there has been no review on this subject although an enormous number of works have already been done. Thus, now is an appropriate time to collect these works. We hope that this paper becomes a good starting point for new studies.

In our understanding, all known thick brane solutions can be classified into two large groups: *static* solutions (presented in Sections II-IV) and solutions *depending on time* (presented in Sections V-VI). In turn, the static solutions could be classified as:

1. Topologically nontrivial thick branes: These solutions are based on the assumption of existence of some topological defects in spacetime. Solutions in this class are composed of thick branes made of either a single scalar field or non-interacting multi-scalar fields (and so on).
2. Topologically trivial thick branes: These solutions can exist for interacting scalar fields but they are absent in the case of one scalar field.

Several representative approaches to formulate cosmological equations on thick branes will be reviewed (section V). Finally, the regular S-brane solutions will also be explained (section VI).

As a final remark, we will not discuss applications of thick brane models to high-energy physics and the problem of localization of field with various spins. In this review, we are going to focus on their geometrical and topological properties only.

## II. TOPOLOGICALLY NONTRIVIAL THICK BRANES

### A. Definition of topologically nontrivial solutions

In this review, we use the classification of thick brane solutions given in section ID. According to this classification, one type of solution is topologically nontrivial. In this section we give the definition of topologically nontrivial solutions. This classification is based on definitions of kink-like and monopole-like solutions.

Kink-like, topologically nontrivial solutions are considered in two dimensions (one space and one time dimensions). Such solutions come up in considerations of problems with one scalar field  $\phi$  with a potential energy having two or more degenerate vacuum states. One example is the well-known Mexican hat potential [52]

$$U(\phi) = \frac{1}{4}\lambda \left( \phi^2 - \frac{m^2}{\lambda} \right)^2,$$

where  $\lambda$  and  $m^2$  are coupling and mass parameters respectively. In this case, there exist localized solutions which go to  $\phi_{\pm\infty} = \pm m/\sqrt{\lambda}$  (the so-called kink) asymptotically. The solution with an asymptotic behavior  $\phi_{\pm\infty} = \mp m/\sqrt{\lambda}$  is



known as the anti-kink. In the papers reviewed below the asymptotic values of the scalar field  $\phi_{\pm\infty}$  can be shifted by  $\phi_0$ , i.e.  $\phi_{\pm\infty} = \pm m/\sqrt{\lambda} + \phi_0$ .

We now consider the case when the dimensionality of space is  $n > 1$ , and when the number of scalar fields is  $m > 1$ . We take the following mapping of a sphere  $S_{n-1}$  on a sphere  $S_{m-1}$ : sphere  $S_{n-1}$  is an infinitely remote sphere in a space of extra dimensions, i.e. it is a sphere at  $r \rightarrow \infty$  in the metric

$$dl_{ed}^2 = b(r)(dr^2 + r^2 d\Omega_{n-1}^2). \quad (18)$$

Here  $dl_{ed}^2$  is the metric in the extra dimensions; the metric  $b(r)r^2 d\Omega_{n-1}^2$  is the metric on a sphere  $S_{n-1}$ . The sphere  $S_{m-1}$  is a sphere in the abstract space of the scalar fields  $\phi^a, a = 1, 2, \dots, m$ .

Using this set up the mapping is defined by

$$\vec{n} = \frac{\vec{r}}{r} \rightarrow \left. \frac{\phi^a}{|\phi^a|} \right|_{\vec{r} \rightarrow \infty}. \quad (19)$$

A picture of this mapping is given schematically in Fig. 2. This mapping maps the point  $\frac{\phi^a}{|\phi^a|} \in S_{m-1}$  in the space of the scalar fields to the vector  $\vec{n} = \frac{\vec{r}}{r} \in S_{n-1}$  in a space of the extra dimensions.

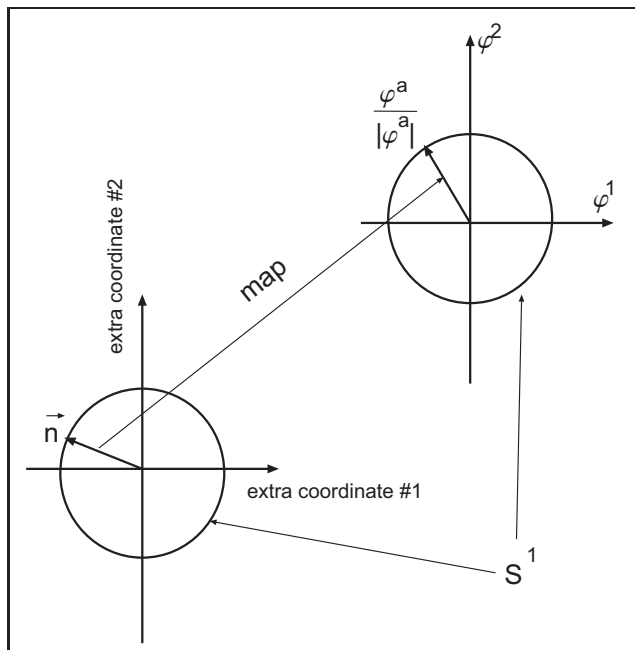


FIG. 2: Map  $\vec{n} = \frac{\vec{r}}{r} \rightarrow \left. \frac{\phi^a}{|\phi^a|} \right|_{\vec{r} \rightarrow \infty}$  for a case  $n = m = 2$

From the topological point of view, we have the following map

$$\pi_{n-1}(S_{m-1}) = S_{n-1} \rightarrow S_{m-1}, \quad (20)$$

where  $\pi_{n-1}(S_{m-1})$  is a homotopy group of the map  $S_{n-1} \rightarrow S_{m-1}$ . It is known from topology that all nonsingular maps from one sphere  $S_{n-1}$  into another one  $S_{m-1}$  can be divided into homotopic sectors. Maps inside of one sector can pass continuously each into other, but maps of two different sectors cannot pass continuously each into other.

For example [53]

$$\pi_n(S_n) = Z, \quad (21)$$

$$\pi_n(S_m) = 0 \quad \text{by} \quad n < m, \quad (22)$$

$$\pi_n(S_1) = 0 \quad \text{by} \quad n > 1, \quad (23)$$

where  $Z$  is a group of integer numbers, i.e. the maps  $S_n \rightarrow S_n$  are divided into a discrete set of homotopic classes which are characterized by integers. In (22) and (23), zero on the right hand side means that the group is trivial, i.e. in this case all maps can be deformed each into other.

We will call a thick brane solution topologically nontrivial if the corresponding homotopy group  $\pi_{n-1}(S_{m-1})$  is nontrivial. Note that the case  $n = 1$  and  $m = 1$  is topologically nontrivial.

## B. Thick brane solutions with a single scalar field

In this subsection, as the simplest example of a topologically nontrivial solution, we review the five-dimensional thick brane solutions with a single scalar field.

### 1. General properties

First, we review the general properties of thick brane solutions of a single scalar field which have the four-dimensional Poincare symmetry, following Ref. [54, 55]. We consider a five-dimensional Einstein-scalar theory

$$S = \frac{1}{2} \int d^5x \sqrt{-g} [R - (\partial\phi)^2 - 2V(\phi)]. \quad (24)$$

In general, the metric of a static five-dimensional spacetime with the four-dimensional Poincare symmetry can be written as

$$ds^2 = e^{2F(z)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{8F(z)} dz^2. \quad (25)$$

For simplicity, it is assumed that the bulk geometry is  $Z_2$ -symmetric across the center of the brane, hence  $F(-z) = F(z)$ ,  $\phi(-z) = -\phi(z)$ , and the imposed boundary conditions are

$$F(0) = F'(0) = 0, \quad \phi(0) = 0. \quad (26)$$

In Ref. [54, 55], the properties of thick brane solutions were investigated and it was shown that the only possible asymptotic geometry of the thick brane spacetime is asymptotically AdS, and is only possible if the potential  $V(\phi)$  has an alternating sign and satisfies the fine-tuning condition

$$\bar{V}(\infty) = 0, \quad \bar{V}(z) := \int_0^z \sqrt{-g} V(\phi(z)) dz = \int_0^z e^{8F(z)} V(\phi(z)) dz. \quad (27)$$

The zero thickness limit is well-defined and the solution reduces to the Randall-Sundrum model if the asymptotic value of  $V(\phi)$  approaches a constant, which is the bulk cosmological constant. Note that if the brane has the four-dimensional de Sitter or anti-de Sitter symmetry, the fine-tuning condition (27) may not be satisfied.

### 2. Single scalar thick brane solutions

In this subsection, we review the various thick brane solutions whose geometry can be Minkowski, de Sitter or anti-de Sitter, although in terms of the cosmological applications we focus on the first two cases. Many of previously known solutions are discussed in the context of the  $Z_2$ -symmetric bulk spacetime but non- $Z_2$  symmetric solutions have also been reported. The thick brane solutions of more than two scalar fields or of other fields will be reported, separately.

*a. Set-up* We focus on the Einstein-scalar theory given in Eq. (24). Most of solutions reviewed here have the following form of the metric

$$ds^2 = dy^2 + a(y)^2 \gamma_{\mu\nu} dx^\mu dx^\nu, \quad \phi = \phi(y) \quad (28)$$

with the four-dimensional Ricci curvature tensor being  $R[\gamma_{\mu\nu}] = 12K$ . The constant  $K = 0, 1, -1$  represents the four-dimensional Minkowski, de Sitter and AdS sections, respectively. It is also assumed that the scalar field depends only on the extra space coordinate  $y$ :  $\phi = \phi(y)$ . If the  $Z_2$ -symmetry is imposed, namely  $a(y) = a(-y)$  and  $\phi(y) = -\phi(-y)$  (the center of the brane is placed at  $y = 0$ ), the boundary conditions for metric and scalar field are given by

$$a'(0) = 0 \quad \phi(0) = 0, \quad (29)$$

where ‘‘prime’’ denotes a derivative with respect to  $y$ . The warp factor is normalized so that  $a(0) = 1$ . At the infinity  $y \rightarrow \pm\infty$ , the scalar field approaches some constant value  $\phi = \pm\phi_\infty$ , where for a kink  $\phi_\infty > 0$  and for an anti-kink  $\phi_\infty < 0$ . By minimizing the action Eq. (24), the Einstein and scalar equations are explicitly given by

$$\begin{aligned} 6\frac{a'^2}{a^2} - 6\frac{K}{a^2} &= \frac{1}{2}\phi'^2 - V, \\ 3\frac{a''}{a} + 3\frac{a'^2}{a^2} - 3\frac{K}{a^2} &= -\frac{1}{2}\phi'^2 - V, \\ \phi'' + \frac{4a'}{a}\phi' - \frac{dV}{d\phi} &= 0. \end{aligned} \quad (30)$$

Sometimes, it is useful to reduce the above Einstein-scalar equations into first order equations. This can be achieved by introducing the auxiliary superpotential  $W(\phi)$  [56–61], in terms of which the scalar field potential is given as

$$V = -6W(\phi)^2 + \frac{9}{2\gamma^2} \left( \frac{dW}{d\phi} \right)^2, \quad \gamma := \sqrt{1 + \frac{K}{a^2 W^2}}. \quad (31)$$

Using this, the above equations become

$$\phi' = \frac{3}{\gamma} \frac{dW}{d\phi}, \quad \frac{a'}{a} = -\gamma W(\phi). \quad (32)$$

For a given superpotential  $W$  the scalar field profile is obtained by integrating the first equation. Then, the metric function is determined by integrating the second equation.

More directly, for a given metric function, one can determine the scalar field profile and potential, using the procedure of [51]. But sometimes, it is impossible to give the potential as an analytical function of the scalar field, rather than as a function of the bulk coordinate. Another method [51, 62] is that first one gives the profile of the scalar field, and then the metric function and the potential are determined through the field equations.

*b. Minkowski brane solutions* Here the simplest cases are solutions which contain the four-dimensional section which has the Poincare symmetry  $K = 0$ . The solutions which are obtained from the simple superpotential

$$W = c \sin(b\phi) \quad (33)$$

were discussed in [57–59, 63–65] (See also the textbook [50]). The corresponding scalar potential has the sine-Gordon form

$$V(\phi) = \frac{3}{2} c^2 [3b^2 \cos^2(b\phi) - 4 \sin^2(b\phi)]. \quad (34)$$

The solution is given by

$$a(y) = \left[ \frac{1}{\cosh(cb^2 y)} \right]^{1/3b^2}, \quad \phi(y) = \frac{2}{b} \arctan \left[ \tanh \left( \frac{3cb^2 y}{2} \right) \right]. \quad (35)$$

In the asymptotic limit  $y \rightarrow \infty$ ,  $\phi(\pm\infty) = \pm\pi/(2b)$  the potential approaches the negative values  $V(\pm\infty) = -6c^2$ . The bulk is asymptotically AdS. This class of solutions also contains the solution derived in [51],

$$a(y) = \left[ \frac{1}{e^{-2n(y+y_0)} + e^{2n(y+y_0)}} \right]^{1/2n},$$

$$\phi(y) = \pm \sqrt{\frac{6}{n}} \arctan(e^{2ny}), \quad (36)$$

whose potential is given by

$$W(\phi) = \cos \left( \frac{\sqrt{6n}}{3} \phi \right), \quad V(\phi) = -6 \cos^2 \left( \frac{\sqrt{6n}}{3} \phi \right) + 3n \sin^2 \left( \frac{\sqrt{6n}}{3} \phi \right). \quad (37)$$

The dimensionless parameter  $n$  represents the brane thickness. The solution is stable against perturbations.

In Ref. [66], another type of thick brane solution was discussed, which was obtained from the superpotential

$$W(\phi) = \frac{av}{3} \phi \left( 1 - \frac{\phi^2}{3v^2} \right), \quad (38)$$

leading to the potential

$$V(\phi) = \frac{\lambda}{4} (\phi^2 - v^2)^2 - \frac{\lambda}{27M^2} \phi^2 \left( \phi^2 - \frac{v^2}{3} \right)^2, \quad (39)$$

where  $\lambda := 2a^2/v^2$  is scalar self-coupling constant and the gravitational scale  $M$  is shown explicitly. In the absence of gravity  $M \rightarrow \infty$ , the theory reduces to the  $\lambda\phi^4$  scalar field model. The scalar and metric functions are given by

$$\phi(y) = v \tanh(ay),$$

$$M^3 \ln a(y) = C_0 - \frac{8}{27} v^2 \ln [\cosh(ay)] - \frac{v^2}{54} \tanh^2(av), \quad (40)$$

where  $C_0$  is an integration constant which represents the overall scaling of our four-dimensional world (See also the textbook [50]). It is trivial to check that the geometry reduces to the five-dimensional Minkowski geometry in the decoupling limit  $M \rightarrow \infty$ . In Ref. [67] the potential

$$V(\phi) = -V_0 + \frac{\lambda}{4} (\phi^2 - v^2)^2$$

was considered. Here  $V_0 > 0$ . This potential gave rise to thick Minkowski brane solution with the asymptotically AdS bulk.

*c. de Sitter brane solutions* Here, we review some known thick de Sitter brane solutions with  $K = +1$ .

In Ref. [51], the stable de Sitter brane version of the solution Eq. (36) was discussed, whose metric function is given by

$$a(y) = \left[ \frac{1}{\sinh^{-2n}(y + y_0) + \sinh^{2n}(y + y_0)} \right]^{1/2n}. \quad (41)$$

But there is no analytic form for  $\phi(y)$  and  $V(\phi)$  [51].

In Ref. [60, 61], for the scalar field potential

$$V(\phi) = \frac{9H^2(\beta^2 - 1)}{2\beta^2} + \frac{15H^2(\beta^2 + 1)}{2\beta^2} \cos \left( \sqrt{\frac{2}{3}} \phi \right), \quad (42)$$

the stable solutions

$$\begin{aligned} a(y) &= -\text{sn}(Hy, i\beta^{-1}), \\ \phi(y) &= \sqrt{6} \arctan \left[ \frac{\text{cn}(Hy, i\beta^{-1})}{\beta \text{dn}(Hy, i\beta^{-1})} \right] \end{aligned} \quad (43)$$

were derived, where the elliptic functions are defined by

$$\text{sn}^{-1}(z, k) = \int_0^z \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad (44)$$

$\text{cn}(z) = (1 - \text{sn}^2(u, k))^{1/2}$ , and  $\text{dn}(z) = (1 - k^2 \text{sn}^2(u, k))^{1/2}$ .

Other interesting solutions were given for an axionic scalar field potential [68–73]

$$V(\phi) = V_0 \cos^{2(1-\sigma)} \left( \frac{\phi}{\phi_0} \right), \quad \phi_0 := \sqrt{3\sigma(1-\sigma)}, \quad (45)$$

where  $0 < \sigma < 1$ . The  $Z_2$ -symmetric solution [68] is given by

$$a(z) = \left[ \cosh \left( \frac{Hz}{\sigma} \right) \right]^{-\sigma}, \quad \phi = \phi_0 \arcsin \left[ \tanh \left( \frac{Hz}{\sigma} \right) \right], \quad (46)$$

where  $dy = a(z)dz$  and the brane expansion rate  $H$  is related to the thickness and potential parameter through

$$H^2 = \frac{2\sigma V_0}{3(1+3\sigma)}. \quad (47)$$

The physical thickness of the brane is given by  $\sigma/H$ . In the limit  $\sigma \rightarrow 0$ , the solution smoothly approaches a thin de Sitter brane embedded in a five-dimensional Minkowski bulk.

Extensions of Eq. (46) to the non- $Z_2$  symmetric bulk were considered in Ref. [71–73]. Other thick de Sitter brane solutions with more complicated metric functions were reported in Ref. [62].

### C. Thick brane solutions in generalized scalar field theories

In this subsection, we review the thick brane solutions in more general theories of a single scalar field, i.e., a scalar field with a non-standard kinetic term [74–77], a scalar field non-minimally coupled to gravity [78], a phantom scalar field [79, 80], and a tachyon field [81].

1. *Scalar field with a non-standard kinetic term*

The paper [74] discussed the case of a scalar field with non-standard kinetic term. Such a model was initially considered as the kinetic generalization of the single field inflation models [82, 83] and extensively studied for modeling inflation and dark energy. Here it is applied to a five-dimensional space-time to find new thick brane solutions. The starting point is the action

$$S = \int d^5x \sqrt{g} \left( -\frac{1}{4}R + \mathcal{L}(\phi, X) \right), \quad (48)$$

where  $g_{ab}$  is the five-dimensional metric with the signature  $(+, -, -, -, -)$ , and the five-dimensional gravitational constant is chosen  $G^{(5)} = 1/(4\pi)$ . The indices  $M, N, \dots$  and  $\mu, \nu, \dots$  run over the five- and four-dimensional space-time, respectively. We now define  $X := (1/2)g^{MN}\nabla_M\phi\nabla_N\phi$ . Then, the equations of motion are given by

$$G^{MN}\partial_M\partial_N\phi + 2X\mathcal{L}_{X\phi} - \mathcal{L}_\phi = 0, \quad (49)$$

where  $G^{MN} = \mathcal{L}_X g^{MN} + \mathcal{L}_{XX}\nabla^M\phi\nabla^N\phi$ . The Minkowski thick brane solutions whose metric is given by

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \quad (50)$$

are considered. The equations of motion are

$$\begin{aligned} (\mathcal{L}_X + 2X\mathcal{L}_{XX})\phi'' - (2X\mathcal{L}_{X\phi} - \mathcal{L}_\phi) &= -4\mathcal{L}_X\phi'A', \\ A'' &= \frac{4}{3}X\mathcal{L}_X, \\ (A')^2 &= \frac{1}{3}(\mathcal{L} - 2X\mathcal{L}_X), \end{aligned} \quad (51)$$

where the prime is derivative with respect to  $y$ . For convenience, the (superpotential) function  $W(\phi)$ , such that  $\mathcal{L}_X\phi' = W_\phi$ , is introduced. The scalar field Lagrangian is assumed to be

$$\mathcal{L} = F(X) - V(\phi), \quad (52)$$

where  $F$  can be an arbitrary function of  $X$ . Note that  $F(X) = X$  for the standard canonical scalar field. The equations of motion now reduce to a set of the first order equations that

$$\begin{aligned} F'(X)\phi' &= \frac{1}{2}W_\phi, \\ F - 2F'(X)X - V(\phi) &= \frac{1}{3}W^2. \end{aligned} \quad (53)$$

In Ref. [74], two specific examples of the choice of the function  $F$  were considered: (I)  $F_I = X + \alpha|X|X$ , and (II)  $F_{II} = -X^2$ .

*a. Case (I)* In case (I),  $F_I = X + \alpha|X|X$ , it is difficult to derive solutions for general values of  $\alpha$ . Thus, Ref. [74] considered the case of small  $\alpha$  and discussed perturbations around the case of a canonical scalar field. Then, the behaviors of scalar and metric functions are given by

$$\phi(y) = \phi_0(y) - \frac{\alpha}{2}\phi'_0(y)W(\phi_0(y)), \quad A(y) = A_0(y) + \frac{\alpha}{12}W(\phi_0(y))^2, \quad (54)$$

where  $\phi_0(y)$  is the solution in the case of  $\alpha = 0$  (the canonical case). For the specific choice that  $W(\phi) = 3a \sin(b\phi)$ , one finds the scalar potential

$$V = \frac{9}{8}a^2b^2 \cos^2(b\phi) - 3a^2 \sin^2(b\phi) - \frac{81\alpha}{64}a^4b^4 \cos^4(b\phi) + O(\alpha^2). \quad (55)$$

The scalar and metric functions were found as

$$\begin{aligned} \phi(y) &= \frac{1}{b} \arcsin \left[ \tanh \left( \frac{3}{2}ab^2y \right) \right] - \frac{9\alpha a^2b}{4} \frac{\sinh \left( \frac{3}{2}ab^2y \right)}{\cosh^2 \left( \frac{3}{2}ab^2y \right)} + O(\alpha^2), \\ A(y) &= -\frac{2}{3b^2} \ln \left( \cosh \left( \frac{3}{2}ab^2y \right) \right) + \frac{3a^2\alpha}{4} \tanh^2 \left( \frac{3}{2}ab^2y \right) + O(\alpha^2). \end{aligned} \quad (56)$$

For larger value of  $\alpha$ , the peak of the warp factor  $A$  is spread out and the energy density at the center of the brane decreases.

b. *Case (II)* In case (II),  $F_{II} = -X^2$ , the choice of the superpotential,

$$W(\phi) = 9a^3b^2 \sin(b\phi) \left(2 + \cos^2(b\phi)\right), \quad (57)$$

leads to

$$V(\phi) = \frac{243}{8}a^4b^4 \cos^4(b\phi) - 27a^6b^4 \sin^2(b\phi) \left(2 + \cos^2(b\phi)\right)^2. \quad (58)$$

An exact solution was found

$$\phi(y) = \frac{1}{b} \arcsin(\tanh(3ab^2y)), \quad A(y) = -\frac{a^2}{6} \tanh^2(3ab^2y) - \frac{2}{3}a^2 \ln(\cosh(3ab^2y)). \quad (59)$$

The effective brane thickness is of order  $(ab^2)^{-1}$ , which can become larger for smaller  $a$  and  $b$ .

The work of [75, 76] considered the model composed of the non-linear kinetic term  $F(X) = X|X|$  and the self-interacting potential  $V(\phi) = \lambda^4(\phi^2 - \phi_0^2)^2$ , where  $\phi = \pm\phi_0$  corresponds to the global minima and  $\lambda$  is the self-coupling of the scalar field. A solitonic solution coupled to gravity, which can be interpreted as a thick brane, was obtained by employing a series expansion method and a numerical one in [76], while an exact solution without gravity was originally derived in [75]. The solution is stable against the linear perturbations. In addition, this model has an appealing property that the propagation of perturbations of the scalar outside the thick brane is completely suppressed and the scalar field is automatically restricted to the brane.

The recent work of [77] discussed the de Sitter or AdS thick brane solutions in more general class of models that

$$\mathcal{L} = F(X, \phi) - V(\phi). \quad (60)$$

The line elements of the de Sitter and AdS brane solutions are assumed to be

$$ds^2 = e^{2A(y)}(dt^2 - e^{2\sqrt{\Lambda}t}(dx_1^2 + dx_2^2 + dx_3^2)) - dy^2, \quad (61)$$

and

$$ds^2 = e^{2A(y)}(e^{-2\sqrt{-\Lambda}x_3}(dt^2 - dx_1^2 - dx_2^2) + dx_3^2) - dy^2, \quad (62)$$

respectively. The choice of  $F(X, \phi) = X\phi^m$ , where  $m = 0, 1, 2, \dots$  is considered. The solutions are given in assuming that  $A = \ln(\cos(by))$ . Note that in this ansatz there are naked singularities at  $y = \pm y^* := \pm\pi/(2b)$ .

The de Sitter brane solutions are obtained as follows: For  $m = 2n$  where  $n = 0, 1, 2, \dots$ ,

$$\phi(y) = [(n+1)\beta \operatorname{arcsinh}(\tan(by))]^{1/(n+1)}, \quad (63)$$

where  $\beta = \frac{1}{b}\sqrt{(3/2)(b^2 - \Lambda)}$ , for the potential

$$V(\phi) = 3b^2 - \frac{9}{4}(b^2 - \Lambda) \cosh^2\left[\frac{\phi^{n+1}}{(n+1)\beta}\right]. \quad (64)$$

Note that for an odd  $n$  the solution of  $\phi(y)$  is invalid for a negative  $y$ , while for an even  $n$  the solution of  $\phi(y)$  is valid everywhere in  $-y^* < y < y^*$ . For  $m = 2n+1$  where  $n = 0, 1, 2, \dots$ , the de Sitter brane solutions are given by

$$\phi(y) = \left[\left(n + \frac{1}{2}\right)\beta \operatorname{arcsinh}(\tan(by))\right]^{2/(2n+1)}, \quad (65)$$

where  $\beta = \frac{1}{b}\sqrt{(3/2)(b^2 - \Lambda)}$ , for the potential

$$V(\phi) = 3b^2 - \frac{9}{4}(b^2 - \Lambda) \cosh^2\left[\frac{2\phi^{(2n+1)/2}}{(2n+1)\beta}\right]. \quad (66)$$

The solution is invalid for a negative  $\phi$ . The de Sitter brane solutions exist only for  $0 < \Lambda < b^2$ . The AdS brane solutions are obtained by reversing the sign of  $\Lambda$ . Then,  $\Lambda$  must take a value in the range of  $-b^2 < \Lambda < 0$ .

The case of  $m = 0$  recovers the solutions with naked singularities in the bulk obtained in the case of a canonical scalar field [84].

## 2. Non-minimally coupled scalar field

Ref. [78] considered thick brane solutions of a scalar field non-minimally coupled to the scalar curvature:

$$S = \int d^5x \sqrt{-G} \left[ f(\phi)R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right], \quad (67)$$

where  $f(\phi)$  denotes the direct coupling of the scalar field to gravity and the five-dimensional metric  $G_{ab}$  has the signature  $(-, +, +, +, +)$ . The above action is conformally related to the Einstein frame action with the Ricci scalar term  $(2M^3R)$  via the conformal transformation  $G_{ab} \rightarrow \tilde{G}_{ab}f(\phi)/(2M^3)$ , where  $M$  is the five-dimensional gravitational scale. The metric ansatz was taken to be

$$ds^2 = e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2. \quad (68)$$

The coupling function  $f(\phi)$  was chosen to be  $f(\phi) = 2M^3 - \frac{1}{2}\xi\phi^2$ , where the cases of  $\xi = 0$  and  $\xi = 3/16$  represent the minimal and conformal couplings, respectively. The Einstein equations are given by

$$\begin{aligned} V(\phi) &= -\frac{3}{2}\left(2M^3 - \frac{\xi}{2}\phi^2\right)\left(2\dot{A}^2 + \ddot{A}\right) + \frac{7}{2}\xi\dot{A}\dot{\phi}\phi + \xi\xi^2, \\ \frac{1}{2}\dot{\phi}^2 &= -\frac{3}{2}\left(2M^3 - \frac{\xi}{2}\phi^2\right)\ddot{A} - \frac{\xi}{2}\dot{A}\dot{\phi}\phi + \xi\ddot{\phi}\phi + \xi\dot{\phi}^2. \end{aligned} \quad (69)$$

In the examples below, the scalar field potential is obtained once the scalar and metric profiles are found.

In the simplest case,  $\xi = 0$ , the solution was given by

$$\phi(y) = \phi_0 \tanh(ay), \quad e^A = (\cosh(ay))^{-\gamma} e^{-\gamma \tanh^2(ay)/4}, \quad (70)$$

where  $\gamma = \phi_0^2/(9M^3)$ . For a non-zero coupling constant  $\xi \neq 0$ , by choosing

$$e^{A(y)} = (\cosh(ay))^{-\gamma} \quad (71)$$

the solution  $\phi(y) = \phi_0 \tanh(ay)$  was obtained with

$$\gamma = 2\left(\frac{1}{\xi} - 6\right), \quad \phi_0 = a^{-1}\dot{\phi}(0) = (2M^3)^{1/2} \sqrt{\frac{6(1-6\xi)}{\xi(1-2\xi)}}. \quad (72)$$

The above solution exists for  $0 < \xi < 1/6$ . The scalar curvature of this solution approaches  $R \rightarrow -20a^2(\xi^{-1} - 6)^2$  for  $y \rightarrow \infty$ , namely the bulk is asymptotically AdS. Eq. (71) is also the metric function of the solution whose scalar field profile is given by  $\phi(y) = \phi(0)(\cosh(ay))^{-1}$ , with  $\phi^2(0) = 12M^3(\xi^{-1} - 6)/(3 - 16\xi)$ .

Another solution with  $\phi(y) = \phi_0 \tanh(ay)$ , where  $\phi_0 = 2\sqrt{\xi^{-1}M^3}$ , and

$$A(y) = -4 \ln \left( \cosh(ay) \right) + \frac{1}{3}(8 - \xi^{-1}) \tanh^2(ay) F_{P F Q} \left( \left\{ 1, 1, 7/6 \right\}, \left\{ 3/2, 2 \right\}, \tanh^2(ay) \right), \quad (73)$$

where  $F_{P F Q}$  represents the generalized hypergeometric function, was also found. Note that in all the above nontrivial solutions there is no smooth limit to the minimally coupled case  $\xi = 0$ .

## 3. Phantom scalar field

Ref. [79] considered a model of a bulk phantom scalar field:

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa_5^2} (R - 2\Lambda) + \frac{1}{2}(\nabla\phi)^2 - V(\phi) \right], \quad (74)$$

where  $g_{ab}$  is the five dimensional metric with signature  $(-, +, +, +, +)$  and  $V(\phi)$  is the potential of the scalar field  $\phi$ . Note that sign of the kinetic term is now negative, i.e., *phantom*. The thick brane solutions for the phantom scalar field with the ansatz

$$ds^2 = dy^2 + e^{-2f(y)}\eta_{\mu\nu}dx^\mu dx^\nu \quad (75)$$

and  $\phi = \phi(y)$  are considered. The equations of motion are given by

$$\begin{aligned} f'' &= -a(\phi')^2, \\ (f')^2 &= \frac{a}{4} \left( -(\phi')^2 - 2V \right) - \frac{\Lambda}{6}, \\ \phi'' - \frac{4f'}{f} \phi' &= -\frac{dV}{d\phi}, \end{aligned} \quad (76)$$

where  $a := \kappa_5^2/3$ . An explicit solution was obtained for the sine-Gordon potential

$$V(\phi) = B \left( 1 + \cos \left( \frac{2\phi}{A} \right) \right) \quad (77)$$

as

$$\begin{aligned} f(y) &= -\frac{a}{\kappa_1} \sqrt{\frac{|\Lambda|}{6}} \ln \cosh \left( \frac{\kappa_1}{a} y \right), \\ \phi(y) &= 2A \arctan \left[ \exp \left( \frac{\kappa_1}{a} y \right) \right] - \frac{\pi A}{2}, \end{aligned} \quad (78)$$

where  $\kappa_1 := (1/A^2) \sqrt{|\Lambda|/6}$ . Here the constant  $B$  has to be chosen as  $B = (|\Lambda|/6a^2) (a - 1/(4A^2))$ . The solution has a growing warp factor for the larger value of  $|y|$ , in contrast to the case of the normal kinetic term.

In [80], by using a massless scalar phantom/ghost field, an analytic time-dependent warped solution forming a standing wave in the bulk was found. The nodes of the standing wave are interpreted as the different four-dimensional Minkowski vacua, called *islands* in Ref. [80], having different physical parameters such as gravitational and cosmological constants. This model possesses the following characteristic features: 1) The ordinary four-dimensional matter can be localized on the tensionless branes, i.e. on the nodes. 2) It is possible to realize a new gravitational localization mechanism when matter fields can reside only on Minkowski *islands*, where the background space-time does not oscillate.

#### 4. Tachyon field

As a final example of a generalized scalar field theory, consider a model of a tachyon field [81] whose action is given by

$$S = \int d^5x \sqrt{-g} \left[ \frac{1}{2\kappa_5^2} (R - 2\Lambda) + V(T) \sqrt{1 + g^{MN} \partial_M T \partial_N T} \right], \quad (79)$$

where  $\Lambda_5$  and  $V(T)$  are the bulk cosmological constant and the tachyon potential, respectively.  $g$  is the determinant of the five-dimensional metric  $g_{ab}$  whose signature is  $(-, +, +, +, +)$ . It is assumed that the metric and tachyon field take the following form

$$ds^2 = dy^2 + e^{2f(y)} \eta_{\mu\nu} dx^\mu dx^\nu \quad (80)$$

and  $T = T(y)$ .

A thick brane solution is obtained for non-zero cosmological constant  $\Lambda_5 \neq 0$ . The potential is explicitly given as a function of  $y$ , as

$$V(y) = \frac{1}{\kappa_5^2} \frac{1}{\cosh^2(by)} \sqrt{(\Lambda_5 + 6b^2) \sinh^2(by) + \Lambda_5 + 6(b^2 - H^2)} \sqrt{(\Lambda_5 + 6b^2) \sinh^2(by) + (\Lambda_5 - 6H^2)}, \quad (81)$$

where  $b$  is an arbitrary constant. The warp and tachyon functions are given by

$$\begin{aligned} f(y) &= \ln \cosh(by), \\ T(y) &= -\frac{i}{b} \sqrt{\frac{3(b^2 + H^2)}{\Lambda_5 - 6H^2}} \text{EllipticF} \left[ iby, \frac{\Lambda_5 + 6b^2}{\Lambda_5 - 6H^2} \right]. \end{aligned} \quad (82)$$

The warp factor grows for large  $y$ .



## D. Other types of thick branes

### 1. Multi-scalar

The brane models with two interacting scalar fields considered in section III A refer to the so-called non-topological solutions in the terminology of Ref. [52]. At the same time, there exist other types of solutions - topological ones. These solutions describe our 4-dimensional Universe as a global topological defect in a multidimensional spacetime. Such a topological defect can be created by a set of scalar fields  $\phi^a$  describing some ‘‘hedgehog’’ configuration [85–87].

The main idea of the research in this direction is illustrated by the example of the paper [85] where the  $(p-1)$ -brane solutions (the number  $p$  refers to the coordinates on the brane) are considered. It is assumed that there is a global defect in  $n$  extra dimensions which is described by a multiplet of  $n$  scalar fields,  $\phi^a$ , whose Lagrangian is <sup>2</sup>

$$L = -\frac{1}{2}\partial_A\phi^a\partial^A\phi^a - V(\phi), \quad (83)$$

where the potential  $V(\phi)$  has its minimum on the  $n$ -sphere  $\phi^a\phi^a = \eta^2$ . As an example, the potential

$$V(\phi) = \frac{\lambda}{4}(\phi^a\phi^a - \eta^2)^2 \quad (84)$$

was used. Using the following *ansatz* for the metric:

$$ds^2 = A(\xi)^2 d\xi^2 + \xi^2 d\Omega_{n-1}^2 + B(\xi)^2 \hat{g}_{\mu\nu} dx^\mu dx^\nu, \quad (85)$$

where  $d\Omega_m^2$  stands for the metric on the unit  $m$ -sphere, and the spherical coordinates in the extra dimensions are defined by the usual relations,  $\xi^a = \{\xi \cos \theta_1, \xi \sin \theta_1 \cos \theta_2, \dots\}$ .

Using a different metric *ansatz*, a few different types of solutions were found. The first type, without a cosmological constant, exists for all  $n \geq 3$ , and is very similar to the global monopole solutions from [88]. The brane worldsheet is flat, and there is a solid angle deficit in the extra dimensions.

For a positive cosmological constant,  $\Lambda > 0$ , the solutions describe spherical branes in an inflating multidimensional universe. In the limit  $\eta \rightarrow 0$ , when the gravitational effect of the defect can be neglected, the universe can be pictured as an expanding  $(p+n-1)$ -dimensional sphere with a brane wrapping around it in the form of a sphere of lower dimensionality  $(p-1)$ . Another class of solutions has curvature singularities even in the absence of a defect ( $\eta = 0$ ), and these solutions were considered as unphysical. Finally, there are solutions having the geometry of a  $(p+1)$ -dimensional de Sitter space, with the remaining  $(n-1)$  dimensions having the geometry of a cylinder.

For a negative cosmological constant,  $\Lambda < 0$ , three classes of solutions were found. The first two are essentially analytic continuations of the positive- $\Lambda$  solutions. The third class is similar to the Randall-Sundrum ( $n=1$ ) and the Gregory ( $n=2$ ) solutions, and exhibit an exponential warp factor.

In [87] a model of a global topological defect with the potential (84) was considered. The authors were looking for solutions decaying far from the brane as  $B(\xi) \sim \exp(-c\xi)$ . It was shown that there exist solutions with the spherical radius  $R \rightarrow \text{const}$  at large  $\xi$ , i.e., the extra dimensions form an  $n$ -dimensional cylinder  $R_+ \times S^{n-1}$ . Such solutions exist only for  $n \geq 3$ .

The branes considered above refer to Minkowski branes with some non-zero bulk cosmological constant. In the paper [89] a model with some extra 4-dimensional  $\Lambda$ -term is considered (i.e. a de Sitter brane). This work investigated global monopole solutions in the Goldstone model with  $n$  scalar fields with the Lagrangian (83). Within the framework of this model, it was shown that new types of solutions exist (mirror symmetric solutions). These solutions have periodic matter and metric functions, and require a fine tuning of two cosmological constants.

### 2. Vortex

If one replaces the domain wall with some more complicated topological defect, such as string-vortices or monopoles (see above), new possibilities for gravitational localization to the brane appear. Braneworld models using a 6-dimensional vortex were proposed in Ref. [90]. In this model gravity was localized on an Abrikosov-Nielsen-Olsen vortex in the context of the Abelian Higgs model in 6 dimensions. In this model, the conservation of a topological

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<sup>2</sup> In the original paper [85], the sign of the kinetic term is different but it is wrong.

charge guarantees the stability of the configuration. In principle the presence of gravity can change the situation, but one can hope that “gravitating” topological defects can provide potential candidates for stable configurations.

For this purpose, the total action of the gravitating Abelian Higgs model in six dimensions can be chosen as

$$S = S_{\text{brane}} + S_{\text{grav}} , \quad (86)$$

where  $S_{\text{brane}}$  is the gauge-Higgs action and  $S_{\text{grav}}$  is the six-dimensional generalization of the Einstein-Hilbert action. More specifically,

$$S_{\text{brane}} = \int d^6x \sqrt{-G} \mathcal{L}_{\text{brane}}, \quad \mathcal{L}_{\text{brane}} = \frac{1}{2} (\mathcal{D}_A \phi)^* \mathcal{D}^A \phi - \frac{1}{4} F_{AB} F^{AB} - \frac{\lambda}{4} (\phi^* \phi - v^2)^2 , \quad (87)$$

where  $\mathcal{D}_A = \nabla_A - ieA_A$  is the gauge covariant derivative, while  $\nabla_A$  is the generally covariant derivative, and the bulk action is

$$S_{\text{grav}} = - \int d^6x \sqrt{-G} \left( \frac{R}{2\chi} + \Lambda \right) , \quad (88)$$

where  $\Lambda$  is a bulk cosmological constant,  $\chi = 8\pi G_6 = 8\pi/M_6^4$  and  $M_6$  denotes the six-dimensional Planck mass. In Eq. (87),  $v$  is the vacuum expectation value of the Higgs field which determines the masses of the Higgs boson and of the gauge boson

$$m_H = \sqrt{2\lambda} v, \quad m_V = e v. \quad (89)$$

Choosing the six-dimensional metric as

$$ds^2 = G_{AB} dx^A dx^B = M^2(\rho) g_{\mu\nu} dx^\mu dx^\nu - d\rho^2 - L(\rho)^2 d\theta^2, \quad (90)$$

where  $\rho$  and  $\theta$  are, respectively, the bulk radius and the bulk angle,  $g_{\mu\nu}$  is the four-dimensional metric and  $M(\rho)$ ,  $L(\rho)$  are the warp factors. The Abrikosov-Nielsen-Olsen ansatz for the gauge-Higgs system reads:

$$\begin{aligned} \phi(\rho, \theta) &= v f(\rho) e^{in\theta}, \\ A_\theta(\rho, \theta) &= \frac{1}{e} [n - P(\rho)], \end{aligned} \quad (91)$$

where  $n$  is the winding number.

Choosing the corresponding regular boundary conditions, one can find that outside the core of the string all source terms vanish and a general solution to the equations can be easily found for the case  $\Lambda \leq 0$  (solutions for the case of a positive cosmological constant in the bulk were studied in [91]) as follows

$$m(x) = -c \frac{1 - \epsilon e^{5cx}}{1 + \epsilon e^{5cx}}, \quad (92)$$

where  $\epsilon$  is an integration constant;  $c = \sqrt{-\mu/10} > 0$ ;  $\mu$  is some combination of the 6-dimensional Einstein gravitational constant,  $\Lambda$ -term and the mass of the Higgs;  $m(x)$  and  $x$  are the rescaled variables  $M(\rho)$  and  $\rho$ , respectively. Once  $m(x)$  is known, the function  $\ell(x)$  (the rescaled  $L(\rho)$ ) can also be determined.

To clarify the question about gravitational localization on the brane, one can demand the finiteness of the four-dimensional Planck mass

$$M_P^2 = \frac{4\pi M_6^4}{m_H^2} \int dx M^2(x) \mathcal{L}(x) < \infty . \quad (93)$$

It was shown in [90] that if  $\epsilon > 0$  or  $\epsilon \leq -1$ , then for  $x \rightarrow \infty$  one has  $M \sim \mathcal{L} \sim e^{cx}$  so that the integral in (93) diverges and gravity cannot be localized.

If  $\epsilon = 0$  the solution is simply

$$m(x) = \ell(x) = -c, \quad (94)$$

and the warp factors will be exponentially decreasing as a function of the bulk radius:

$$\begin{aligned} M(x) &= M_0 e^{-cx}, \\ \mathcal{L}(x) &= \mathcal{L}_0 e^{-cx}. \end{aligned} \quad (95)$$

The solution of Eqs. (94)–(95) leads to the localization of gravity and to a smooth AdS geometry far from the string core.

If  $-1 < \epsilon < 0$ ,  $M(x_0) = 0$  where  $x_0 = \frac{1}{5c} \log 1/|\epsilon|$ . The geometry is singular at  $x = x_0$  thus  $x < x_0$  should be required. In spite of the fact that  $\mathcal{L}$  diverges at  $x = x_0$ , the integral (93) defining the four-dimensional Planck mass is finite. So, these geometries can be potentially used for the localization of gravity, provided the singularity at  $x_0$  is resolved in some way, e.g. by string theory. These types of singular solutions were discussed in [91] for the case of positive cosmological constant.

Finally, if the bulk cosmological constant is zero, the solutions have a power-law behavior, namely

$$M(x) \sim x^\gamma, \quad \mathcal{L}(x) \sim x^\delta, \quad (96)$$

with

$$d\gamma + \delta = 1, \quad d\gamma^2 + \delta^2 = 1, \quad (97)$$

where  $d = 4$  is the number of dimensions of the metric  $g_{\mu\nu}$ . These solutions belong to the Kasner class. The Kasner conditions leave open only two possibilities: either  $\delta = 1$  and  $\gamma = 0$  or  $\gamma = 2/5$  and  $\delta = -3/5$ . None of them lead to the localization of gravity.

Summarizing the results of the paper [90], it was shown that the localization of gravity is possible on a “thick” string, and a fine-tuning condition was found which led to a set of physically interesting solutions. Since the described geometries are regular, gravity can be described in classical terms both in the bulk and on the vortex.

Similar considerations allowed the authors of [92] to conclude that within the framework of an anomaly-free Abelian Higgs model coupled to gravity in a 6-dimensional spacetime, it is possible to construct an effective  $D = 4$  electrodynamics of charged particles interacting with photons and gravitons. In this work both gauge field and gravity are localized near the core of an Abrikosov-Nielsen-Olsen vortex configuration.

Further considerations of questions concerning localization of gauge fields on a gravitating vortex in a 6-dimensional spacetime can be found in the papers [93, 94]. Note that most of the calculations are performed numerically or asymptotically, and there are no general analytic solutions for 6-dimensional vortices. However, in order to address some questions (for example, stability analysis) the availability of exact solutions is desirable. In Ref. [95] some exact solutions for 6-dimensional vortices were found. In particular, using the special boundary conditions, the several kink solutions were found. The first solution was

$$f = f_0 \arctan\left(\sinh \frac{\beta r}{\delta}\right), \quad f_0 = 2\sqrt{\delta}, \quad (98)$$

with the corresponding metric “warp” factor

$$M(r) = \cosh^{-\delta}\left(\frac{\beta r}{\delta}\right), \quad \delta > 0, \quad \beta > 0, \quad (99)$$

where  $\delta$  could be seen as the wall thickness and  $\beta$  as a parameter depending on the cosmological constant. The effective potential for this case was

$$V(f) = \frac{2\beta^2}{\delta} [(1 + 5\delta) \cos^2(f/f_0) - 5\delta]. \quad (100)$$

The second and third solutions were, respectively

$$f = f_0 \tanh(ar), \quad \text{and} \quad (101)$$

$$f = f_0 \arctan(ar), \quad (102)$$

with the metric “warp” factors, respectively

$$M(r) = \exp\left[\frac{1}{24} f_0^2 \operatorname{sech}(ar)\right] \cosh^{-f_0^2/6}(ar), \quad \text{and} \quad (103)$$

$$M(r) = \exp\left[-\frac{a}{2} f_0^2 r \arctan(ar)\right], \quad (104)$$

and with a polynomial potential identical to the Mexican hat potential, or a potential containing polynomial and non-polynomial terms

$$V(f) = \frac{a^2}{2} \left[ (f^2 - f_0^2)^2 - \frac{5}{9} f^2 (f^2 - 3f_0^2) \right], \quad \text{and} \quad (105)$$

$$V(f) = \frac{a^2 f_0^2}{2} \left\{ \cos^4\left(\frac{f}{f_0}\right) - \frac{5}{16} f_0^2 \left[ \frac{f}{f_0} + \frac{1}{2} \sin\left(\frac{f}{f_0}\right) \right] \right\}^2. \quad (106)$$

Using these exact solutions, it was shown that the zero mode of the linearized gravity spectrum is localized on the 3-brane and the massive modes are not bounded at all.

### 3. Weyl gravity model

In Refs. [96–98], thick brane solutions in a pure geometric Weyl 5D spacetime, which constitutes a non-Riemannian generalization of Kaluza-Klein gravity, are considered. A Weyl geometry is an affine manifold specified by  $(g_{MN}, \omega_M)$  with  $M, N = 0, 1, 2, 3, 5$ , where  $g_{MN}$  is the metric tensor, and  $\omega_M$  is a “gauge” vector involved in the definitions of the affine connections of the manifold.

Besides the scalar curvature  $R$ , the action used in this approach also contains terms describing a Weyl scalar  $\omega$

$$S_5^W = \int_{M_5^W} \frac{d^5x \sqrt{|g|}}{16\pi G_5} e^{\frac{3}{2}\omega} [R + 3\tilde{\xi}(\nabla\omega)^2 + 6U(\omega)], \quad (107)$$

where  $M_5^W$  is a Weyl manifold specified by the pair  $(g_{MN}, \omega)$ . The Weylian Ricci tensor reads

$$R_{MN} = \Gamma_{MN,A}^A - \Gamma_{AM,N}^A + \Gamma_{MN}^P \Gamma_{PQ}^Q - \Gamma_{MQ}^P \Gamma_{NP}^Q, \quad (108)$$

where

$$\Gamma_{MN}^C = \{^C_{MN}\} - \frac{1}{2} (\omega_{,M} \delta_N^C + \omega_{,N} \delta_M^C - g_{MN} \omega^{,C}) \quad (109)$$

are the affine connections on  $M_5^W$ ,  $\{^C_{MN}\}$  are the Christoffel symbols and  $M, N = 0, 1, 2, 3, 5$ ; the constant  $\tilde{\xi}$  is an arbitrary coupling parameter, and  $U(\omega)$  is a self-interaction potential for the scalar field  $\omega$ .

The 5-dimensional metric describing a thick brane is assumed to have the form

$$ds_5^2 = e^{2A(y)} \eta_{mn} dx^m dx^n + dy^2, \quad (110)$$

where  $e^{2A(y)}$  is the warp factor depending on the extra coordinate  $y$ , and  $m, n = 0, 1, 2, 3$ ,  $\eta_{mn}$  is the 4D Minkowski metric.

Further, the conformal transformation  $\hat{g}_{MN} = e^\omega g_{MN}$ , mapping the Weylian action (107) into the Riemannian one is carried out via

$$S_5^R = \int_{M_5^R} \frac{d^5x \sqrt{|\hat{g}|}}{16\pi G_5} [\hat{R} + 3\xi(\hat{\nabla}\omega)^2 + 6\hat{U}(\omega)], \quad (111)$$

where  $\xi = \tilde{\xi} - 1$ ,  $\hat{U}(\omega) = e^{-\omega} U(\omega)$  and all hatted magnitudes and operators are defined in the Riemann frame. Now the metric (110) takes the form

$$\hat{ds}_5^2 = e^{2\sigma(y)} \eta_{mn} dx^n dx^m + e^{\omega(y)} dy^2, \quad (112)$$

where  $2\sigma = 2A + \omega$ . If one introduces new functions  $X = \omega'$  and  $Y = 2A'$  [96] then the Einstein equations read

$$X' + 2YX + \frac{3}{2}X^2 = \frac{1}{\xi} \frac{d\hat{U}}{d\omega} e^\omega, \quad (113)$$

$$Y' + 2Y^2 + \frac{3}{2}XY = \left( -\frac{1}{\xi} \frac{d\hat{U}}{d\omega} + 4\hat{U} \right) e^\omega. \quad (114)$$

As pointed out in [96], this system of equations can be easily solved if one uses the condition  $X = kY$ , where  $k$  is an arbitrary constant parameter, but excluding the value  $k = 1$ .

In this case both field equations in (113) (114) reduce to a single differential equation

$$Y' + \frac{4+3k}{2}Y^2 = \frac{4\lambda}{1+k} e^{(\frac{4k\xi}{1+k}+1)\omega}, \quad (115)$$

where  $\lambda$  is some constant parameter. The authors of [96] first considered the problem for a case of  $Z_2$ -symmetric manifolds. Here, the two following cases are considered:

- (i)  $\xi = -(1+k)/(4k)$  and leaving  $k$  arbitrary except for  $k = -4/3$ ,
- (ii)  $k = -4/3$  and leaving  $\xi$  arbitrary except for  $\xi = -1/16$ .

*The case (i)* For this case, a solution was found for the simplified (115) in the form

$$\omega(y) = bk \ln[\cosh(ay)], \quad (116)$$

$$e^{2A(y)} = [\cosh(ay)]^b, \quad (117)$$

where

$$a = \sqrt{\frac{4+3k}{1+k}} 2\lambda, \quad b = \frac{2}{4+3k}.$$

*The case (ii)* The authors of [97, 98] considered another simplified case with  $k = -4/3$  and  $\xi$  an arbitrary parameter. Then Eq. (115) takes the form

$$Y' + 12\lambda e^{p\omega} = 0 \quad \text{or} \quad \omega'' - 16\lambda e^{p\omega} = 0, \quad (118)$$

where  $p = 1 + 16\xi$ . Note that the choice  $\xi = -1/16$  leads to a constant potential and thus the case  $\xi \neq -1/16$  is assumed. For such a choice of the parameter, one can find the following solution for  $\omega$  and  $A$

$$\omega = -\frac{2}{p} \ln \left\{ \frac{\sqrt{-8\lambda p}}{c_1} \cosh [c_1(y - c_2)] \right\}, \quad (119)$$

$$e^{2A} = \left\{ \frac{\sqrt{-8\lambda p}}{c_1} \cosh [c_1(y - c_2)] \right\}^{\frac{3}{2p}}. \quad (120)$$

The following physically interesting problems were considered in detail:

- $\lambda > 0, p < 0, c_1 > 0$ . In this case the solution is a thick brane, and the fifth coordinate changes within the limits  $-\infty < y < \infty$ .
- $\lambda > 0, p > 0, c_1 = iq_1$ . In this case the solution describes a compact manifold along the extra dimension with  $-\pi \leq q_1(y - c_2) \leq \pi$  and consequently this solution is not a thick brane.

If one calculates the function  $\omega'(y)$  for both the cases (i) and (ii), then one has

$$\omega'(y) = ab \tanh(ay), \quad (i) \quad (121)$$

$$\omega'(y) = -\frac{2c_1}{p} \tanh [c_1(y - c_2)], \quad (ii) \quad (122)$$

respectively. As one can see, both solutions are kink-like. Thus these solutions are topologically nontrivial solutions.

The de Sitter thick brane solution in Weyl gravity was investigated in Ref. [99]. To show the analytic form of the solution, it is convenient to transform the fifth coordinate to the conformal coordinate  $z$ , defined by  $dz = e^{-A(z)} dy$ ,

$$ds_5^2 = e^{2A(z)} (-dt^2 + e^{2\beta t} d\mathbf{x}^2 + dz^2), \quad (123)$$

where  $\beta$  is the expansion rate of the de Sitter spacetime. For the potential

$$U(\omega) = \frac{1+3\delta}{2\delta} \beta^2 e^{-\omega} \left( \cos \frac{\omega}{\omega_0} \right)^{2(1-\delta)}, \quad (124)$$

where  $\omega_0 = \sqrt{3\delta(1-\delta)}$  ( $0 < \delta < 1$ ), the solutions are given by

$$\omega = \omega_0 \arctan \left( \sinh \frac{\beta z}{\delta} \right), \quad e^{2A} = \cosh^{-2\delta} \left( \frac{\beta z}{\delta} \right) e^{2\omega_0 \arctan \left( \frac{\beta z}{\delta} \right)}. \quad (125)$$

After transforming back to the proper coordinate  $y$ , the geometry is asymmetric across the  $y = 0$  surface. But the solution still has a kink-like profile and connects two Minkowski vacua.

#### 4. Bloch brane

It is possible to borrow ideas from condensed matter physics and apply them to brane world models. Of interest is the situation which occurs in ferromagnetic systems, in which one has an Ising or Bloch type wall. An Ising wall is a simple interface with no internal structure whereas a Bloch wall is an interface which has a nontrivial internal structure. One can find thick brane solutions with internal structures like that of the Bloch wall.

In the paper [100], the thick brane solution with two scalar fields  $(\phi, \chi)$ , called a Bloch brane, was considered. The solution has an internal structure because of the dependence on  $\chi$ . For this case, the action is taken to have the form

$$S_c = \int d^4x dy \sqrt{|g|} \left[ -\frac{1}{4}R + \frac{1}{2}\partial_a\phi\partial^a\phi + \frac{1}{2}\partial_a\chi\partial^a\chi - V(\phi, \chi) \right]. \quad (126)$$

The thick brane metric is

$$ds^2 = g_{ab}dx^a dx^b = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu - dy^2, \quad (127)$$

where  $a, b = 0, 1, 2, 3, 4$ , and  $e^{2A}$  is the warp factor.  $x^4 = y$  is the extra dimension coordinate. The potential is given by

$$V_c(\phi, \chi) = \frac{1}{8} \left[ \left( \frac{\partial W_c}{\partial \phi} \right)^2 + \left( \frac{\partial W_c}{\partial \chi} \right)^2 \right] - \frac{1}{3}W_c^2, \quad (128)$$

$$W_c = 2 \left( \phi - \frac{1}{3}\phi^3 - r\phi\chi^2 \right), \quad (129)$$

where  $r$  is a constant. It is readily seen that this potential is unbounded below at  $\chi = \text{const}$

$$V_c(\phi, \chi) \xrightarrow{|\phi| \rightarrow \infty} -\infty. \quad (130)$$

The corresponding equations describing two gravitating scalar fields are

$$\phi'' + 4A'\phi' = \frac{\partial V(\phi, \chi)}{\partial \phi}, \quad (131)$$

$$\chi'' + 4A'\chi' = \frac{\partial V(\phi, \chi)}{\partial \chi}, \quad (132)$$

$$A'' = -\frac{2}{3}(\phi'^2 + \chi'^2), \quad (133)$$

$$A'^2 = \frac{1}{6}(\phi'^2 + \chi'^2) - \frac{1}{3}V(\phi, \chi), \quad (134)$$

where prime stands for derivative with respect to  $y$ . One can reduce these equations to the form

$$\phi' = \frac{1}{2} \frac{\partial W_c}{\partial \phi}, \quad (135)$$

$$\chi' = \frac{1}{2} \frac{\partial W_c}{\partial \chi}, \quad (136)$$

$$A' = -\frac{1}{3} W_c. \quad (137)$$

*a. Bloch walls* The solution of these equations is

$$\phi(y) = \tanh(2ry), \quad (138)$$

$$\chi(y) = \pm \sqrt{\frac{1}{r} - 2} \operatorname{sech}(2ry). \quad (139)$$

$$A(y) = \frac{1}{9r} [(1 - 3r) \tanh^2(2ry) - 2 \ln \cosh(2ry)]. \quad (140)$$

The brane has a typical width as large as  $1/r$ . In the case  $r < 1/2$ , the two-field solution represents a Bloch-type brane, where  $(\phi, \chi)$  represents an elliptic trajectory in the field space. For  $r > 1/2$ , the two-field solution is changed to a one-field solution, i.e., an Ising-type brane, connecting two minima  $(\phi, \chi) = (\pm 1, 0)$  straight along the line  $\chi = 0$  in the field space. Equation (138) shows that this solution is a topologically nontrivial one.

*b. General method* In Ref. [101], the method to obtain more general (often numerical) Bloch brane solutions was given. The starting point was the general class of the potential

$$W_c(\phi, \chi) = 2\phi \left[ \lambda \left( \frac{\phi^2}{3} - a^2 \right) + \mu \chi^2 \right], \quad (141)$$

which includes the previous case, for the choice that  $a = 1$ ,  $\lambda = -1$  and  $\mu = -r$ . From Eq. (137),  $d\phi/d\chi = W_{c,\phi}/W_{c,\chi}$ , where  $W_{c,\phi} := \partial W_c/\partial\phi$ , one obtains

$$\frac{d\phi}{d\chi} = \frac{\lambda(\phi^2 - a^2) + \mu\chi^2}{2\mu\phi\chi}. \quad (142)$$

Introducing the new variable  $\rho = \phi^2 - a^2$ , one can replace Eq. (142) with an ordinary differential equation for  $\rho$  with respect to  $\chi$ , which can be analytically solved as

$$\begin{aligned} \rho(\chi) &= \phi^2 - a^2 = c_0 \chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu} \chi^2, \quad (\lambda \neq 2\mu), \\ \rho(\chi) &= \phi^2 - a^2 = \chi^2 \left( \ln(\chi) + c_1 \right), \quad (\lambda = 2\mu), \end{aligned} \quad (143)$$

where  $c_0$  and  $c_1$  are integration constants. Substituting Eq. (143) into the differential equation for  $\chi$ ,  $d\chi/dy = (1/2)W_{c,\chi}$ ,

$$\begin{aligned} \frac{d\chi}{dy} &= \pm 2\mu\chi \sqrt{a^2 + c_0 \chi^{\lambda/\mu} - \frac{\mu}{\lambda - 2\mu} \chi^2}, \quad (\lambda \neq 2\mu), \\ \frac{d\chi}{dy} &= \pm 2\mu\chi \sqrt{a^2 + \chi^2 \left( \ln(\chi) + c_1 \right)}, \quad (\lambda = 2\mu). \end{aligned} \quad (144)$$

For the metric function  $A$ ,

$$\frac{dA}{dy} = \frac{dA}{d\chi} \frac{d\chi}{dy} = \frac{dA}{dy} W_{c,\chi} = -\frac{1}{3} W_c, \quad (145)$$

which leads to  $dA/d\chi = -(1/3)(W_c/W_{c,\chi})$ . This can be solved as

$$\begin{aligned} A(\chi) &= \alpha_0 + \frac{2a^2}{9} \ln \chi - \frac{1}{9} \frac{\lambda - 3\mu}{\lambda - 2\mu} \chi^2 - \frac{c_0}{9} \chi^{\lambda/\mu}, \quad (\lambda \neq 2\mu) \\ A(\chi) &= \alpha_1 + \frac{2a^2}{9} \ln \chi - \frac{3\mu + \lambda c_1}{18\mu} \chi^2 - \frac{\chi^2}{6\mu} \chi^2 \left( \ln \chi - \frac{1}{2} \right), \quad (\lambda = 2\mu), \end{aligned} \quad (146)$$

where  $\alpha_0$  and  $\alpha_1$  are integration constants which are chosen so that  $A(y=0) = 0$ .

To summarize, the general method to obtain solutions is as follows: First, for a given set of parameters  $\lambda$ ,  $a$  and  $\mu$ , one solves Eq. (144) and determines the profile of  $\chi$  as a function of  $y$ . Then, from Eq. (143) and (146), one can determine the profile of  $\phi$  and  $A$  as the function of  $y$ , through  $\chi(y)$ . In general, Eq. (144) does not have exact solutions except for the limited cases discussed below. Note that the solutions discussed in Ref. [100] correspond to  $c_0 = 0$ . It should be stressed that the model admits a particular class of solutions which cannot be obtained from the method given above. A set of solutions such that  $\chi = 0$  (for which Eq. (142) does not make any sense) do not have any internal structure for the brane, although the system admits a solution given by  $\phi = \pm a \tanh(\lambda ay)$ .

According to the method shown above, in Ref. [101], several new analytic thick brane solutions were found for  $c_0 \neq 0$ , which were called *degenerate Bloch walls* and *critical Bloch walls*. Here, by ‘‘degenerate’’ we mean that one cannot specify the property of the solutions only by fixing the potential parameters  $\lambda$ ,  $a$  and  $\mu$ , but one also needs to fix the integration constant  $c_0$ . In fact, the solutions still have remarkable variety for different values of  $c_0$ .

*c. Degenerate Bloch walls* For the case  $\lambda = \mu$  and  $c_0 < -2$ , the following solution was obtained

$$\chi(y) = \frac{2a}{\sqrt{c_0^2 - 4} \cosh(2\mu ay) - c_0}, \quad \phi(y) = \frac{\sqrt{c_0^2 - 4}}{\sqrt{c_0^2 - 4} \cosh(2\mu ay) - c_0}, \quad (147)$$

and

$$e^{2A(y)} = N \left[ \frac{2a}{\sqrt{c_0^2 - 4} \cosh(2\mu ay) - c_0} \right]^{(4a^2)/9} \exp \left[ \frac{2a(c_0^2 - c_0 \sqrt{c_0^2 - 4} \cosh(2\mu ay) - 4a)}{9(\sqrt{c_0^2 - 4} \cosh(2\mu ay) - c_0)^2} \right], \quad (148)$$

where  $N$  is chosen so that  $A(y=0) = 0$ .

For the choice  $\lambda = 4\mu$  and  $c_0 < 1/16$ , a similar solution was obtained

$$\chi(y) = -\frac{2a}{\sqrt{\sqrt{1-16c_0}\cosh(4a\mu y)+1}}, \quad \phi(y) = \sqrt{1-16c_0}a \frac{\sinh(4\mu ay)}{\sqrt{\sqrt{1-16c_0}\cosh(4a\mu y)+1}}, \quad (149)$$

and

$$e^{2A} = N \left[ \frac{-2a}{\sqrt{\sqrt{1-16c_0}\cosh(4a\mu y)+1}} \right]^{16a^2/9} \exp \left\{ -\frac{4a^2}{9} \left[ \frac{1+32a^2c_0+\sqrt{1-16c_0}\cosh(4\mu ay)}{(\sqrt{1-16c_0}\cosh(4\mu ay)+1)^2} \right] \right\} \quad (150)$$

and again  $N$  was chosen so that  $A(y=0) = 0$ . An interesting feature of these solutions is that, for some value of  $c_0$ ,  $\phi$  exhibits a double kink profile. This represents the formation of a double wall structure, extended along the spatial dimension.

*d. Critical Bloch walls* For the case that  $\lambda = \mu$  and  $c_0 = -2$ , the solution

$$\chi = \frac{a}{2} [1 \pm \tanh(\mu ay)], \quad \phi = \frac{a}{2} [\tanh(\mu ay) \mp 1], \quad (151)$$

and

$$e^{2A} = N \left[ \frac{a}{2} (1 \pm \tanh(\mu ay)) \right]^{2a^2/9} \exp \left\{ -\frac{a^2}{9} \left[ (1 \pm \tanh(\mu ay))^2 - \frac{2}{a} (1 \pm \tanh(\mu ay))^2 \right] \right\}, \quad (152)$$

was obtained. Similarly, for the case that  $c_0 = 1/16$  and  $\lambda = 4\mu$ , another solution was obtained of the form

$$\chi = -\sqrt{2}a \frac{\cosh(\mu ay) \pm \sinh(\mu ay)}{\sqrt{\cosh(2\mu ay)}}, \quad \phi = \frac{a}{2} (1 \mp \tanh(2\mu ay)), \quad (153)$$

and

$$e^{2A} = N \left[ \frac{-ae^{\pm\mu ay}}{\sqrt{\cosh(2\mu ay)}} \right]^{16a^2/9} \times \exp \left\{ -\frac{2a^2}{9} \tanh(2a\mu y) [a^2 \tanh(2a\mu y) \mp (1+2a^2)] \right\}. \quad (154)$$

The novelty in these cases is the fact that both,  $\phi$  and  $\chi$  fields have a kink-like profile and the warp factor has the remarkable feature of having two Minkowski-type regions.

### III. TOPOLOGICALLY TRIVIAL THICK BRANES

#### A. Thick branes of two strongly interacting scalar fields

Here, we review the topologically trivial solutions. For example, one possibility is to consider two nonlinear gravitating scalar fields which can create a 4-dimensional brane in a multidimensional spacetime. Such problems have been investigated for the 5-dimensional case in [102, 103], for the 6-dimensional case in [104], and for the 7- and the 8-dimensional cases in [105]. From the physical point of view, the situation is as follows: an interaction potential of these fields has two local and two global minima. Thus there are two different vacuums. The multidimensional space is filled with these scalar fields which are located at the vacuum in which the scalar fields are at the local minimum. Therefore there is a defect in the form of a 4-dimensional brane on the background of this vacuum.

The general approach to such solutions is as follows: the action of  $D = 4 + n$  dimensional gravity for all models can be written as [106]:

$$S = \int d^D x \sqrt{|g|} \left[ -\frac{M^{n+2}}{2} R + L_m \right], \quad (155)$$

where  $M$  is the fundamental mass scale, and  $n$  is a number of extra dimensions. The Lagrangian,  $L_m$ , for the two interacting scalar fields  $\varphi, \chi$  takes the form:

$$L_m = \epsilon \left[ \frac{1}{2} \partial_A \varphi \partial^A \varphi + \frac{1}{2} \partial_A \chi \partial^A \chi - V(\varphi, \chi) \right], \quad (156)$$



where the potential energy  $V(\varphi, \chi)$  is:

$$V(\varphi, \chi) = \frac{\Lambda_1}{4}(\varphi^2 - m_1^2)^2 + \frac{\Lambda_2}{4}(\chi^2 - m_2^2)^2 + \varphi^2\chi^2 - V_0. \quad (157)$$

(This potential was used in [107] as an effective description of a condensate of gauge field in SU(3) Yang-Mills theory, i.e. the scalar fields were taken as effective fields which described a condensate of Yang-Mills fields.) Here the capital Latin indices run over  $A, B = 0, 1, 2, 3, \dots, D$  and the small Greek indices  $\alpha, \beta = 0, 1, 2, 3$  refer to four dimensions;  $\Lambda_1, \Lambda_2$  are the self-coupling constants,  $m_1, m_2$  are the masses of the scalar fields  $\varphi$  and  $\chi$ , respectively;  $V_0$  is an arbitrary normalization constant which can be chosen based of physical motivations;  $\epsilon = +1$  for usual scalar fields, and  $\epsilon = -1$  for phantom scalar fields.

The use of two fields ensures the presence of two global minima of the potential (157) at  $\phi = 0, \chi = \pm m_2$  and two local minima at  $\chi = 0, \phi = \pm m_1$  for the values of the parameters  $\Lambda_1, \Lambda_2$  used in the above papers. The conditions for existence of the local minima are:  $\Lambda_1 > 0, m_1^2 > \Lambda_2 m_2^2 / 2$ , and for the global minima they are:  $\Lambda_2 > 0, m_2^2 > \Lambda_1 m_1^2 / 2$ . Because of these minima there were solutions localized on the brane for the 5-, 6-, 7- and 8-dimensional cases [102]-[105]. In these different cases the solutions go asymptotically to one of the local minima.

Variation of the action (155) with respect to the  $D$ -dimensional metric tensor  $g_{AB}$  leads to Einstein's equations:

$$R_B^A - \frac{1}{2}\delta_B^A R = \frac{1}{M^{n+2}}T_B^A, \quad (158)$$

where  $R_B^A$  and  $T_B^A$  are the  $D$ -dimensional Ricci and the energy-momentum tensors, respectively. The corresponding scalar field equations can be obtained from (155) by variation with respect to the field variables  $\varphi, \chi$ . These equations are

$$\frac{1}{\sqrt{D}g} \frac{\partial}{\partial x^A} \left[ \sqrt{D}g g^{AB} \frac{\partial(\varphi, \chi)}{\partial x^B} \right] = -\frac{\partial V}{\partial(\varphi, \chi)}. \quad (159)$$

### 1. General equations

Using the generalized  $D$ -dimensional metric [106]

$$ds^2 = \phi^2(r)\eta_{\alpha\beta}(x^\nu)dx^\alpha dx^\beta - \lambda(r)(dr^2 + r^2 d\Omega_{n-1}^2), \quad (160)$$

where  $d\Omega_{n-1}^2$  is the solid angle for the  $(n-1)$  sphere, one can rewrite the Einstein equations [106] in the form

$$\begin{aligned} & 3 \left( 2 \frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} \right) + 6 \frac{(\phi')^2}{\phi^2} + (n-1) \\ & \times \left[ 3 \frac{\phi'}{\phi} \left( \frac{\lambda'}{\lambda} + \frac{2}{r} \right) + \frac{\lambda''}{\lambda} - \frac{1}{2} \frac{\lambda'}{\lambda} \left( \frac{\lambda'}{\lambda} - \frac{6}{r} \right) + \frac{n-4}{4} \frac{\lambda'}{\lambda} \left( \frac{\lambda'}{\lambda} + \frac{4}{r} \right) \right] \\ & = -\frac{2\lambda}{M^{n+2}} T_\alpha^\alpha, \end{aligned} \quad (161)$$

$$\begin{aligned} & 12 \frac{(\phi')^2}{(\phi)^2} + (n-1) \left[ 4 \frac{\phi'}{\phi} \left( \frac{\lambda'}{\lambda} + \frac{2}{r} \right) + \frac{n-2}{4} \frac{\lambda'}{\lambda} \left( \frac{\lambda'}{\lambda} + \frac{4}{r} \right) \right] \\ & = -\frac{2\lambda}{M^{n+2}} T_r^r, \end{aligned} \quad (162)$$

$$\begin{aligned} & 4 \left( 2 \frac{\phi''}{\phi} - \frac{\phi' \lambda'}{\phi \lambda} \right) + 12 \frac{(\phi')^2}{\phi^2} + (n-2) \\ & \times \left[ 4 \frac{\phi'}{\phi} \left( \frac{\lambda'}{\lambda} + \frac{2}{r} \right) + \frac{\lambda''}{\lambda} - \frac{1}{2} \frac{\lambda'}{\lambda} \left( \frac{\lambda'}{\lambda} - \frac{6}{r} \right) + \frac{n-5}{4} \frac{\lambda'}{\lambda} \left( \frac{\lambda'}{\lambda} + \frac{4}{r} \right) \right] \\ & = -\frac{2\lambda}{M^{n+2}} T_\theta^\theta, \end{aligned} \quad (163)$$

where  $T_\alpha^\alpha, T_r^r$  and  $T_\theta^\theta$  are  $(\alpha), (r)$  and  $(\theta)$  components of the energy-momentum tensor, respectively. The scalar field equations become

$$\varphi'' + \left( \frac{n-1}{r} + 4 \frac{\phi'}{\phi} + \frac{n-2}{2} \frac{\lambda'}{\lambda} \right) \varphi = \lambda \varphi [2\chi^2 + \Lambda_1 (\varphi^2 - m_1^2)], \quad (164)$$

$$\chi'' + \left( \frac{n-1}{r} + 4 \frac{\phi'}{\phi} + \frac{n-2}{2} \frac{\lambda'}{\lambda} \right) \chi = \lambda \chi [2\varphi^2 + \Lambda_2 (\chi^2 - m_2^2)]. \quad (165)$$

The set of equations (161)-(165) is a set of nonlinear equations with solutions whose behavior very much depends on values of the parameters  $m_1, m_2, \Lambda_1, \Lambda_2$ . As shown in Refs. [102]-[105], by specifying some values of the self-coupling constants  $\Lambda_1, \Lambda_2$ , the problem of searching for *regular* solutions to systems similar to the system given by (161)-(165) reduced to the evaluation of eigenvalues of the parameters  $m_1, m_2$ . Only for specific values of these parameters one was able to find *regular* solutions with *finite* energy.

## 2. 5-dimensional problems

The usual scalar field case with  $\epsilon = +1$  was considered in [102], and the phantom field case with  $\epsilon = -1$  was considered in [103]. These problems were solved by the numerical shooting method. The results are presented in Figs. 3-4. These solutions describe a thick brane in a 5-dimensional bulk. As pointed out in Ref. [103], there are two distinctive features of the solutions for the phantom versus the usual scalar fields: a) they have different asymptotic behavior; b) the solutions with phantom scalar fields are stable, and the solutions with the usual scalar fields are not (at least for the parameters  $\Lambda_1, \Lambda_2, \varphi(0), \chi(0)$  used in [103]).

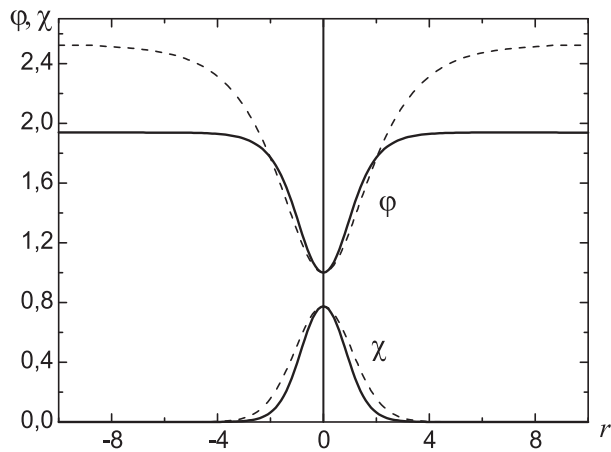


FIG. 3: The scalar fields profiles for the cases  $\epsilon = -1$  (solid lines) and  $\epsilon = +1$  (dashed lines). 5D case.

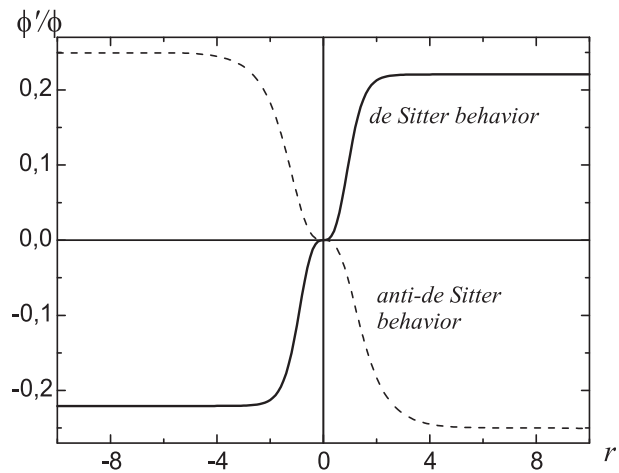


FIG. 4: The profiles of the functions  $\mathcal{H} = \phi'/\phi$  for the cases  $\epsilon = -1$  (solid line) and  $\epsilon = +1$  (dashed line). 5D case.

## 3. 6,7,8-dimensional problems

These cases were considered in the papers [104, 105]. The results are presented in Figs. 5-6. All the results obtained in [102, 103] for the 5-dimensional case and in [104] for the 6-dimensional case, and also in [105] for the 7- and 8-dimensional cases show that it is in principle possible to localize scalar fields with the potential (157) to a brane in any dimensions. This lends support to the idea that there exist similar regular solutions is possible in a larger number (any number) of extra dimensions (a discussion of this question is given in [105]).

## 4. Phantom thick de Sitter brane solutions in multidimensions

Thick de Sitter brane solutions for phantom scalar fields were considered in [108] for 5-, 6- and 7-dimensional cases. For this purpose the following metric was used

$$ds^2 = a^2(r)\gamma_{\alpha\beta}(x^\nu)dx^\alpha dx^\beta - \lambda(r)(dr^2 + r^2 d\Omega_{n-1}^2), \quad (166)$$

where  $d\Omega_{n-1}^2$  is the solid angle of the  $(n-1)$  sphere.  $\gamma_{\alpha\beta}$  is the metric of the four-dimensional de Sitter space whose Ricci curvature tensor is given by  $R_{\mu\nu}[\gamma] = 3H^2\gamma_{\mu\nu}$ . Then one has Einstein equations similar to equations (161)-(163) with the following extra terms on the left hand side:  $-6H^2\lambda/\phi^2$  in the  $(\alpha)$  component, and  $-12H^2\lambda/\phi^2$  in the  $(r)$  and  $(\theta)$  components. The scalar field equations remain the same. The results of numerical calculations are presented in Figs. 7-10.

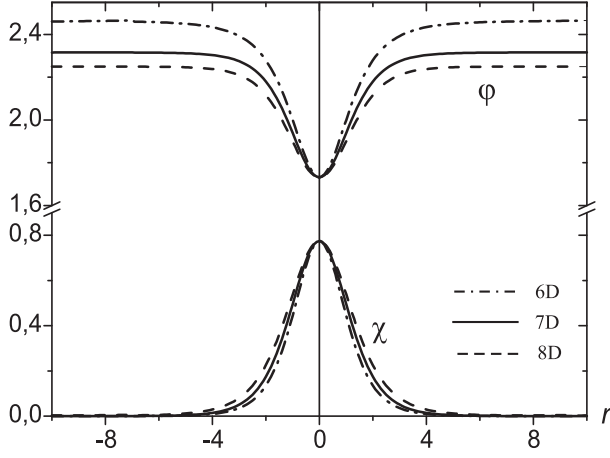


FIG. 5: The scalar fields profiles for the 6,7,8-dimensional problems.

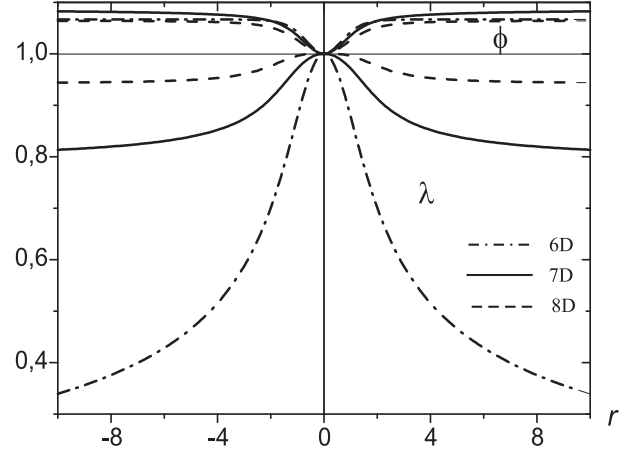


FIG. 6: The profiles of the metric functions  $\phi$  and  $\lambda$  for the 6,7,8-dimensional problems.

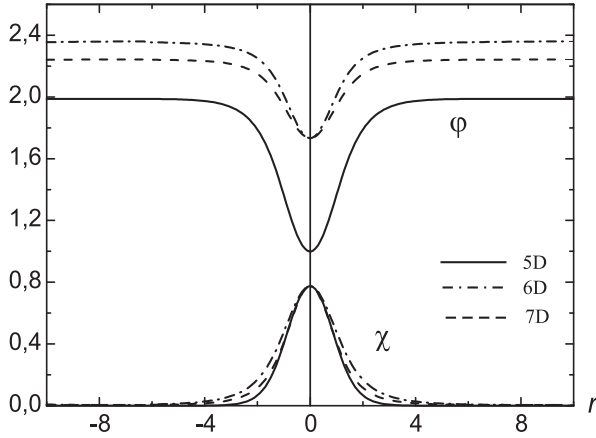


FIG. 7: The scalar field configurations  $\phi$  and  $\chi$  are shown as functions of the dimensionless  $r$ .

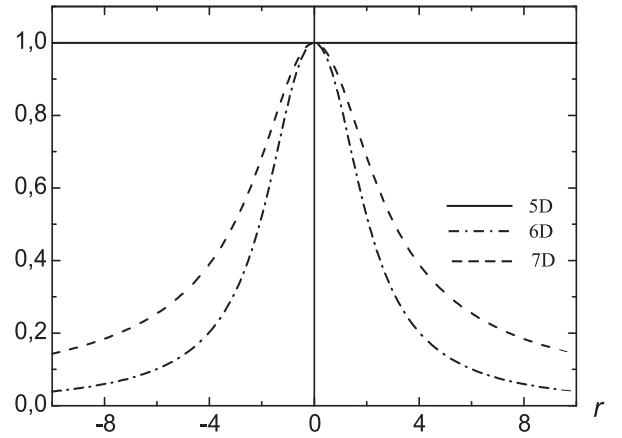


FIG. 8: The metric function  $\lambda(r)$  is shown as a function of the dimensionless  $r$ .

As one can see from these figures, there exist five-, six- and seven-dimensional thick de Sitter brane solutions of two interacting (*phantom*) scalar fields. It was also shown in [108] that asymptotically there exist anti-de Sitter spacetimes for the five-dimensional case and flat spacetimes for the six- and seven-dimensional cases. For the five-dimensional case, it was shown that the solutions are stable. It is also quite reasonable to expect that the thick brane solutions in multidimensions are also stable since the spacetime and vacuum structures are essentially the same as in the case of five spacetime dimensions.

#### IV. BRANES WITH UNUSUAL SOURCE FUNCTIONS

Here we present the thick brane solutions created by special source functions [109, 110]. It was shown that by going from 5D to 6D it is possible to trap fields of all spins (i.e. spins 0,  $\frac{1}{2}$ , 1, 2) to the 4D spacetime using only gravity. The “trapping” provided by the 6D solution of [110] is automatic since the proper distance from the brane at  $r = 0$  is finite and any field can only be distributed within some finite distance from the brane.

The 6D action was chosen as

$$S = \int d^6x \sqrt{-g} \left[ \frac{M^4}{2} R + \Lambda + L_m \right], \quad (167)$$

where  $M$ ,  $R$ ,  $\Lambda$  and  $L_m$  are the fundamental scale, the scalar curvature, the cosmological constant and the matter

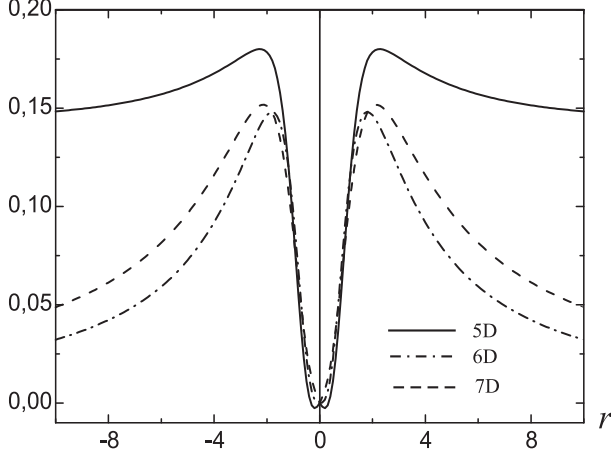


FIG. 9: The function  $\phi'/\phi$  is shown a function of dimensionless  $r$ .

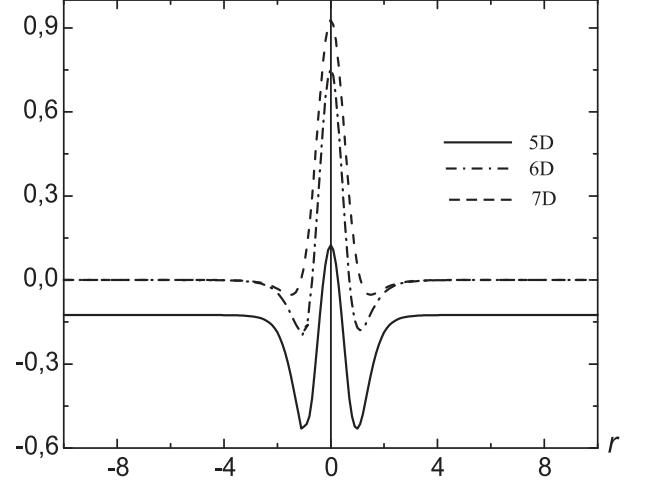


FIG. 10: The energy density  $T_0^0 = \epsilon \left[ \frac{1}{2\lambda} (\varphi'^2 + \chi'^2) + V \right]$  is shown a function of dimensionless  $r$ .

field Lagrangian, respectively. All these physical quantities refer to the 6-dimensional spacetime with the signature of  $(+ - - - -)$ . The variation of the action (167) with respect to the 6-dimensional metric tensor  $g_{AB}$  leads to the Einstein equations

$$R_{AB} - \frac{1}{2}g_{AB}R = \frac{1}{M^4} (g_{AB}\Lambda + T_{AB}) , \quad (168)$$

where  $R_{AB}$  and  $T_{AB}$  are the Ricci and the energy-momentum tensors, respectively. The capital Latin indices run  $A, B, \dots = 0, 1, 2, 3, 5, 6$ .

The 4-dimensional Einstein equations are  $R_{\mu\nu}^{(4)} - \frac{1}{2}\eta_{\mu\nu}R^{(4)} = 0$ , i.e., the ordinary vacuum equations without no cosmological constant. The Greek indices  $\alpha, \beta, \dots = 0, 1, 2, 3$  refer to the ordinary four dimensions. The ansatz functions for the source are

$$T_{\mu\nu} = -g_{\mu\nu}F(r), \quad T_{ij} = -g_{ij}K(r), \quad T_{i\mu} = 0 . \quad (169)$$

The small Latin indices refer to the extra coordinates  $i, j = 5, 6$ . Finally the metric takes the form

$$ds^2 = \phi^2(r)\eta_{\mu\nu}dx^\mu dx^\nu - \lambda(r)(dr^2 + r^2d\theta^2) . \quad (170)$$

The metric of the ordinary 4-space,  $\eta_{\alpha\beta}$ , has the signature of  $(+, -, -, -)$ . The 4D and 2D “warp” factors, namely the ansatz functions  $\phi(r)$  and  $\lambda(r)$ , depend only on the extra radial coordinate,  $r$ . The Einstein field equations (168) in terms of the ansatz functions are [110]

$$3\frac{\phi''}{\phi} + 3\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} + \frac{1}{2}\frac{\lambda''}{\lambda} - \frac{1}{2}\frac{(\lambda')^2}{\lambda^2} + \frac{1}{2}\frac{\lambda'}{r\lambda} = \frac{\lambda}{M^4}[F(r) - \Lambda] , \quad (171)$$

$$\frac{\phi'\lambda'}{\phi\lambda} + 2\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} = \frac{\lambda}{2M^4}[K(r) - \Lambda] , \quad (172)$$

$$2\frac{\phi''}{\phi} - \frac{\phi'\lambda'}{\phi\lambda} + 3\frac{(\phi')^2}{\phi^2} = \frac{\lambda}{2M^4}[K(r) - \Lambda] , \quad (173)$$

where the prime  $= \partial/\partial r$ . These equations are the  $(\alpha, \alpha)$ ,  $(r, r)$ , and  $(\theta, \theta)$  components, respectively. Two of the three equations (171) – (173) are independent, and they can be reduced to a set of two equations for  $\phi(r)$ ,  $\lambda(r)$ . An analytic solution to these equations is found with the ansatz functions of the form

$$\phi(r) = \frac{c^2 + ar^2}{c^2 + r^2} \xrightarrow{r \rightarrow \infty} a, \quad \lambda(r) = \frac{c^4}{(c^2 + r^2)^2} \xrightarrow{r \rightarrow \infty} \frac{c^4}{r^4} , \quad (174)$$

where  $c$  and  $a$  are constants. The source functions are

$$F(r) = \frac{f_1}{2\phi(r)^2} + \frac{3f_2}{4\phi(r)} \xrightarrow{r \rightarrow \infty} \frac{f_1}{2a^2} + \frac{3f_2}{4a}, \quad K(r) = \frac{f_1}{\phi(r)^2} + \frac{f_2}{\phi(r)} \xrightarrow{r \rightarrow \infty} \frac{f_1}{a^2} + \frac{f_2}{a}, \quad (175)$$

where the constants  $f_1 = -\frac{3\Lambda}{5}a$  and  $f_2 = \frac{4\Lambda}{5}(a+1)$  are determined by the 6D cosmological constant  $\Lambda$ , and the constant  $a$ , from the 4D warp function  $\phi(r)$ .

A drawback of this solution is that the matter sources are put in by hand via the ansatz functions  $F(r)$  and  $K(r)$  rather than given on some realistic foundation of the field theory. Also from  $T_{\mu\nu}$  from (169) and  $F(r)$  from (175) (as well as using the asymptotic values of  $F(r)$  at  $r=0$  and  $r=\infty$ ) one sees that the energy density is negative on the brane and decreases to a negative value at  $r=\infty$ .

In the paper [111] these solutions were generalized to the case of  $n > 2$  extra dimensions. As with the 6D solution, the obtained solutions provide the universal, gravitational trapping mechanism of fields with various spins of from 0 to 2. Considering the magnitude of the ansatz functions for the stress-energy which are necessary to realize this trapping solution one finds physically reasonable properties for a range of parameters  $n$ ,  $\Lambda$ ,  $\epsilon$  (brane width) and  $a$ . In addition, for  $n=2$  and  $n=3$ , it is possible to have the stress-energy tensor which approaches zero in the limit of  $r \rightarrow \infty$ . For  $n=4$ , either  $F(r)$  or  $K(r)$  goes to zero, but not both do.

## V. COSMOLOGICAL THICK BRANES

In the previous sections, thick brane solutions which contain Minkowski or de Sitter four-dimensional geometry were considered. For such models, the bulk metric functions depend only on the radial bulk coordinate and solutions can be obtained, just by solving the ordinary differential equations. But if a thick brane which has a less symmetric, Friedmann-Robertson-Walker (FRW) cosmological geometry, then in most of cases the metric functions also depend on time, and to find a solution, one has to solve partial differential equations. The cosmological thick brane solutions were not explored much, except for works e.g., [112].

In this section, several approaches to formulate the cosmological equations, in the presence of arbitrary matter inside the brane, are reviewed. Because of the subtlety about how to define and handle the effective four-dimensional quantities, there is no unique formulation. Thus, we review several representative approaches both in five- and multidimensional spacetimes.

### A. Codimension-one model

#### 1. Thin brane cosmology

The interest in the localized matter distributions in the context of gravity has a long history. Because of the difficulty to treat a thick wall, in the early times it had been considered to idealize a wall as an infinitesimally thin object [113–115]. Then, the interest had come again with the studies on cosmological phase transitions and formation of topological defects. Again, these defects were mainly assumed to be infinitesimally thin [116, 117]. The formulation of the equations of motion of a singular domain wall was summarized by Israel [118]. In the context of the Lanczos-Darmois-Israel formalism, a thin shell is regarded as an idealized object with zero thickness.

Recent developments of string and M-theory have motivated us to study cosmology in brane world. The concept of a domain wall was applied very frequently to treat a self-gravitating brane mathematically in the multidimensional general relativity. Cosmological equations on a thin brane distribution embedded into a five-dimensional bulk spacetime is easily derived by integrating the five-dimensional Einstein equations with a  $\delta$ -functional contribution into the energy-momentum tensor [119, 120] or by employing the Israel junction conditions [121, 122]. The thin brane cosmology for a given bulk geometry is uniquely determined. As the simplest example, in an AdS-Schwarzschild bulk spacetime with the  $Z_2$  symmetry with respect to the brane, the effective Friedmann equation on the brane is given by

$$\left(\frac{\dot{a}}{2}\right)^2 + \frac{K}{a^2} = \frac{1}{3}\Lambda_4 + \frac{8\pi G_4}{3}\rho\left(1 + \frac{\rho}{2\sigma}\right) + \frac{M}{a^4}, \quad (176)$$

where  $a$  and  $\rho$  are the scale factor and energy density of the brane, respectively.  $\sigma$  represents the brane tension. The AdS curvature radius is given by  $\ell^2 = -6/\Lambda$ , where  $\Lambda$  is a (negative) bulk cosmological constant. The constant  $M$  is related to the mass of a black hole sitting in the bulk. The four-dimensional Newton and cosmological constants are given by

$$8\pi G_4 = \frac{\kappa_5^2}{\ell}, \quad \Lambda_4 = \frac{\kappa_5^4}{12}\sigma^2 - \frac{3}{\ell^2} = \frac{\kappa_5^4}{12}\sigma^2 + \frac{1}{2}\Lambda_5. \quad (177)$$

In the effective Friedmann equation (176), two corrections to the conventional one appear. The first correction is the term which is proportional to the square of energy density on the brane, which may be important in the early

universe. The second contribution is the so-called dark radiation which is proportional to the mass of the black hole  $M$  and a purely geometrical effect.

## 2. Thickness of domain walls in general relativity

In contrast to the case of a thin brane, the thickness brings new subtlety. Here, we see the brief history on the treatment of the thickness of a domain wall. Early attempts to formulate the thickness were mainly motivated in modeling the late time cosmological phase transition [123]. Silveira [124] studied the dynamics of a spherical thick domain wall by defining an average of radius  $R$ , weighted by the energy density. It was also unclear whether the Israel conditions are applicable to the domain wall problem and the zero thickness limit of a thick domain wall is really equivalent to a thin domain wall dominated by the Israel conditions. There were several works related to this issue [125–127]. The work of [128] discussed a modification of the Israel thin shell equations in the Einstein-scalar theory and concluded that the effect of thickness at first approximation was effectively to reduce the energy density of the wall in comparison with the thin case, leading to a faster collapse of a spherical wall in vacuum. The work of [129] applied the expansion method of the wall action in terms of the small thickness parameter and showed that the effective action for a thick domain consists of the usual Nambu-Goto term and a contribution proportional to the induced Ricci scalar curvature.

### B. Thick brane cosmology

Here, several representative approaches to discuss cosmology on a thick brane are reviewed.

*a. Averaging approach* In the approach employed in Ref. [130], the effective brane quantities are obtained by integrating the five-dimensional ones over the brane thickness. The brane has a finite thickness between  $y = -y_0/2$  and  $y = y_0/2$ , where  $y$  is the proper coordinate along the extra dimension. For simplicity, it is assumed that the brane thickness is time-independent.

There is ambiguity of the definition of the effective four-dimensional quantities. Here, the averaging prescription that one integrates a five-dimensional quantity  $Q(t, y)$  over the brane thickness

$$\langle Q(t) \rangle = \frac{1}{y_0} \int_{-y_0/2}^{y_0/2} Q(t, y) dy, \quad (178)$$

is adopted. For convenience, the dimensionless quantities

$$\begin{aligned} \epsilon &:= \frac{\kappa_5^2}{6} y_0^2 \langle \rho \rangle, & \alpha &:= \frac{a(y_0/2)}{\langle a \rangle}, \\ \eta &:= \frac{\langle \rho a^2 \rangle}{\langle \rho \rangle \langle a^2 \rangle}, & \tilde{\eta} &= \frac{\langle a^2 \rangle}{\langle a \rangle^2} \end{aligned} \quad (179)$$

are introduced. The parameter  $\eta$  represents the homogeneity of the matter distribution over the brane. By imposing  $p_r(\pm y_0/2) = 0$ , where  $p_r$  is the pressure along the extra direction, and by integrating the Einstein equations, the effective Friedmann equation is found to be

$$H^2 = \frac{2}{y_0^2} \left( \alpha^2 + \epsilon \eta - \frac{y_0^2}{\ell^2} \tilde{\eta} \right) \left[ 1 \pm \sqrt{1 - \frac{(\epsilon \eta - \tilde{\eta} \frac{y_0^2}{\ell^2})^2 + \frac{C y_0^2}{\langle a \rangle^4} - \frac{\alpha^4 y_0^2}{\ell^2}}{(\alpha^2 + \epsilon \eta - \tilde{\eta} \frac{y_0^2}{\ell^2})^2}} \right], \quad (180)$$

where  $C$  is an integral constant. The constant  $C$  must satisfy the inequality that

$$C \frac{y_0^2}{\langle a \rangle^2} \leq 2\alpha^2 \left( \eta \epsilon - \tilde{\eta} \frac{y_0^2}{\ell^2} \right) + \alpha^4 \left( 1 - \frac{y_0^2}{\ell^2} \right). \quad (181)$$

In the thin brane limit  $y_0 \rightarrow 0$ , we find

$$H^2 \approx \frac{1}{\alpha^2} \left( \frac{\kappa_5^2}{36} \langle \rho \rangle^2 \eta^2 + \frac{C}{a^4} - \frac{\alpha^4}{\ell^2} \right), \quad (182)$$

which corresponds to the well-known thin brane cosmology with the quadratic dependence on the energy density of a brane. In the opposite limit  $y_0 \rightarrow \infty$ , the generalized Friedmann equation reduces to

$$H^2 \simeq \frac{2\epsilon\eta}{y_0^2} - \frac{2\tilde{\eta}}{\ell^2} + \frac{2}{y_0} \sqrt{\frac{\alpha^2(\alpha^2 - \eta)}{\ell^2} - \frac{C}{\langle a \rangle^4}}, \quad (183)$$

up to  $O(y_0^{-1})$ . In the case of a Minkowski bulk  $y_0 \rightarrow 0$ , the inequality yields  $C \rightarrow 0$  and

$$H^2 \approx \frac{2\epsilon\eta}{y_0^2} = \frac{\kappa_5^2 \eta}{3y_0} \langle \rho \rangle, \quad (184)$$

which corresponds to the standard four-dimensional cosmology with effective Newton's constant  $\kappa_4^2 := \kappa_5^2 \eta / y_0$ . In the homogeneous bulk  $\eta = 1$ , the result is in agreement with the Kaluza-Klein picture. Thus, the above equations smoothly interpolate between the Kaluza-Klein and thin brane pictures of cosmology. The extension of this method to the DGP model was discussed in Ref. [131].

*b. Two thin-shell approximation* Another different approach based on the gluing of the regular thickened brane with two different spacetime geometries along two regular hypersurfaces was considered in [132, 133]. The purpose was to understand the dynamics of a localized matter distribution of any kind, confined within two different spacetimes or matter phases. It would be enable one to have any topology and any spacetime on each side of a thick wall or brane. By construction, the matching is regular and there is no singular surface whatsoever in this formulation. Therefore, the junction conditions for the induced metric extrinsic curvature tensors on the thick wall boundaries with the two embedding spacetimes can be applied as

$$[h_{ab}]_{\Sigma_i} = 0, \quad [K_{ab}]_{\Sigma_i} = 0, \quad (185)$$

where  $\Sigma_i$  ( $i=1,2$ ) represents each regular boundary located between the core brane region and each bulk region. By applying the formalism developed in [134], in Ref. [135] cosmology on the brane in an AdS-Sch bulk was investigated. It turns out that the first modified Friedmann equation becomes

$$H^2 + \frac{K}{a^2} = \frac{8\pi G_4}{\rho} + \frac{\kappa_5^4 \rho^2}{36} + \frac{\Lambda_4}{3} + \frac{C}{a^4}, \quad (186)$$

where the effective four-dimensional cosmological and gravitational constants can be read off

$$\frac{\Lambda_4}{3} = \frac{\Lambda}{6} + \frac{w^2 \Lambda^2}{9}, \quad 8\pi G_4 = \frac{\kappa_5^2 w (-\Lambda)}{3}. \quad (187)$$

$2w$  is the proper separation between two regular walls. In contrast to the thin brane case, the linear dependence on the brane energy density is obtained even without the tension. Newton's constant is proportional to the brane thickness. Other than the energy density equation, one also obtains the pressure equation which depends on the bulk pressure in a nontrivial way. These two cosmological equations together with the energy conservation law determine the system completely. The time-variation of the brane thickness was discussed in Ref. [136]. It was shown that in the absence of the pressure along the extra dimension in the brane energy-momentum tensor the thickness of the brane decreases with time, while at a late time the negative transverse pressure can lead to an increase in the brane thickness. According to Eq. (187), this suggests the time variation of the constants  $G_4$  and  $\Lambda_4$ .

*c. Quasi-static approximation* The authors of Ref. [137] considered a thick codimension-one brane model in which the brane energy-momentum tensor includes the pressure component along the extra dimension. It is assumed that there is no matter outside the brane. By integrating the four-dimensional time and spatial components of the five-dimensional Einstein equations along the fifth dimension, the first-order transverse derivatives of the metric functions at the brane boundaries are related to the components of the brane energy-momentum tensor integrated over the brane. Here, the assumption that the derivatives parallel to the brane are negligible inside the thick brane in comparison with the transverse ones, is made. These *matching conditions* are plugged into the remaining extra-dimensional component of the Einstein equations and the cosmological acceleration equations become

$$3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \frac{1}{12M^6} \left[ 2(\sigma + \sigma_r)^2 + (\sigma + \sigma_r)(1 - 3w - 4w_r)\rho_m \right] + \frac{\Lambda}{M^3},$$

$$\dot{\rho}_m + 3\frac{\dot{a}}{a}(1+w) - w_r \dot{\rho}_m, \quad (188)$$

where  $\rho = \sigma + \rho_m$ ,  $p = -\sigma + w\rho_m$  and  $p_r = -\sigma_r + w_r\rho_m$  are energy density, pressure and bulk pressure in the brane, which are defined by the integration of the bulk stress tensor over the thick brane, respectively.  $\sigma$  and  $\sigma_r$  are constant parts of the four-dimensional matter, i.e., tension and the bulk pressure, and  $(w, w_r)$  specify the equations of state of the time-dependent parts. The presence of the bulk pressure violates the energy conservation on the brane. The effective equation of state of the brane matter is given by

$$w_{\text{eff}} = \frac{w + w_r}{1 - w_r}. \quad (189)$$

An accelerating universe can be obtained for  $w_{\text{eff}} < -1/3$ . It implies that for example even for cosmic dust  $w = 0$ , the cosmic acceleration can be realized as long as  $w_r < -1/2$  [137].

This method was originally developed in the codimension-two case and also generalized to the cases of higher codimensions, which will be reviewed in the next subsection.

### C. Higher-codimensional cases

#### 1. UV behavior of higher-codimensional branes

In general, a 3-brane with more than two codimensions is a singular object because the gravitational potential sourced by the brane is divergent in approaching it. This is understood as follows. Imagine a codimension- $n$  ( $n = 1, 2, 3, \dots$ ) self-gravitating object. Gauss's law shows that the gravitational potential becomes  $V(r) \propto 1/r^{n-2}$ , where  $r$  is the radial distance from the object. In the case of  $n = 1$ , which is a singular domain wall case, the jump across the object is finite and the matching conditions are available. The case of  $n = 2$ , where the object is a string-like defect, is marginal and only the tensional matter can be put on the object and induces the deficit angle of the bulk. In the case of  $n > 3$ , for any kind of the localized matter, the potential becomes singular and a naked singularity is usually formed. In such a case, one needs some prescription, such as UV regularization, to put matter there.

Several prescriptions for such divergence were suggested in the field theory approaches. In Refs. [138–140], an UV regularization of the self-forces of various fields (scalar, graviton and form fields), for extended objects with codimensions more than two, were discussed in the linearized approximation. The regularization was performed by replacing a singular delta functional source with a regular profile function (An IR cut-off is also required for a codimension two brane to cure the logarithmic divergence). For the gravitational force alone, the force is proportional to the extrinsic curvature vector, i.e., the trace of the second fundamental tensor. For the special case of a codimension-two brane, the force is actually zero and so the self-interactions cancel, which is a well-known result for cosmic strings in four dimensions, i.e., the tension determines the conical deficit angle and does not affect the local geometry both inside and outside the string. When a dilaton and a form field are included, certain combinations of couplings result in the vanishing total self-force. It was also shown that these regularized self-interactions can be expressed as renormalizations of the action. In Ref. [141, 142], the classical renormalization schemes for a local divergence at a codimension-two (conical) brane were discussed, mainly for a scalar theory. It was shown that the logarithmic (classical) divergence of a bulk field in the thin brane limit can be renormalized into the brane localized mass and couplings, at the tree and one-loop levels, and the renormalization group equations for them were obtained. In Ref. [141], the brane dynamics was not taken into account and in Ref. [142] it was done. The philosophy behind the procedure is that the low-energy physics on the brane should be independent of the high-energy regularization mechanism: 1) bulk fields evaluated away from the brane should not depend on the regularization mechanism and thus be finite in the thin-brane limit, 2) bulk fields evaluated on the brane itself would be sensitive to the regularization procedure and thus it would not be required that bulk fields evaluated on the brane are finite in the thin-brane limit, 3) brane fields should have a well defined low-energy theory independent of the brane regularization.

#### 2. Cosmology on regularized higher-codimensional branes

To discuss nonlinear gravity and cosmology on a codimension-two brane, various ways of regularization were derived, in e.g., Refs. [143–150]. In the works Ref. [143, 144], the authors considered a way of regularizations to smooth out the matter over the brane with a finite thickness and investigated the effective cosmology in a class of six-dimensional models with stable and compact extra dimensions. The time-dependence of the brane matter was taken into consideration perturbatively around a static solution. By integrating over the brane, the effective cosmological equations are obtained and coincide with the conventional one at the low energy scales.



Another way of regularization of a codimension-two brane has been discussed in Refs. [145–149], such that a codimension-two brane was replaced with a regular bulk geometry, i.e., a capped region. Between the original bulk and the capped region, a ring-like 4-brane with the compact fifth dimension, wrapping around the axis of symmetry of the bulk, appears. This 4-brane is a codimension-one object and the Israel junction conditions can be applied. A motion of the brane realizes cosmology, but as long as the bulk is static, the effective cosmology is not the realistic one. An appropriate inclusion of the time-dependence of the bulk helps the recovery of it on the brane.

The work [150] also discussed cosmology on a thick codimension-two brane. Apart from the previous approaches, the discussion of [150] was given in the general background and may be able to be applied to various codimension-two brane models, as long as several assumptions are satisfied. The authors of Ref. [151] extended the method employed in [150] to the case of an arbitrary number of codimensions. The discussion starts from the general metric ansatz that

$$ds^2 = -N(t, r)^2 dt^2 + A(t, r)^2 g_{ij} dx^i dx^j + dr^2 + \alpha(t, r)^2 \gamma_{ab} dy^a dy^b, \quad (190)$$

where  $\gamma_{ab}$  is the metric of  $(m - 1)$ -sphere. Following Ref. [138–140], regularization of the brane is performed by replacing the delta function with a regular profile function. As the profile function, here, it is assumed that all the brane matter is uniformly distributed over the brane region  $0 < r < \epsilon$ .

Here several assumptions are going to be made: The important one is that the derivatives tangential to the brane are small enough in comparison with those normal to the brane. It is also assumed that  $A(t, 0) \approx A(t, \epsilon) = a(t)$  and the energy momentum tensor averaged over the brane takes the form of  $\tilde{T}_A^B = \text{diag}(-\rho, p, p, p, p_r, p_b, \dots)$ , where  $\rho$ ,  $p$ ,  $p_r$  and  $p_b$  are the energy density and pressures along the three-dimensional space, bulk radial direction and bulk angular directions, respectively.

In the case of a codimension-two brane of  $m = 2$ , in Ref. [150], the pressure equation is obtained as

$$\begin{aligned} 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) &= -\frac{\Lambda}{M^4} + \frac{1}{2\pi\alpha^2 M^4} (p + p_r) \frac{\sqrt{-g}|_0}{\sqrt{-g}|_\epsilon} \\ &\quad - \frac{1}{32\pi^2 \alpha^2 M^8} \left[ 3(\rho + p)^2 + 3p_b^2 + 2(p_b + p_r)(\rho - 3p) - 2pp_r - 5p_r^2 \right]. \end{aligned} \quad (191)$$

Finally, in the codimension  $m > 3$  cases,

$$3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = \omega_1 + \omega_2(\rho + p)^2 + \omega_3 p(p - \rho) + \omega_4(\rho - 3p_r), \quad (192)$$

where

$$\begin{aligned} \omega_1 &= -\frac{\Lambda}{M^{m+2}} - \frac{A_m^2}{8(m+2)} \left[ 3(m-1)p_b^2 - 2(m-1)p_r p_b - (m+3)p_r^2 \right] \\ &\quad + \frac{(m-1)(m-2)}{\epsilon^2} - \frac{(m-1)A_m}{2\epsilon} (p_r + p_b), \\ \omega_2 &= -\frac{A_m^2(1+m)}{8(m+1)}, \\ \omega_3 &= -\frac{A_m^2(m-2)}{4(m+2)}, \\ \omega_4 &= -\frac{A_m^2}{4(m+2)} [(m-1)p_b + p_r], \end{aligned} \quad (193)$$

and,  $A_m = 2/(\Omega_{m-1}\alpha^{m-1}M^{m+2})$  and  $B_m = -2 - m$ . As in the five-dimensional case, the brane matter is decomposed into the constant (tension) part and time-dependent part as  $\rho = \sigma + \rho_m$  and  $p = -\sigma + w\rho_m$ . In the low energy regime, expanding the equation of motion up to the linear order  $\rho_m$ ,

$$3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = (\omega_1 + 2\sigma^2\omega_3 + 4\omega_4\sigma) + (\omega_4 + \sigma\omega_3)(1 - 3w)\rho_m + O(\rho_m^2), \quad (194)$$

which has the form in the conventional cosmology.

In the case of higher-codimensions that  $m > 2$ , under several assumptions such that the time evolution of the brane thickness is negligible,  $p_r$  is constant, and the energy flow into the bulk is negligible  $T_{\mu r}|_\epsilon = 0$ , the standard energy conservation law is satisfied. The form is similar to the Randall-Sundrum one. But the effective gravitational constant

is proportional to  $\omega_4$  which explicitly depends on the bulk pressures  $p_b$  and  $p_r$ . However, in the higher codimensional case, there is the subtlety on the presence of a localized zero mode.

In the case of a codimension-two brane  $m = 2$ , the difference from the cases of  $m > 2$  comes from the term proportional to  $\sqrt{-g}|_{r=\epsilon}/\sqrt{-g}|_{r=0}$ . But by using the matching condition, as long as  $|p_r|$  and  $|p_\theta|$  are small enough in comparison with  $|\rho|$  and  $|p|$ ,  $\sqrt{-g}|_{r=\epsilon}/\sqrt{-g}|_{r=0}$  can be approximated to be unity. Under such an approximation, the cosmological behavior has no essential difference from the cases of  $m > 2$ , but there is no problem on the localization of a graviton zero mode.

## VI. S-BRANES

Usually, a brane is considered as a time-like manifold. However, it is also possible to consider time-dependent solutions in multidimensional gravitational theories and to interpret them as spacelike branes (S-branes). Such an approach was suggested in the paper [152]. S-branes which are thick and regular everywhere are interesting cosmological solutions for solving the cosmological singularity problem.

The authors of the paper [152] support their interest for investigations in this direction by the following arguments: SD-branes (S-branes with Dirichlet boundary conditions) may holographically reconstruct a time dimension just as in the AdS/CFT correspondence the D-brane field theory holographically reconstructs a spatial dimension.

In [152], the physical situations giving the solutions of an S2-brane domain wall, an S1-brane vortex, and a charged S-brane are considered. The arguments in favor of existence of S-branes in string theory are presented.

Several solutions of the supergravity equations corresponding to S-branes with odd codimensions are obtained. The following S0-brane solution in the D=4 Einstein-Maxwell gravity is obtained. The starting action is

$$S = \int d^4x \sqrt{-g} (R - F^2). \quad (195)$$

The corresponding Einstein-Maxwell equations have the following solution

$$ds^2 = -\frac{Q^2}{\tau_0^2} \frac{\tau^2}{\tau^2 - \tau_0^2} d\tau^2 + \frac{\tau_0^2}{Q^2} \frac{\tau^2 - \tau_0^2}{\tau^2} dz^2 + \frac{Q^2 \tau_0^2}{\tau_0^2} dH_2^2, \quad (196)$$

where  $Q$  and  $\tau_0$  are constants and  $dH_2^2$  is the unit metric on a 2-dimensional space with the constant negative curvature, respectively. The solution describes a charged S0-brane.

An S5-brane in the D=11 supergravity is obtained as well. The starting action is

$$S = \int d^{11}x \sqrt{-g} \left( R - \frac{1}{2 \cdot 4!} F^2 \right). \quad (197)$$

The solution of corresponding field equations is

$$ds^2 = -\left(\frac{q^2}{24}\right)^{1/3} \frac{[\cosh \sqrt{24}(t-t_0)]^{2/3}}{(\sinh 3t)^{8/3}} [-dt^2 + (\sinh 3t)^2 dH_4^2] + \left(\frac{q^2}{24}\right)^{-1/6} \frac{1}{[\cosh \sqrt{24}(t-t_0)]^{1/3}} dx_6^2, \quad (198)$$

where  $q, t_0$  are constants and  $dH_4^2$  is the unit metric on a 4-dimensional space with the constant negative curvature, respectively. It was shown that there exists a singularity of this metric near the brane for a large  $t$ .

In Ref. [153] S-branes containing a graviton,  $q$ -form field strength,  $F_{[q]}$ , and a dilaton scalar,  $\phi$ , coupled to the form field with the coupling constant  $a$ , are discussed. In the Einstein frame, the action is given by

$$S = \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2q!} e^{a\phi} F_{[q]}^2 \right). \quad (199)$$

The corresponding equations of motion are

$$R_{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{e^{a\phi}}{2(q-1)!} \left[ F_{\mu\alpha_2 \dots \alpha_q} F_{\nu}{}^{\alpha_2 \dots \alpha_q} - \frac{q-1}{q(d-2)} F_{[q]}^2 g_{\mu\nu} \right] = 0, \quad (200)$$

$$\partial_\mu (\sqrt{-g} e^{a\phi} F^{\mu\nu_2 \dots \nu_q}) = 0, \quad (201)$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) - \frac{a}{2q!} e^{a\phi} F_{[q]}^2 = 0. \quad (202)$$

A solution describing an S-brane is sought in the form

$$ds^2 = -e^{2A} dt^2 + e^{2B} (dx_1^2 + \cdots + dx_p^2) + e^{2C} d\Sigma_{k,\sigma}^2 + e^{2D} (dy_1^2 + \cdots + dy_{q-k}^2). \quad (203)$$

The hyperspace  $\Sigma_{k,\sigma}$  is a maximally symmetric space with a constant scalar curvature

$$d\Sigma_{k,\sigma}^2 = \bar{g}_{ab} dz^a dz^b = \begin{cases} d\psi^2 + \sinh^2 \psi d\Omega_{k-1}^2, & \sigma = -1, \\ d\psi^2 + \psi^2 d\Omega_{k-1}^2, & \sigma = 0, \\ d\psi^2 + \sin^2 \psi d\Omega_{k-1}^2, & \sigma = +1. \end{cases} \quad (204)$$

The corresponding field equations can be written in the following form

$$\dot{h}^2 + \frac{(q-1)b^2}{(d-2)\chi} e^{\chi h} = \alpha^2, \quad (205)$$

$$\dot{g}^2 + \sigma e^{2(k-1)g} = \beta^2, \quad (206)$$

where the following notations are introduced

$$f(t) = h(t) - \frac{ac_1}{\chi} t - \frac{ac_2}{\chi}, \quad (207)$$

$$\chi = 2p + \frac{a^2(d-2)}{q-1}, \quad (208)$$

$$A(t) = kg(t) - \frac{p}{q-1} f(t), \quad B(t) = f(t), \quad C(t) = g(t) - \frac{p}{q-1} f(t), \quad D(t) = -\frac{p}{q-1} f(t). \quad (209)$$

The constants  $\alpha$  and  $\beta$  satisfy

$$\frac{p c_1^2}{\chi} + \frac{(d-2)\chi\alpha^2}{2(q-1)} - k(k-1)\beta^2 = 0. \quad (210)$$

The equations (205) (206) have the following solution

$$f(t) = \frac{2}{\chi} \ln \left\{ \frac{\alpha}{\cosh \left[ \frac{\chi\alpha}{2} (t - t_0) \right]} \right\} + \frac{1}{\chi} \ln \left[ \frac{(d-2)\chi}{(q-1)b^2} \right] - \frac{ac_1}{\chi} t - \frac{ac_2}{\chi}, \quad (211)$$

$$g(t) = \left\{ \begin{array}{ll} \frac{1}{k-1} \ln \left\{ \frac{\beta}{\sinh[(k-1)\beta(t-t_1)]} \right\}, & \sigma = -1. \\ \pm \beta(t-t_1), & \sigma = 0. \\ \frac{1}{k-1} \ln \left\{ \frac{\beta}{\cosh[(k-1)\beta(t-t_1)]} \right\} & \sigma = +1. \end{array} \right\}, \quad (212)$$

which describes S-branes.

The S-brane solution obtained in Ref.[154] is based on the use of two strongly interacting scalar fields giving regular solutions over all spacetime. The solution is based on the fact that the potential of the scalar fields has two global and two local minima. As  $|t| \rightarrow \infty$  the scalar fields go to one of the local minima. This is the necessary condition for existence of such type of a solution. The numerical analysis showed that, apparently, there is no other solution which tends toward one of the global minima. From the mathematical point of view, the obtained S-brane solution in Ref.[154] is very similar to the usual (time-like) brane solution considered in the section III A.

The Lagrangian for gravitating phantom scalar fields  $\phi, \chi$  is:

$$L = -\frac{R}{16\pi G} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \partial_\mu \chi \partial^\mu \chi - V(\phi, \chi), \quad (213)$$

where the potential  $V(\phi, \chi)$  describing strong interaction between scalar fields is given by

$$V(\phi, \chi) = \frac{\lambda_1}{4} (\phi^2 - m_1^2)^2 + \frac{\lambda_2}{4} (\chi^2 - m_2^2)^2 + \phi^2 \chi^2 - V_0 \quad (214)$$

where  $V_0$  is defined by initial conditions. The metric describing an S-brane is

$$ds^2 = dt^2 - a(t)^2 (dx^2 + dy^2 + dz^2). \quad (215)$$

After finding the corresponding equations for the gravitating scalar fields  $\phi$  and  $\chi$ , one can see, from the physical point of view, that the metric describes the bounce of the universe at the moment  $t = 0$ . Corresponding equations (in the Planck units, i.e. at  $c = \hbar = G = 1$ ) are

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{6} [-\dot{\phi}^2 - \dot{\chi}^2 + V(\phi, \chi)] \quad (216)$$

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{4} (\dot{\phi}^2 + \dot{\chi}^2), \quad (217)$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \phi [\chi^2 + \lambda_1 (\phi^2 - m_1^2)], \quad (218)$$

$$\ddot{\chi} + 3\frac{\dot{a}}{a}\dot{\chi} = \chi [\phi^2 + \lambda_2 (\chi^2 - m_2^2)], \quad (219)$$

and

$$V_0 = \frac{\lambda_1}{2} (\phi_0^2 - m_1^2)^2 + \frac{\lambda_2}{2} (\chi_0^2 - m_2^2)^2 + \phi_0^2 \chi_0^2. \quad (220)$$

The numerical solution with the boundary conditions

$$a(0) = a_0, \quad \dot{a}(0) = 0, \quad (221)$$

$$\phi(0) = \phi_0, \quad \dot{\phi}(0) = 0, \quad (222)$$

$$\chi(0) = \chi_0, \quad \dot{\chi}(0) = 0 \quad (223)$$

gives the result presented in Figs. 11 and 12. It is necessary to note that, as well as for time-like branes described

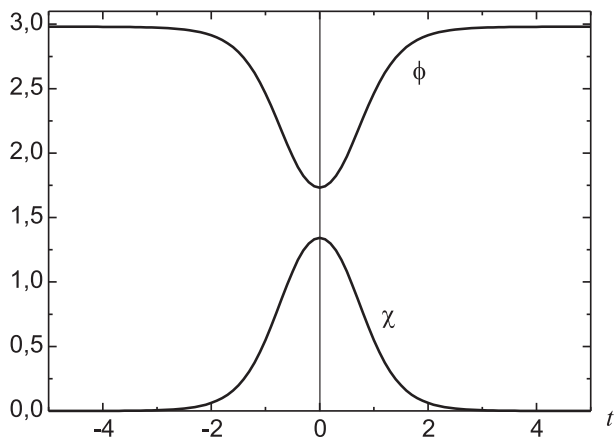


FIG. 11: The graphs for the scalar fields  $\phi^*(t), \chi^*(t)$ . Both functions tend asymptotically to the local minimum at  $\phi^*(t) = m_1^*, \chi^*(t) = 0$ .

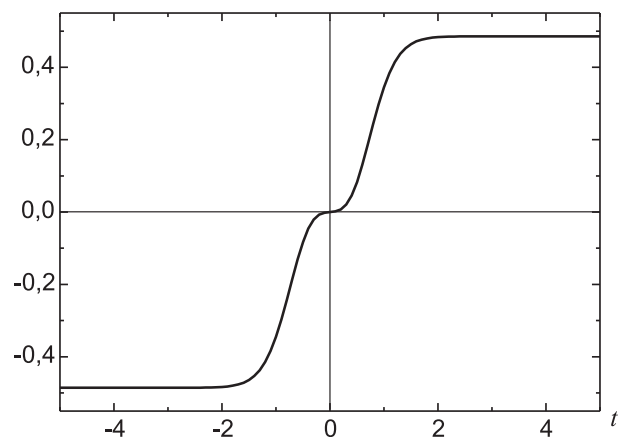


FIG. 12: The Hubble parameter  $H = \dot{a}/a$  tends asymptotically to the de Sitter stages: if  $H < 0$  - the stage of contraction, if  $H > 0$  - the stage of expansion.

in the section III A, regular solutions (presented in Figs. 11 and 12) exist only for definite values of the parameters  $m_1 = m_1^*, m_2 = m_2^*$ . In this sense, the solution of the equations (216)-(219) is a solution of a problem for two eigenvalues  $m_1$  and  $m_2$  and two eigenfunctions  $\phi^*(t)$  and  $\chi^*(t)$ .

## VII. CONCLUSION

In this paper, we gave a review of works devoted to study the thick brane solutions. Our main objective is to introduce the recent progress. As mentioned in the introduction (section I), there are the important motivations to consider the thick brane models: The first motivation comes from the interesting feature of a multidimensional topological defect that can trap the various fields inside it. The second one comes from the fact that along the developments of the (thin) brane cosmology, in particular in considering models of a spacetime with more than six dimensions, it turned out that even at the classical level a thin brane exhibits a singular behavior, due to the stronger self-gravity of the brane. To cure it, it would be crucial to take the effects of brane thickness into consideration.

We showed the previously known solutions and reviewed their basic properties. But as mentioned in the introduction, in this article, little was referred to about the applications of these solutions to the high energy physics and the localizability of the fields with various spins, because they concern both the thin and thick brane models and should be reviewed in separate publications. We tried to introduce classifications of the thick branes on the basis of their division into some classes by combining these models on topological and physical properties. They are summarized below (see also tables I and II).

*Topological feature:* The first important classification concerns whether a solution is topologically trivial or non-trivial. This is the division of the models in terms of topological properties. The detailed explanation about the concept of the topological triviality was given in the subsection II A.

The presence of topologically nontrivial vacuum states is typical for the topologically nontrivial solutions (section II). Here there are two cases:

1. codimension is equal to one
2. codimensions are more than two.

In the first case, there are kink-like solutions. Such solutions were found in the theory with scalar field(s) with a non-linear potential. In the second case, there are hedgehog-like solutions. Such a topological structure provides the stability of the solutions (at least at the classical level) that, of course, is a great advantage of such models.

In the case of the topologically trivial solutions, there is only one vacuum state (see section III): a solution starts from a vacuum and returns to the same vacuum. The solutions were constructed by employing two strongly coupled scalar fields. A question concerning the stability still remains. The stability analysis should be done in case by case.

*Brane geometry:* The geometry of the four-dimensional section on the brane is also an important point. The most works were devoted to search for a solution with a maximally symmetric four-dimensional section, i.e., Minkowski, de Sitter or anti-de Sitter(AdS) spacetime. This is because the system can be described by a set of the coupled ordinary differential equations with respect to the (radial) bulk coordinate. In particular, in the five-dimensional problem, the solution generation techniques as the superpotential method, discussed in the subsection II B, were developed. In a Minkowski or a de Sitter thick brane background, the trapping gravitons are possible, but in the AdS one, it may be impossible. The (stable) thick de Sitter brane solutions could be used for modeling inflation or dark energy.

In terms of obtaining a realistic cosmological model, a solution with the cosmological four-dimensional section is more desired (see section V). They would be used for describing the evolution of a brane configuration changing with time. In this case, generically, the problem depends both on extra space-like coordinates and time. It is obvious that both the search and interpretation of solutions would be much more complicated. Therefore, the research in this field was performed by introducing the brane thickness by hand. Although there is always ambiguity on how to define and handle the thickness, several procedures were proposed, such as the averaging approach, the thin-shell approximation and the quasi-static approximation. In such models, generically, new effects due to the thickness of the brane are induced onto the effective Friedmann-like equations, which for instance could accelerate the Universe even though there is only the ordinary matter inside the brane. But the nature of these effects depends on the regularization procedure.

*Bulk geometry:* In known solutions in five-dimensional models, the asymptotic bulk geometry can be either flat, de Sitter or AdS. In the model with a canonical scalar field, to obtain an asymptotically AdS thick brane solution, the fine-tuning condition Eq. (27) must be satisfied. In a multidimensional spacetime, the asymptotic geometry and regularity of the spacetime crucially depend on the boundary conditions and the choice of parameters. It is not only the case that the bulk spacetime is regular, but in some cases the bulk spacetime contains a naked singularity at a finite distance from the brane.

*Special models :* The model with an unusual source function put in by hand provides another special kind of solutions (see section IV). In this case, the aim is to simplify a model as much as possible to obtain analytic solutions. It allows analyzing properties of the model in the most evident form. However, this advantage of the model is smoothed over by the fact that the source is put in by hand.

The next special kind of solutions is the so-called S-brane model, in which it is assumed that the warp factor is some function of time (see section VI). It allows one to describe a cosmological evolution of multidimensional models which are both regular in a whole multidimensional space and do not have the initial cosmological singularity.

Within the framework of multidimensional theories including string and Kaluza-Klein theories, the possibility of extra time-like dimensions may not be excluded. This may provide the disappearance of the cosmological constant

in the four-dimensional spacetime [155, 156]. However, using the extra time-like dimensions leads to considerable difficulties in fundamental properties, like appearance of tachyons and ghosts [157, 158], and violation of causality [159]. A model of a thick brane with extra time-like coordinate can be obtained, for example, with the use of two interacting scalar fields from section III A. In the case of a 5-dimensional model from III A 2, one can choose the metric in the form

$$ds^2 = \phi^2(r)\eta_{\alpha\beta}(x^\nu)dx^\alpha dx^\beta - \delta dr^2, \quad (224)$$

where  $\delta = +1$  corresponds to the problem with one time-like dimension considered in III A, and  $\delta = -1$  refers to the problem with the additional time-like coordinate. In the latter case, the same thick brane solutions like in section III A will exist if one takes the reverse sign for the potential (157).

TABLE I: Classification of the thick brane models with scalar fields

Class of brane models	Matter sources	Basic feature
Topologically non-trivial thick branes	Scalar field(s) with non-linear potentials	The presence of topologically nontrivial vacuum states provide stability of the solutions at the classical level. A question about the quantum stability requires further studies.
Topologically trivial thick branes	Two strongly interacting usual or phantom scalar fields	A question about the classical and quantum stability requires further studies.

TABLE II: Special thick brane models

	Matter sources	Basic feature
Branes with unusual source functions	Sources put in by hand via the special ansatz of functions	The advantage is that it is possible to make use of analytical studies. The main drawback is that the source is put in by hand.
S-branes	Scalar fields, including fields with spontaneous symmetry breaking	The multidimensional time-dependent solutions can be both topologically trivial and nontrivial ones.
Thick branes with additional time coordinates	Two strongly interacting scalar fields	They are not excluded but possess the considerable difficulties such as appearance of tachyons, ghosts and violation of the causality.

We shall close this review article after suggesting the possible perspective directions of investigations:

1. To search new thick brane solutions, with the maximally symmetric four-dimensional section, in more generalized models of the field theory and the modified gravity.
2. To search self-consistent solutions describing objects localized on the brane (monopoles, black holes and so on) depending both on extra space-like coordinates and 4-dimensional coordinates.
3. To search the cosmological thick brane solutions which depend both on time and extra space-like coordinates. Having explicit solutions of such cosmological thick branes, the ambiguity mentioned above would be somewhat resolved.

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