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RESEARCH ARTICLE

On the ΛCDM Universe in $f(G)$ gravity

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Abstract In the context of the so-called Gauss–Bonnet gravity, where the gravitational action includes function of the Gauss–Bonnet invariant, we study cosmological solutions, especially the well-known ΛCDM model. It is shown that the dark energy contribution and even the inflationary epoch can be explained in the frame of this kind of theories with no need of any other kind of component. Other cosmological solutions are constructed and the rich properties that this kind of theories provide are explored.

Keywords Dark energy · Cosmology · Modified gravity · Gauss–Bonnet gravity

1 Introduction

It is widely accepted by the scientific community that our Universe is currently in an accelerated phase. General Relativity in its standard form can not explain the accelerated expansion without extra terms or components, which have been gathered under the name of dark energy. Since the discovery of the acceleration, a large number of

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possible mechanisms have been proposed to explain the origin of the dark energy, from the cosmological constant, scalar fields, to modifications of general relativity as well as other alternatives (for reviews on unified inflation-dark energy modified gravities, see Refs. [1,2], and for comparison with observational data see Refs. [3,4]). The most popular idea is represented by the cosmological constant, whose origin may be explained by means of the vacuum energy density, although its value does not match the one predicted by quantum field theories. Even in such a case, the Equation of State (EoS) parameter is a constant equal to -1, while some recent observations suggest that the EoS parameter is a dynamical variable and it could have crossed the phantom barrier, with the EoS parameter being less than -1. Such a dynamical behavior of the EoS could be well explained by scalar fields with quintessence or phantom behavior (see Ref. [5]). Another alternative is the modification of the gravity law. Several ways have been suggested to perform such a modification of General Relativity by means of the modification of the Hilbert-Einstein action, which obviously modifies the field equations. In this framework, the so-called $F(R)$ gravity has been explored, which introduces a more complex function of the Ricci scalar in the action and explains well the cosmic history via cosmological reconstruction (see Refs. [6–17]).

In this work we study the so-called Gauss–Bonnet gravity, where the gravitational action includes functions of the Gauss–Bonnet invariant. This kind of theories has been investigated and may reproduce the cosmic history (see Refs. [18–38]). Here, we show that the ΛCDM model can be well explained with no need of a cosmological constant but with the inclusion of terms depending on the Gauss–Bonnet invariant in the action. Even more, it is shown that the extra terms in the action coming from the modification of gravity could behave relaxing the vacuum energy density, represented by a cosmological constant, and may resolve the so-called cosmological constant problem. To study and reconstruct the theory that reproduces such a model as well as other kind of solutions studied, we shall use a method proposed in Ref. [10] for $f(R)$ gravity and implemented for $f(G)$ gravity in Ref. [38], where the FLRW equations are written as functions of the so-called number of e-foldings instead of the cosmic time. The possible phantom epoch produced by this kind of theories is also explored, as well as other interesting cosmological solutions, where the inclusion of other contributions as perfect fluids with inhomogeneous EoS are studied.

2 $[R + f(G)]$ gravity

We consider the following action, which describes General Relativity plus a function of the Gauss–Bonnet term (see Refs. [18] and [20]):

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + f(G) + L_m\right], \tag{1}$$

where $\kappa^2 = 8\pi G_N$, G_N being the Newton constant, and the Gauss–Bonnet invariant is defined as usual:

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}. \tag{2}$$

By varying the action over $g_{\mu\nu}$, the following field equations are obtained:

$$\begin{aligned}0 = &\frac{1}{2k^2}\left(-R^{\mu\nu}+\frac{1}{2}g^{\mu\nu}R\right)+T^{\mu\nu}+\frac{1}{2}g^{\mu\nu}f(G)-2f_G RR^{\mu\nu}+4f_G R^{\mu}_{\rho}R^{\nu\rho}\\ &-2f_G R^{\mu\rho\sigma\tau}R^{\nu}_{\rho\sigma\tau}-4f_G R^{\mu\rho\sigma\nu}R_{\rho\sigma}+2(\nabla^{\mu}\nabla^{\nu}f_G)R-2g^{\mu\nu}(\nabla^2 f_G)R\\ &-4(\nabla_{\rho}\nabla^{\mu}f_G)R^{\nu\rho}\\ &-4(\nabla_{\rho}\nabla^{\nu}f_G)R^{\mu\rho}+4(\nabla^2 f_G)R^{\mu\nu}+4g^{\mu\nu}(\nabla_{\rho}\nabla_{\sigma}f_G)R^{\rho\sigma}\\ &-4(\nabla_{\rho}\nabla_{\sigma}f_G)R^{\mu\rho\nu\sigma},\end{aligned} \tag{3}$$

where we made the notations $f_G = f'(G)$ and $f_{GG} = f''(G)$. We shall assume throughout the paper a spatially-flat FLRW Universe, whose metric is given by

$$ds^2 = -dt^2 + a(t)^2\sum_{i=1}^{3}\left(dx^i\right)^2, \tag{4}$$

where $a(t)$ is the scale factor at cosmological time t. For the metric (4), the field equations give the FLRW equations, with the form

$$\begin{aligned}0 &= -\frac{3}{\kappa^2}H^2 + Gf_G - f(G) - 24\dot{G}H^3 f_{GG} + \rho_m,\\ 0 &= 8H^2\ddot{f}_G + 16H\left(\dot{H}+H^2\right)\dot{f}_G + \frac{1}{\kappa^2}\left(2\dot{H}+3H^2\right) + f - Gf_G + p_m.\end{aligned} \tag{5}$$

The Hubble rate H is here defined by $H = \dot{a}/a$, while the matter energy density ρ_m satisfies the standard continuity equation:

$$\dot{\rho_m} + 3H(1+w)\rho_m = 0, \tag{6}$$

while the Gauss–Bonnet invariant G and the Ricci scalar R can be defined as functions of the Hubble parameter as

$$G = 24\left(\dot{H}H^2 + H^4\right), \quad R = 6\left(\dot{H}+2H^2\right). \tag{7}$$

Let us now rewrite Eq. (5) by using a new variable, $N = \ln\frac{a}{a_0} = -\ln(1+z)$, i.e. the number of e-foldings, instead of the cosmological time t, where z is the redshift (this method has been implemented in Ref. [10] for $f(R)$ gravity). The following expressions are then easily obtained

$$\begin{aligned}&a = a_0 e^N, \quad H = \dot{N} = \frac{dN}{dt}, \quad \frac{d}{dt} = H\frac{d}{dN}, \quad \frac{d^2}{dt^2} = H^2\frac{d^2}{dN^2} + HH'\frac{d}{dN},\\ &H' = \frac{dH}{dN}.\end{aligned} \tag{8}$$

Equation (5) can thus be expressed as follows

$$0 = -\frac{3}{\kappa^2}H^2 + 24H^3(H' + H)f_G - f - 576H^6\left(HH'' + 3H'^2 + 4HH'\right)f_{GG} + \rho_m, \tag{9}$$

where G and R are now

$$G = 24\left(H^3H' + H^4\right), \quad \dot{G} = 24\left(H^4H'' + 3H^3H'^2 + 4H^4H'\right),$$
$$R = 6\left(HH' + 2H^2\right). \tag{10}$$

By introducing a new function x as $x = H^2$, we have

$$H' = \frac{1}{2}x^{-1/2}x', \quad H'' = -\frac{1}{4}x^{-3/2}x'^2 + \frac{1}{2}x^{-1/2}x''. \tag{11}$$

Hence, Eq. (9) takes the form

$$0 = -\frac{3}{\kappa^2}x + 12x(x' + 2x)f_G - f - 24^2x\left[\frac{1}{2}x^2x'' + \frac{1}{2}xx'^2 + 2x^2x'\right]f_{GG} + \rho_m, \tag{12}$$

where we have used the expressions

$$G = 12xx' + 24x^2, \quad \dot{G} = 12x^{-1/2}\left[x^2x'' + xx'^2 + 4x^2x'\right] \quad \text{and} \quad R = 3x' + 12x. \tag{13}$$

Then, by using the above reconstruction method, any cosmological solution can be achieved, by introducing the given Hubble parameter in the FRW equations, which leads to the corresponding Gauss–Bonnet action.

3 Reconstructing ΛCDM model in $R + f(G)$ gravity

We are now interested to reconstruct ΛCDM solution in $R + f(G)$ gravity for different kind of matter contributions. It is well known that such a solution can be achieved in GR by introducing a cosmological constant (cc) in the action. Nevertheless, we show that in GB gravity there is no need of a cc. The cosmological models coming from the different versions of modified GB gravity considered will be carefully investigated with the help of several particular examples where calculations can be carried out explicitly.

In this paper, we restrict ourself to explore some "classical" modified gravity models for the ΛCDM case, as well as other interesting and important cosmological solutions.

For the ΛCDM model, the Hubble rate is given by

$$H^2 = \frac{\Lambda}{3} + \frac{\rho_0}{3a^3}, \tag{14}$$

where ρ_0 is the matter density (which consists of barionic matter and cold dark matter) and Λ is the cosmological constant. In the rest of this section we put $\kappa^2 = 1$.

For the ΛCDM model, described by the Hubble parameter (14), we can write the derivatives of the scale factor as well as the Hubble parameter in the following useful way:

$$\begin{aligned} &\dot{a} = \sqrt{\frac{\Lambda a^2}{3} + \frac{\rho_0}{3a}}, \quad \ddot{a} = \frac{2\Lambda a^3 - \rho_0}{6a^2}, \\ &\dot{H} = -\frac{\rho_0}{2a^3} = \frac{3}{2}\left(\frac{\Lambda}{3} - H^2\right), \quad \ddot{H} = \frac{3\rho_0}{2a^3}\sqrt{\frac{\Lambda}{3} + \frac{\rho_0}{3a^3}} = \frac{9}{2}\left(H^2 - \frac{\Lambda}{3}\right)H. \end{aligned} \tag{15}$$

Using these formulas we get

$$\begin{aligned} &R = 4\Lambda + \frac{\rho_0}{a^3}, \quad G = 24\left(\frac{\rho_0}{3a^3} + \frac{\Lambda}{3}\right)\left(\frac{\Lambda}{3} - \frac{\rho_0}{6a^3}\right), \\ &\dot{R} = \frac{-3\rho_0}{a^3}\sqrt{\frac{\rho_0}{3a^3} + \frac{\Lambda}{3}}, \quad \dot{G} = \frac{4\rho_0}{a^3}\left(\frac{2\rho_0}{a^3} - \Lambda\right)\sqrt{\frac{\rho_0}{3a^3} + \frac{\Lambda}{3}}. \end{aligned} \tag{16}$$

Then, the following relation between R and G holds:

$$G = -\frac{4}{3}\left(R^2 - 9\Lambda R + 18\Lambda^2\right). \tag{17}$$

Let us recall that $x = H^2$. Then, some of the above formulas take the form:

$$\begin{aligned} &\dot{H} = \frac{1}{2}(\Lambda - 3x) \quad \ddot{H} = \frac{3}{2}(3x - \Lambda)\sqrt{x}, \quad R = 3(\Lambda + x), \\ &G = 12x(\Lambda - x), \quad \dot{R} = 3(\Lambda - 3x)\sqrt{x}, \quad \dot{G} = 12(3x - \Lambda)(2x - \Lambda)\sqrt{x}. \end{aligned} \tag{18}$$

Note that the variable x can be expressed in terms of R or G as

$$x = \frac{R}{3} - \Lambda \quad \text{or} \quad x = \frac{3\Lambda \pm \sqrt{9\Lambda^2 - 3G}}{6}, \tag{19}$$

respectively. The above formulas will be useful to reconstruct the ΛCDM model as well as other cosmological solutions in the context of Gauss–Bonnet gravity, as it is shown below.

We write the first Friedmann Eq. (5) in the form

$$0=-3H^2+12H^2(\Lambda-H^2)f_G-f-288H^4\left(3H^2-\Lambda\right)\left(2H^2-\Lambda\right)f_{GG}+\rho_m, \tag{20}$$

or

$$0=-3x+12x(\Lambda-x)f_G-f-288x^2(3x-\Lambda)(2x-\Lambda)f_{GG}+\rho_m. \tag{21}$$

For further algebra more convenient is the following form of this equation

$$\begin{aligned}0=&(\rho_m-3x-f)(\Lambda-2x)+\left[48x^2(3x-\Lambda)+x(\Lambda-x)\right]f_x\\&+24x^2(3x-\Lambda)(\Lambda-2x)f_{xx}.\end{aligned} \tag{22}$$

Now we wish to construct some particular exact solutions of this equation.

3.1 Case I: $\rho_m=0$

First of all, let us consider the simple case in absence of matter, $\rho_m=0$. Then, the Eq. (22) takes the form

$$\begin{aligned}0=&-(3x+f)(\Lambda-2x)+\left[48x^2(3x-\Lambda)+x(\Lambda-x)\right]f_x\\&+24x^2(3x-\Lambda)(\Lambda-2x)f_{xx}.\end{aligned} \tag{23}$$

We can analyze the cases where the cosmological constant term in the solution (14) vanishes and where it is non-zero.

– Let $\Lambda=0$. Then Eq. (23) reads as

$$0=-2(3x+f)-x(144x-1)f_x+144x^3f_{xx}. \tag{24}$$

The general solution of (24) is given by

$$f(x)=C_2x^2+C_1x(144x-1)e^{\frac{1}{144x}}+v_1(x), \tag{25}$$

where

$$v_1(x)=-864x\left[\frac{1}{144x}+x\ln x+\left(\frac{1}{144x}-x\right)Ei\left(1,\frac{1}{144x}\right)e^{\frac{1}{144x}}\right]. \tag{26}$$

Here

$$Ei(a,z)=z^{a-1}\Gamma(1-a,z)=\int_1^\infty e^{-zs}s^{-a}ds. \tag{27}$$

The two integrals of motion are

$$I_1 = \frac{144}{x}\left[x(1-144x)(f_x - v_{1x}) + \left(288x - 2 + \frac{1}{144x}\right)(f - v_1)\right], \quad (28)$$

$$I_2 = \frac{1}{x(144x-1)e^{\frac{1}{144x}}}\left\{f - v_1 - 144\left[x(1-144x)(f_x - v_{1x}) + \left(288x - 2 + \frac{1}{144x}\right)(f - v_1)\right]\right\}. \quad (29)$$

– Let $\Lambda \neq 0$. Then Eq. (23) has a complex solution, which has no physical meaning as it gives a complex action.

Hence, it appears that the ΛCDM model (14) can not be reproduced by $R + f(G)$ gravity in the absence of matter. The only solution found restricts the Hubble rate to give a decelerated Universe.

3.2 Case II: $\rho_m \neq 0$ and $iLambda = 0$

We now explore the case when some kind of matter with a particular EoS is present in the Universe, but with no cosmological constant term in the Hubble parameter described in (14). We explore several examples where different kind of matter contributions are considered.

3.2.1 Example 1

Let us now consider the case when $\Lambda = 0$ and the evolution of the matter density behaves as

$$\rho_m = 3H^2 = 3x. \quad (30)$$

In this case the modified Friedmann Eq. (5) reads as

$$0 = 2f + x(144x-1)f_x - 144x^3 f_{xx}. \quad (31)$$

This equation has two integrals of motion

$$I_1 = \frac{144}{x}\left[x(1-144x)f_x + \left(288x - 2 + \frac{1}{144x}\right)f\right], \quad (32)$$

$$I_2 = \frac{1}{x(144x-1)e^{\frac{1}{144x}}}\left\{f - 144\left[x(1-144x)f_x + \left(288x - 2 + \frac{1}{144x}\right)f\right]\right\}. \quad (33)$$

The general solution of the Eq. (31) is given by

$$f(x) = C_1 x^2 + C_2 x(144x - 1)e^{\frac{1}{144x}}. \tag{34}$$

This function reproduces the solution (14) under the conditions imposed above.

3.2.2 Example 2

Now we consider a more general case, where the energy density is given by,

$$\rho_m = u(x), \tag{35}$$

where $u(x)$ is some function of x. In this case the modified Friedmann Eq. (5) reads as

$$0 = 2[3x - u(x) + f] + x(144x - 1) f_x - 144x^3 f_{xx}. \tag{36}$$

Its general solution is

$$f(x) = C_1 x^2 + C_2 x(144x - 1)e^{\frac{1}{144x}} + v_2(x), \tag{37}$$

with

$$v_2(x) = 288x\left[\left(x - \frac{1}{144}\right)e^{\frac{1}{144x}} J_1 - \frac{x}{144} J_2\right], \tag{38}$$

where

$$J_1 = \int \frac{3x - u}{x^2} e^{\frac{1}{144x}} dx, \quad J_2 = \int \frac{(3x - u)(144x - 1)}{x^3} dx. \tag{39}$$

Note that the two integrals of motion are

$$I_1 = \frac{144}{x}\left[x(1 - 144x)(f_x - v_{2x}) + \left(288x - 2 + \frac{1}{144x}\right)(f - v_2)\right], \tag{40}$$

$$I_2 = \frac{1}{x(144x - 1)e^{\frac{1}{144x}}}\left\{f - v_2 - 144\left[x(1 - 144x)(f_x - v_{2x}) + \left(288x - 2 + \frac{1}{144x}\right)(f - v_2)\right]\right\}. \tag{41}$$

In fact, we can directly verify that

$$I_{1x} = I_{2x} = 0. \tag{42}$$

Then, the solution (37) gives the function of the Gauss–Bonnet invariant that reproduces this model for any kind of EoS matter fluid.

3.3 Case III: $\rho_m \neq 0$ and $\Lambda \neq 0$

Let us now explore the most general case for the solution (14) in $R + f(G)$ gravity with a non vanishing matter fluid with a given EoS parameter.

3.3.1 Example 1

We consider

$$\rho_m = 3x + \beta. \tag{43}$$

Then Eq. (5) takes the form

$$0 = (\beta - f)(\Lambda - 2x) + \left[48x^2(3x - \Lambda) + x(\Lambda - x)\right] f_x + 24x^2(3x - \Lambda)(\Lambda - 2x) f_{xx} \tag{44}$$

and has the following particular solution:

$$f(x) = \gamma x^2 - \gamma \Lambda x + \beta. \tag{45}$$

If $\Lambda = 0$, then $f(x) = \gamma x^2 + \beta$. Also if $\gamma = 0$, then the solution takes the form $f = \beta$, which corresponds to the cosmological constant. Note that if $\beta = -\Lambda$ then

$$\rho_m = \frac{\rho_{03}}{a^3} = 3x - \Lambda, \quad f(x) = \gamma x^2 - \gamma \Lambda x - \Lambda. \tag{46}$$

This gives a solution where the cosmological constant is corrected by the contribution from $f(G)$, what may resolve the cosmological constant problem.

3.3.2 Example 2

Now we consider the density of the energy which is given by

$$\rho_m = kx^2 + 3x + \beta. \tag{47}$$

The corresponding first Friedmann equation reads as

$$0 = \left(kx^2 + \beta - f\right)(\Lambda - 2x) + \left[48x^2(3x - \Lambda) + x(\Lambda - x)\right] f_x + 24x^2(3x - \Lambda)(\Lambda - 2x) f_{xx}. \tag{48}$$

This equation has the following particular solution:

$$f(x) = \gamma x^2 + \left(\frac{k}{72} - \gamma \Lambda\right) x + \beta, \tag{49}$$

where we must have $\Lambda = -\frac{1}{24}$. As well as in the above example, the given function $f(G)$ produces a relaxation on the cosmological constant, which can be seen as the possible resolution of the cc problem.

4 Cosmology in $R + f(G)$ gravity with the presence of an inhomogeneous fluid

Let us now consider the theory described by the action (1) in the presence of a perfect fluid, whose EoS is given by the general expression:

$$p = w(a)\rho + \eta(a), \tag{50}$$

where $w(a)$ and $\eta(a)$ are arbitrary functions of the scale factor. This kind of EoS could correspond to a dynamical viscous fluid or possibly the effective EoS that accounts for the extra terms in the gravitational action, as curvature terms or scalar and vector fields (see Ref. [15]). Then, the FRW Eq. (5) are written now as

$$3H^2 = \rho + \rho_{f(G)}, \quad 2\dot{H} + 3H^2 = -(p + p_{f(G)}). \tag{51}$$

Here the energy and pressure densities $\rho_{f(G)}$ and $p_{f(G)}$ are properly defined to account for the extra terms introduced by the $f(G)$ function in the action (1), and are given by

$$\begin{aligned} \rho_{f(G)} &= Gf_G - f - 24\dot{G}H^3 f_{GG}, \\ p_{f(G)} &= 8H^2 \ddot{f}_G + 16H(\dot{H} + H^2)\dot{f}_G + f - Gf_G. \end{aligned} \tag{52}$$

Then, by combining the FRW Eq. (51) and using the EoS (50), we could write the inhomogeneous term $\eta(a)$ as a function of the Hubble parameter:

$$\eta(a) = -w(a)\rho_{f(G)} + p_{f(G)} + 2\dot{H} + 3H^2(1 + w(a)). \tag{53}$$

Hence, by specifying a cosmic solution $H(a)$, any cosmological history can be reconstructed in this framework, where the contributions from $f(G)$ and the inhomogeneous fluid described by (50) drives the Universe evolution. To show this, let us consider the example

$$3H^2 = H_0^2 + H_1 a^m, \tag{54}$$

where H_0, H_1 and m are constants. This solution reproduces a Universe dominated by an effective cosmological constant at the current time and which enters a phantom epoch in the future (as some observations suggest). We could consider $w(a) = 0$ and the $f(G)$ function calculated in (25), which reproduces an effective cosmological constant. Then, the effective fluid (50) describes dust-matter at the beginning, when the cosmological constant dominates and then it drives the Universe into a phantom epoch, which probably finishes at the so-called Big Rip singularity.

5 Cosmological solutions in pure $f(G)$ gravity

We have studied so far a theory described by the action (1), which is given by the usual Hilbert-Einstein term plus a function of the Gauss–Bonnet invariant, that is assumed to become important in the dark energy epoch. In this section, we are interested to investigate some important cosmic solutions in the frame of a theory described only by the Gauss–Bonnet invariant, and whose action is given by

$$S = \int d^4x\sqrt{-g}\,[f(G) + L_m]\,. \tag{55}$$

In this case, the FRW equations are:

$$\begin{aligned} 0 &= Gf_G - f - 24\dot{G}H^3 f_{GG} + \rho_m \\ 0 &= 8H^2\ddot{f}_G + 16H\left(\dot{H} + H^2\right)\dot{f}_G + f - Gf_G + p_m. \end{aligned} \tag{56}$$

We are interested to explore some important solutions from the cosmological point of view, as de Sitter and power law expansions.

5.1 De Sitter solutions

De Sitter solutions are described by an exponential expansion of the Universe, where the Hubble parameter and the scale factor are given by

$$H(t) = H_0 \rightarrow a(t) = e^{H_0 t}, \tag{57}$$

where H_0 is a constant. This kind of solutions are very important, as the observations suggest that the expansion of our Universe behaves approximately as de Sitter. It has been shown in Ref. [14] that de Sitter points are critical points in $f(R)$ gravity. It is straightforward to see that this is also the case in $f(G)$ gravity. We can explore the de Sitter points admitted by a general $f(G)$ by introducing the solution (57) in the first FRW equations given in (56), which yields

$$0 = G_0 f_G(G_0) - f(G_0). \tag{58}$$

Here, $G_0 = 24H_0^2$ and we have ignored the contribution of matter. Then, we have reduced the differential equation (56) to an algebraic equation that can be resolved by specifying a function $f(G)$. The de Sitter points are given by the positive roots of this equation, which could explain not just the late-time accelerated epoch but also the inflationary epoch. The stability of these solutions has to be studied in order to achieve a grateful exit in the case of inflation, and future predictions for the current cosmic acceleration.

5.2 Power law solutions

We are now interested to explore power law solutions for a theory described by the action (55). This kind of solutions are very important during the cosmic history as the matter/radiation epochs are described by power law expansions, as well as the possible phantom epoch, which can be seen as a special type of these solutions. Let us start by studying a Hubble parameter given by

$$H(t) = \frac{\alpha}{t} \rightarrow a(t) \sim t^{\alpha}, \tag{59}$$

where we take $\alpha > 1$. Then, by introducing the solution (59) into the first FRW Eq. (56), it yields the differential equation

$$0 = -f(G) + Gf_G + \frac{4G^2}{\alpha - 1} f_{GG}, \tag{60}$$

where we have neglected any contribution of matter for simplicity. The Eq. (60) is a type of Euler equation, whose solution is

$$f(G) = C_1 G + C_2 G^{\frac{1-\alpha}{4}}. \tag{61}$$

Thus, we have shown that power-law solutions of the type (59) correspond to actions with powers on the Gauss–Bonnet invariant, in a similar way as in $f(R)$ gravity, where power-law solutions correspond to an action with powers on the scalar curvature, R (see Ref. [11]).

Let us now explore another kind of power-law solutions, where the Universe enters a phantom phase and ends in a Big Rip singularity. This general class of Hubble parameters may be written as

$$H(t) = \frac{\alpha}{t_s - t}, \tag{62}$$

where t_s is the so-called Rip time, i.e. the time when the future singularity will take place. By inserting the solution (62) into the first FRW Eq. (56), the equation yields

$$0 = -f(G) + Gf_G(G) - \frac{4\alpha^2 G^2}{1 + \alpha}, \tag{63}$$

which is also a Euler equation, whose solution is given by,

$$f(G) = C_1 G + C_2 G^{\frac{1+\alpha}{4\alpha^2}}. \tag{64}$$

Thus, we have showed that power law solution of the type radiation/matter dominated epochs on one side and phantom epochs on the other, are well reproduced in pure $f(G)$ gravity, in a similar way as it in $f(R)$ gravity.

6 Conclusions

We have explored in this paper several cosmological solutions in the frame of Gauss–Bonnet gravity, considering specially the case of an action composed of the Hilbert-Einstein action plus a function on the Gauss–Bonnet invariant. Also pure $f(G)$ gravity has been considered, as well as the possibility of the implication of inhomogeneous terms in the EoS of a perfect fluid, which could contribute together with modified gravity to the late-time acceleration. We have shown that the ΛCDM model can well be explained in this kind of theories, which may give an explanation to the cosmological constant problem as the modified gravity terms may act relaxing the vacuum energy density. Other kinds of solutions in $f(G)$ gravity have been reconstructed. It has been shown that $f(G)$ gravity could explain the dark energy epoch whatever the nature of its EoS, of type quintessence or phantom, and even the inflationary phase. More complex cosmological solutions would require numerical analysis, but our analysis of a few simple cases has already shown that $f(G)$ gravity accounts for the accelerated epochs and may contribute during the radiation/matter dominated eras, and it may explain also the dark matter contributions to the cosmological evolution, what will be explored in future works. This kind of modified gravity models which reproduces dark energy and inflation, can be modeled as an inhomogeneous fluid with a dynamical equation of state, what would be distinguished from other models with a static EoS. Even as perturbations in modified gravity behave different than in General Relativity, it could give a signature of the presence of higher order terms in the gravity action, as the Gauss–Bonnet invariant, when structure formation is studied and simulations are performed, what should be explored in the future.

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