

Stochastic Einstein equations

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Stochastic Einstein equations are considered when 3D space metric γ_{ij} are stochastic functions. The probability density for the stochastic quantities is connected with the Perelman's entropy functional. As an example, the Friedman Universe is considered. It is shown that for the Friedman Universe the dynamical evolution is not changed. The connection between general relativity and Ricci flows is discussed.

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I. INTRODUCTION

In quantum gravity dynamical variables are 3D space metric γ_{ij} which should be quantized, i.e. they have to be fluctuating quantities. Up to now we do not have any quantum theory of gravity. Here we offer a stochastic approach to quantum gravity where the dynamical variables are stochastic quantities and they fluctuate according to some probability density. For the calculation of this density we use the idea proposed in Ref. [1] that Ricci flow (for 3D space metric) is a statistical system and every metric in the Ricci flow is a microscopical state.

There have been a number of investigations which aim to apply Ricci flow techniques in general relativity. In Ref. [2] the evolution of wormhole geometries under Ricci flow is studied: depending on value of initial data parameters, wormhole throats either pinch off or evolve to a monotonically growing state. In Ref. [3] Ricci flows are applied to general relativity, where gradient flow is Ricci flow, and the investigation is focused on the example of 4-dimensional Euclidean gravity with boundary $S^1 \times S^2$, representing the canonical ensemble for gravity in a box. In Ref. [4] Ricci flow techniques is applied to general relativity: the evolution of the area \mathcal{A} and Hawking mass \mathcal{M}_H under the Ricci flow is studied. In Ref. [5] Ricci flow deformation of cosmological initial data sets in general relativity is a technique for generating families of initial data sets which potentially would allow to interpolate between distinct spacetimes. The linear stability of Schwarzschild - Tangherlini spacetimes and their Anti-de Sitter counterparts under Ricci flow for a special class of perturbations in [6] is investigated.

The main goal of this investigation is to show that there is some connection between 3D Ricci flow (for 3D space metric) and Einstein equations (for 4D metric). The connection is based on the idea that the 3D space metric in Einstein equations is a stochastic quantity with the probability density defined by the Ricci flow.

II. STOCHASTIC EINSTEIN EQUATIONS

According to Eq's (A5) - (A7) we will write the stochastic Einstein equations in the form

$$\tilde{R} - K_{ij}K^{ij} + K^2 = 2\rho, \quad (1)$$

$$D_j K^j_i - D_i K = S_i, \quad (2)$$

$$\frac{d}{dt} K_{ij} = -D_i D_j \alpha + \alpha(\tilde{R}_{ij} - 2K_{il}K^l_j + K K_{ij} - M_{ij}) \quad (3)$$

here \tilde{R}_{ij} and \tilde{R} are *the fluctuating 3D Ricci tensor and scalar curvature*. Eg's (1) - (3) are stochastic equations (for details about stochastic equations, see appendix B) since quantities \tilde{R}_{ij} , \tilde{R} are made from a stochastic 3D space metric $\tilde{\gamma}_{ij}$. The probability density ρ for the random functions $\tilde{\gamma}_{ij}$ is the functional $F(\lambda)$ which is connected with the Perelman's entropy functional $\mathcal{W}(\lambda)$ as $F(\lambda) = \frac{d\mathcal{W}(\lambda)}{d\lambda}$ (see, Appendix C for the details).

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These equations describe the time evolution of the stochastic 3D space metric $\tilde{\gamma}_{ij}$. The next values of $\tilde{\gamma}_{ij}(t_0 + \Delta t)$ are defined through Eq's (1) - (3) taking into account the “noise” created by the stochastic terms $\tilde{R}_{ij}, \tilde{R}$.

The solution procedure for Eq's (1) - (3) is following:

1. the initial value of the 3D space metric $\tilde{\gamma}_{ij}$ is chosen;
2. the Ricci flow equations (C1) (C2) are solved in order to obtain the probability density for the “noise” $\tilde{\gamma}_{ij}$;
3. the equation (C4) is used for the definition of the probability density for the “noise” $\tilde{\gamma}_{ij}, \tilde{R}_{ij}$ and \tilde{R} ;
4. the probability density (C4) is normalized;
5. the probability density is used for obtaining of current values of the stochastic quantities $\tilde{\gamma}_{ij}, \tilde{R}_{ij}$ and \tilde{R} that are inserted into Eq's (1) - (3);
6. Eq's (1) - (3) are used to define the metric $\tilde{\gamma}_{ij}(t_0 + \Delta t)$;
7. the steps 1 - 6 are being repeated to obtain $\tilde{\gamma}_{ij}(t_0 + n\Delta t)$.

III. STOCHASTIC FRIEDMAN UNIVERSE

Now we would like to apply Eq's (1) - (3) to describe the Friedman Universe where the 3D space metric is a stochastic quantity. The Einstein equations for the Friedman Universe with metric

$$ds^2 = dt^2 - a^2(t) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)] \quad (4)$$

are

$$3\frac{\dot{a}^2}{a^2} + \frac{3}{a^2} = \varkappa\varepsilon, \quad (5)$$

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{1}{a^2} = 0 \quad (6)$$

here, for simplicity, we consider dust matter with zero pressure $p = 0$. We use equation (5) for the determination of the energy density ε , and equation (6) is the stochastic equation for the determination of the scale factor $a(t)$.

First we will seek for the Ricci flow equations (C1) (C2). The 3D space metric of the Friedman Universe is

$$dl^2 = a^2(t) [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (7)$$

In this case the Ricci flow equations (C1) (C2) are

$$\frac{da^2}{d\lambda} = -4, \quad (8)$$

$$\frac{df}{d\lambda} = -\frac{6}{a^2} + \frac{3}{2} \frac{1}{\lambda_0 - \lambda} \quad (9)$$

where λ is the parameter under which the Ricci flow is evolved. The solution of the system (8) (9) is

$$a = 2\sqrt{(\lambda_0 - \lambda)}, \quad (10)$$

$$f = f_0 = \text{const}. \quad (11)$$

Then the functional (C4) is equal to zero

$$F(\lambda) = 0. \quad (12)$$

We choose the probability density for the fluctuating 3D space metric $\tilde{\gamma}_{ij}$ in the stochastic Einstein equations as the functional (C4). This means that for every moment t the probability density for the “noise” $\tilde{g}_{ij}, \tilde{R}_{ij}, \tilde{R}$ is the functional (C4). Thus, for solving stochastic Einstein equations (6) we have to use a randomly function \tilde{g}_{ij} with the probability density ρ equal to the functional $\rho(\lambda) = F(\lambda) = \frac{W(\lambda)}{a\lambda}$. In order to have the full probability equal to unity we choose

$$\int \rho dV = 1 \quad (13)$$

and complement the function $\rho(\lambda)$ by delta - function at $\lambda = 0$

$$\rho(\lambda) = \delta(\lambda). \quad (14)$$

Then

$$\rho(\lambda) = \begin{cases} \mathcal{W}(\lambda), & \text{if } \lambda \neq 0 \text{ that is consistent with equation (12)} \\ \delta(\lambda), & \text{if } \lambda = 0 \end{cases} \quad (15)$$

The probability distribution (15) means that at any moment t the stochastic function $\tilde{a}(t)$ is realized with the probability equal to unity, i.e. the Friedman equations (5) - (6) do not change. The 3D metric $a(t)$ is the determinate function. One can say that in this sense 4D Einstein equations and 3D Ricci flows are connected in the manner discussed above.

IV. DISCUSSION AND CONCLUSIONS

Here we have offered the idea that the Ricci flows can be applied in general relativity. For every moment the Ricci flow shows a probability for the realization of 3D space metric in the Einstein equations. In such approach the Einstein equations become stochastic equations where dynamical variables (3D space metric) are stochastic functions. The probability density of these stochastic variables is connected with some functional of the Ricci flow of the 3D space metric. In this sense the 4D Einstein equations can be connected with the 3D Ricci flows.

Naturally the question arises: whether such an approach has the connection with quantum gravity? At the moment we cannot give the answer on this question but there are some arguments that such connection may exist. It is well known that in quantum theory the path integral can be approximately calculated if a classical solution does exist (saddle-point method). One can assume that in some way the calculation of a path integral can be connected with Ricci flow.

Here we have shown that in a special case of the Friedman Universe the dynamics of the scale factor have not changed. It is interesting to do similar calculations for another solutions of the Einstein equations: black holes, wormholes, cosmological solutions and so on.

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Appendix A: 3+1 - decomposition

In this section we follow to Ref. [7]. We write the metric in the form

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt), \quad (A1)$$

where α is the lapse function, β^i is the shift vector, and γ_{ij} is 3D spatial metric; Latin indexes are spatial indexes $i, j = 1, 2, 3$; Greek indexes are spacetime indexes $\mu, \nu = 0, 1, 2, 3$. The extrinsic curvature K_{ij} can be defined by the equation

$$\frac{d}{dt}\gamma_{ij} = -2\alpha K_{ij}, \quad (A2)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} - \mathcal{L}_\beta \quad (A3)$$

where \mathcal{L}_β denotes the Lie derivative with respect to β^i .

Einstein equations

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = T_{\mu\nu} \quad (A4)$$

can then be splitted into the Hamiltonian constraint

$$R - K_{ij}K^{ij} + K^2 = 2\rho, \quad (\text{A5})$$

the momentum constraint

$$D_j K_i^j - D_i K = S_i, \quad (\text{A6})$$

and the evolution equation for the extrinsic curvature

$$\frac{d}{dt} K_{ij} = -D_i D_j \alpha + \alpha (R_{ij} - 2K_{il}K_j^l + K K_{ij} - M_{ij}) \quad (\text{A7})$$

here D_i is the covariant derivative associated with γ_{ij} ; $\mathcal{R}_{\mu\nu}$ is the 4D Ricci tensor; \mathcal{R} is the 4D scalar curvature; R_{ij} is the three-dimensional Ricci tensor

$$R_{ij} = \frac{1}{2} \gamma^{kl} \left(\gamma_{kj,il} + \gamma_{il,kj} - \gamma_{kl,ij} - \gamma_{ij,kl} \right) + \gamma^{kl} \left(\Gamma_{il}^m \Gamma_{mkj} - \Gamma_{ij}^m \Gamma_{mkl} \right), \quad (\text{A8})$$

and R is its trace of $R = \gamma^{ij} R_{ij}$. We have also introduced the matter sources ρ , S_i and S_{ij} , which are projections of the stress-energy tensor with respect to the unit normal vector n_α ,

$$\begin{aligned} \rho &= n_\alpha n_\beta T^{\alpha\beta}, \\ S_i &= -\gamma_{i\alpha} n_\beta T^{\alpha\beta}, \\ S_{ij} &= \gamma_{i\alpha} \gamma_{j\beta} T^{\alpha\beta}, \end{aligned} \quad (\text{A9})$$

and have abbreviated

$$M_{ij} \equiv S_{ij} + \frac{1}{2} \gamma_{ij} (\rho - S), \quad (\text{A10})$$

where S is the trace of S_{ij} , $S = \gamma^{ij} S_{ij}$.

The evolution equations (A2) and (A7) together with the constraint equations (A5) and (A6) are equivalent to the Einstein equations, and are commonly referred to as the ADM form of the gravitational field equations [8].

Appendix B: Stochastic equations

The definition of a *stochastic differential equation* is as follows (for details, see Ref. [9])

$$d\mathbf{X}(t) = \mathbf{b}(\mathbf{X}(t)) dt + \mathbf{B}(\mathbf{X}(t)) d\mathbf{W}(t), \quad (\text{B1})$$

$$\mathbf{X}(0) = x_0. \quad (\text{B2})$$

where \mathbf{b} is a given vector field; \mathbf{X} is a solution which is random field and \mathbf{W} is a stochastic "noise".

Appendix C: Ricci flows

Ricci flow on a 3D space with positive metric is defined as follows (for details, see Ref. [10])

$$\frac{\partial \gamma_{ij}}{\partial \lambda} = -2R_{ij}, \quad (\text{C1})$$

$$\frac{\partial f}{\partial \lambda} = -\Delta f + |\nabla f|^2 - R + \frac{3}{2} \frac{1}{(\lambda_0 - \lambda)} \quad (\text{C2})$$

where λ is the parameter describing the evolution of the metric γ_{ij} under Ricci flow and λ_0 is the value of the parameter λ where γ_{ij} becomes singular. For the Ricci flow one can introduce the following functionals

$$\mathcal{W}(\lambda) = \int \left[(\lambda_0 - \lambda) \left(R + |\nabla f|^2 \right) + f - 3 \right] u dV, \quad (\text{C3})$$

$$F(\lambda) = \frac{d\mathcal{W}}{d\lambda} = 2(\lambda_0 - \lambda) \int \left[R_{ij} + \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\gamma_{ij}}{2(\lambda_0 - \lambda)} \right] \left[R^{ij} + \frac{\partial^2 f}{\partial x_i \partial x_j} - \frac{\gamma^{ij}}{2(\lambda_0 - \lambda)} \right] u dV \geq 0, \quad (\text{C4})$$

$$u = \frac{1}{(4\pi)^{3/2}} \frac{1}{(\lambda_0 - \lambda)^{3/2}} e^{-f} \quad (\text{C5})$$

where $\mathcal{W}(\lambda)$ is the Perelman's entropy functional.

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