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SHORT COMMUNICATIONS

On the Sharp Order of Informativeness of All Possible Linear Functionals in the Discretization of Solutions of the Wave Equation

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Abstract—We find sharp lower bounds for the accuracy in the approximation to solutions of the wave equation by computational aggregates constructed on the basis of numerical information obtained from all linear functionals (whose total number is given) applied to two initial conditions in Sobolev classes. We compute the sharp order of approximation by all linear aggregates of numerical analysis and of the theory of approximations.

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1. STATEMENT OF THE PROBLEM AND MAIN RESULTS

In the present paper, we consider the Cauchy problem for the wave equation ($s = 1, 2, \ldots$)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_s^2} \qquad (u = u(x,t), \quad 0 \le t < \infty, \quad x \in R^s), \tag{1}$$

$$u(x,0) = f_1(x) \in F^{(1)}, \qquad \frac{\partial u}{\partial t}(x,0) = f_2(x) \in F^{(2)} \qquad (x \in R^s). \tag{2}$$

The problem is to find coinciding (up to constants) upper and lower bounds for the quantity

$$\delta_N(L_N \times \{\varphi_N\}; F^{(1)}, F^{(2)})_Y = \min_{N_1+N_2=N} \inf_{(l^{(N_1,N_2)},\varphi_N)\in L_N\times\{\varphi_N\}} \sup_{f_1\in F^{(1)},\, f_2\in F^{(2)}} \|u(x,t;f_1,f_2) - \varphi_N(l_1^{(1)}(f_1),\ldots,l_1^{(N_1)}(f_1),l_2^{(1)}(f_2),\ldots,l_2^{(N_2)}(f_2);x,t)\|_Y \tag{3}$$

and indicate a computational aggregate, that is, a function of the form

$$\Phi(x,t;f_1,f_2) \equiv \varphi_N(l_1^{(1)}(f_1),\ldots,l_1^{(N_1)}(f_1),l_2^{(1)}(f_2),\ldots,l_2^{(N_2)}(f_2);x,t)$$

that realizes the upper bound. Here $u(x,t;f_1,f_2)$ is the solution of Eq. (1), (2), and L_N is the set of all possible linear functionals $l_1^{(1)}(f_1),\ldots,l_1^{(N_1)}(f_1)$ and $l_2^{(1)}(f_2),\ldots,l_2^{(N_2)}(f_2)$ defined on the linear spans of the classes $F^{(1)}$ and $F^{(2)}$, respectively.

In what follows, we assume that $\varphi_N(\tau_1,\ldots,\tau_N)$ with arbitrary, in general, complex τ_j ($j = 1,\ldots,s$) is a function of the variables (x,t) in the class

$$L^{q,\infty} \equiv L^{q,\infty}((0,1)^s \times [0,+\infty)),$$

which is defined as the set of all functions $g: \ R^s \times [0,+\infty) \to C$ such that, for each $t \in [0,+\infty)$, $g_t(x) = g(x,t)$ treated as a function of the argument $x \in R^s$ is a measurable function 1-periodic with respect to each of s variables and satisfies the inequality

$$\|g\|_{L^{q,\infty}} \equiv \|g\|_{L^{q,\infty}((0,1)^s\times[0,+\infty))} = \operatorname*{vrai\,sup}_{t\geq 0}\left(\int\limits_{(0,1)^s} |g(x,t)|^q\,dx\right)^{1/q} < +\infty,$$

$$Y = L^{q,\infty} = L^{q,\infty}((0,1)^s \times [0,+\infty)).$$

According to [11], the quantity δ_N, which depends only on L_N, will be called the informativeness of all possible linear functionals L_N.

It is important in problem (1)–(3) that, for a given total amount N of information, one should allocate the amounts N_j of numerical information obtained from $f_j \in F^{(j)}$ $(j = 1, 2)$ so as to ensure that the total error δ_N be as small as possible, i.e., sharp or nearly sharp.

Recall the definition of function classes taken for $F^{(1)}$ and $F^{(2)}$ in this paper.

For s $(s = 1, 2, \ldots)$, a nonnegative integer r, and $1 \leq q \leq \infty$, the Sobolev class $W_q^r(0,1)^s$ is the set of all functions $f(x) = f(x_1, \ldots, x_s)$ that are 1-periodic in each variable and, for $r > 0$ together with their partial derivatives of order $\leq r$, belong to $L^q(0,1)^s$ and satisfy the inequality

$$\|f\|_{W_q^r} \equiv \|f\|_{L^q} + \sum_{j=1}^{s}\left\|\frac{\partial^r}{\partial x_j^r} f\right\|_{L^q} \leq 1.$$

By $c(\alpha, \beta, \ldots)$ we denote some positive numbers, which can be different in different formulas and depend only on the parameters listed in parentheses.

If $\{A_N\}_{N=1}^{\infty}$ is a nonnegative numerical sequence and $\{B_N\}_{N=1}^{\infty}$ is an arbitrary numerical sequence, then we write $B_N \underset{\alpha,\beta,\ldots}{\ll} A_N$ to indicate that there exists a number $c(\alpha, \beta, \ldots)$ such that the inequality $|B_N| \leq c(\alpha, \beta, \ldots)A_N$ holds for each positive integer N. If $\{A_N\}_{N=1}^{\infty}$ and $\{B_N\}_{N=1}^{\infty}$ are two nonnegative numerical sequences, then we write $B_N \underset{\alpha,\beta,\ldots}{\asymp} A_N$ to indicate that $B_N \underset{\alpha,\beta,\ldots}{\ll} A_N$ and $B_N \underset{\alpha,\beta,\ldots}{\gg} A_N$ simultaneously.

Note that the general statement of this problem can be found in [1]; various specific choices of the spaces, classes, operators, and computational aggregates given there lead to various statements of problems, many of which have been studied (e.g., see [1–10] and the bibliography therein).

Now let us proceed to the exposition of main results of the paper.

We denote the solution of problem (1), (2) for the case in which

$$f(x) \equiv f_1(x) \in W_2^{r_1}(0,1)^s, \qquad f_2(x) \equiv 0 \tag{4}$$

by $u(x, t, f, 0)$, for the case in which

$$f_1(x) \equiv 0, \qquad f(x) \equiv f_2(x) \in W_2^{r_2}(0,1)^s \tag{5}$$

by $u(x, t, 0, f)$, and for the case in which

$$f_1(x) \in W_2^{r_1}(0,1)^s, \qquad f_2(x) \in W_2^{r_2}(0,1)^s \tag{6}$$

by $u(x, t, f_1, f_2)$.

Let $l_1, \ldots, l_N$ be linear functionals.

We prove the following assertions.

Theorem 1. *Let s and r_1 be given positive integers, let $2 \leq q \leq \infty$, and let $r_1 > 2 + s/2$. Then the relations*

$$\inf_{\varphi_N}\ \sup_{f\in W_2^{r_1}(0,1)^s} \|u(\cdot; f, 0) - \varphi_N(l_1(f), \ldots, l_N(f); \cdot)\|_{L^{q,\infty}((0,1)^s\times[0,+\infty))} \underset{s,r_1}{\gg} N^{-r_1/s+1/2-1/q}$$

$(N = 1, 2, \ldots)$ hold for problem (1), (4).

Theorem 2. *Let s and r_2 be given positive integers, let $2 \le q \le \infty$, and let $r_2 > 1 + s/2$. Then the relations*

$$\inf_{\varphi_N} \sup_{f \in W_2^{r_2}(0,1)^s} \|u(\cdot;0,f) - \varphi_N(l_1(f),\dots,l_N(f);\cdot)\|_{L^{q,\infty}((0,1)^s\times[0,+\infty))} \underset{s,q,r_2}{\gg} N^{-(r_2+1)/s+1/2-1/q}$$

$(N = 1,2,\dots)$ *hold for problem* (1), (5).

Theorem 3. *Let s, r_1, and r_2 be given positive integers, let $2 \le q \le \infty$, and suppose that $r_1 > 2 + s/2$ and $r_2 > 1 + s/2$.*

Then the following assertions hold.

1. *For problem* (1), (4), *one has the relations* $(N = 1,2,\dots)$

$$\sup_{f \in W_2^{r_1}(0,1)^s} \|u(\cdot;f,0) - \tilde{\varphi}_N^{(1)}(l_1(f),\dots,l_N(f);\cdot)\|_{L^{q,\infty}((0,1)^s\times[0,+\infty))} \underset{s,r_1}{\ll} N^{-r_1/s+1/2-1/q},$$

where

$$\tilde{\varphi}_N^{(1)}(l_1(f),\dots,l_N(f);x,t) = \sum_{\substack{k=(k_1,\dots,k_s):\\ |k_j|\le 2^n\ (j=1,\dots,s)}} \hat{f}(k_1,\dots,k_s)\cos\left(\sqrt{k_1^2+\dots+k_s^2}\,t\right)e^{2\pi i(x_1k_1+\dots+x_sk_s)},$$

$$n = [\log_2(\sqrt[s]{N} - 1)] - 1.$$

2. *For problem* (1), (5), *one has the relations*

$$\sup_{f \in W_2^{r_2}(0,1)^s} \|u(\cdot;0,f) - \tilde{\varphi}_N^{(2)}(l_1(f),\dots,l_N(f);\cdot)\|_{L^{q,\infty}((0,1)^s\times[0,+\infty))} \underset{s,r_2}{\ll} N^{-(r_2+1)/s+1/2-1/q},$$

where

$$\tilde{\varphi}_N^{(2)}(l_1(f),\dots,l_N(f);x,t) = \sum_{\substack{k=(k_1,\dots,k_s)\in Z^s:\\ |k_j|\le 2^n (j=1,\dots,s)}} \hat{f}(k_1,\dots,k_s)\frac{\sin\left(\sqrt{k_1^2+\dots+k_s^2}\,t\right)}{\sqrt{k_1^2+\dots+k_s^2}}\,e^{2\pi i(x_1k_1+\dots+x_sk_s)},$$

$$n = [\log_2(\sqrt[s]{N} - 1)] - 1.$$

Here and throughout the following, $[\cdot]$ is the integer part of a number.

Theorem 4 (the informativeness of all possible linear functionals). *Let s and $2 \le q \le \infty$ be given positive integers, and let r_1 and r_2 be positive numbers such that $r_1 > 2+s/2$ and $r_2 > 1+s/2$. Then for problem* (1), (6), *one has the relation* $(N = 2,3,\dots)$

$$\min_{\substack{N_1+N_2=N\\ N_1\ge1,\ N_2\ge1}} \inf_{\substack{l_1^{(1)},\dots,l_N^{(1)}\\ l_1^{(2)},\dots,l_N^{(2)},\varphi_N}} \sup_{\substack{f_1\in W_2^{r_1}(0,1)^s\\ f_2\in W_2^{r_2}(0,1)^s}} \|u(\cdot;f_1,f_2) - \varphi_N(l_1^{(1)}(f_1),\dots,l_1^{(N_1)}(f_1),l_2^{(1)}(f_2),\dots,l_2^{(N_2)}(f_2);\cdot)\|_{L^{q,\infty}((0,1)^s\times[0,+\infty))} \underset{s,r_1,r_2}{\asymp} N^{-\min\{r_1;r_2+1\}/s+1/2-1/q}.$$

Remark 1. Let $\min\{r_1, r_2+1\} = r$. If $r_1 = r$, then $r_2 + 1 \ge r$, and any increase in the smoothness $r_2 > r-1$ does not affect the error order $\asymp N^{-r/s+1/2-1/q}$ for the discretization of solutions of problem (1)–(6); the same is true for the case in which $r_1 \ge r$ and $r_2 = r-1$. In addition, the widest class of ordered pairs of functions (f_1, f_2), $f_1 \in W_2^{r_1}(0,1)^s$, $f_2 \in W_2^{r_2}(0,1)^s$, for which the above-mentioned optimal order holds is obtained for $r_1 = r_2 + 1 = r$.

Remark 2. One can readily see (see also [9, 10]) that, by specifying linear functionals $l_1,\dots,l_N$ and functions φ_N, as computational aggregates $\varphi_N(l_1,\dots,l_N;x,t)$ in Theorem 4, one can obtain all possible N-term partial sums of Fourier series in all possible orthonormal systems, including wavelet systems, all possible N-term partial sums of expansions in all possible bases, and also difference schemes; that is, one obtains an almost complete spectrum of aggregates considered in the theory of approximations and numerical mathematics. Theorem 4 implies that, under the corresponding assumptions, the entire above-listed set cannot provide an estimate better than $\asymp N^{-\min\{r_1;r_2+1\}/s+1/2-1/q}$; and this estimate is realized on the operator in Theorem 3.

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