Characteristic Equations of Strip-slotted Structures

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Abstract— This paper deals with further development of Wiener-Hopf-Fock method for obtaining the characteristic equation which follows from the solution of the boundary value problem of electromagnetic waves diffraction on the strip-slotted structures. The strip or an infinite grating is chosen in return for strip-slotted structures. The boundary value problem is consecutively solved by reducing to the system of singular boundary integral equations, then to the system of the second kind Fredholm equations, which effectively is solved by reducing to a system of the linear algebraic equations with the help of the etalon integral and of saddle point method. Setting the determinant of a system of the algebraic equations equal to zero, we find a characteristic equation, which determines the eigenfrequencies of the structures.

1. INTRODUCTION

To present time, basically only two rigorous analytical methods for solving diffraction problems are known: the Wiener-Hopf-Fock method (WHF) [1–3] and the method of Riemann-Hilbert [4, 5]. The WHF method is also known as the factorization method [6]. A canonical problem for plane finite structures is the diffraction of electromagnetic waves on a strip or slot. Many works are devoted to an asymptotic solution of diffraction problems on a strip (slot) [8–17]. In [7] the diffraction problem for a strip is considered by the WHF method and reduced to a system of integral equations. Unlike the results of D. S. Jones [8], in this paper the solution of this problem is reduced to a series having an asymptotic form that contains a resonant denominator [9]. Therefore, it is suggested to consider the characteristic equations by the WHF method.

2. REDUCING TO A SYSTEM OF THE INTEGRAL EQUATIONS

Let the plane wave impinges on ideally conducting strip $|z| \leq a$, $y = 0$, $-\infty < x < \infty$:

$$E_x^o = -E_0 e^{ik(y\sin\vartheta_0 + z\cos\vartheta_0)}, \quad H_y^o = E_x^o\sqrt{\varepsilon/\mu}\cos\vartheta_0, \quad H_z^o = -E_x^o\sqrt{\varepsilon/\mu}\sin\vartheta_0, \tag{1}$$
$$H_x^o = 0, \quad E_y^o = E_z^o = 0, \quad k = \omega/c, \quad E_0 = const.$$

The direction of propagation of the incident wave is orthogonal to the x axis and makes an angle ϑ_0 with the z axis (Fig. 1). Further, the harmonic time factor $\exp(-i\omega t)$ is everywhere omitted.

The electromagnetic field

$$E_x = ikcA_x, \quad H_y = \frac{1}{\mu}\frac{\partial}{\partial z}A_x, \quad H_z = -\frac{1}{\mu}\frac{\partial}{\partial y}A_x, \tag{2}$$

is expressed by means:

$$A_x(y, z) = \frac{i\mu}{4\pi}\int_{-\infty}^{\infty}\frac{1}{v}\exp\{i(wz + v|y|)\}F(w)dw, \quad v = \sqrt{k^2 - w^2}. \tag{3}$$

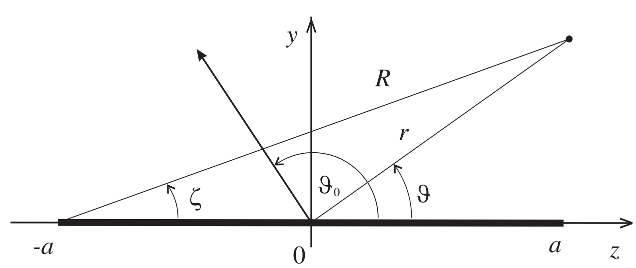

Figure 1: Diraction of plane wave at a strip.

From the boundary condition for the electric field on the strip

$$E_x + E_x^o = 0 \quad \text{at} \quad |z| \le a \quad (y = 0, \, -\infty < x < \infty),$$

and (2) and (3) we obtain an integral equation

$$\int_{-\infty}^{\infty} \exp(iwz) \frac{1}{v} F(w) dw + A_0 \exp(ihz) = 0 \quad \text{at} \quad |z| \le a, \tag{4}$$

where $A_0 = 4\pi E_0/(\omega\mu)$ and $h = k \cos\vartheta_0$.

We have the following integral equation from the continuity condition of the magnetic field (H_z) on the continuation of the strip:

$$\int_{-\infty}^{\infty} \exp(iwz) F(w) dw = 0 \quad \text{at} \quad a < |z|, \tag{5}$$

which follows immediately from absence of currents on the prolongation of the strip.

Let k have a small positive imaginary part that will vanish in the final formulas. Taking into account that the edges of a strip are secondary sources of waves, the Fourier-component of the current density is written as a sum from two analytical sources:

$$\begin{aligned} F(w) &= F_1 + F_2, \qquad (6) \\ F_2(w) &= \sqrt{k - w} \left(A_2(w) + B^+(w)\right) \exp(iwa), \quad F_1(w) = \sqrt{k + w} \left(A_1(w) + B^-(w)\right) \exp(-iwa). \end{aligned}$$

The fields from the analytical sources must satisfy Meixner's condition, i.e., behaving at infinity as $w^{-1/2}$. The terms F_1 and F_2 are constructed by the Wiener-Hopf-Fock method such that A_1 and A_2 correspond to plane wave amplitudes; B^+ and B^- correspond to the amplitudes of the reflected waves from the strip edges. From this follows, that A_1 and A_2 should be analytical functions on the entire complex w plane except for a simple pole at $w = h$. As the singular points in the upper half plane (UHP) correspond to traveling waves to the right along the z axis, B^- should be analytical in the LHP, and B^+ in the UHP.

Thus, a system of singular integral Equations (4), (5) is reduced to a system of Fredholm integral equations of the second kind:

$$B^-(w) = \frac{1}{2\pi i} \int_{C^+} \frac{\exp(i2au)}{u - w} \sqrt{\frac{k - u}{k + u}} \left(A_2(u) + B^+(u)\right) du, \tag{7}$$

$$B^+(w) = \frac{1}{2\pi i} \int_{C^+} \frac{\exp(i2au)}{u + w} \sqrt{\frac{k - u}{k + u}} \left(A_1(-u) + B^-(-u)\right) du, \tag{8}$$

where

$$A_1(w) = \frac{A_o}{2\pi i} \frac{\sqrt{k - h}}{w - h} \exp(iha), \quad A_2(w) = -\frac{A_o}{2\pi i} \frac{\sqrt{k + h}}{w - h} \exp(-iha). \tag{9}$$

By introducing the integral operator

$$\mathbf{I}(w, u) = \frac{1}{2\pi i} \int_{C^+} du \frac{\exp(i2au)}{u - w} \sqrt{\frac{k - u}{k + u}},$$

the system of Equations (7) and (8) may be represented compactly as

$$\begin{aligned} B^+(w) &= \mathbf{I}(-w, u) \left(A_1(-u) + B^-(-u)\right), \\ B^-(w) &= \mathbf{I}(w, u)(A_2(u) + B^+(u)). \qquad (10) \end{aligned}$$

Thus, for the solution of a boundary value problem it is necessary to find $F(w)$ that satisfies the system of integral Equations (4) and (5).

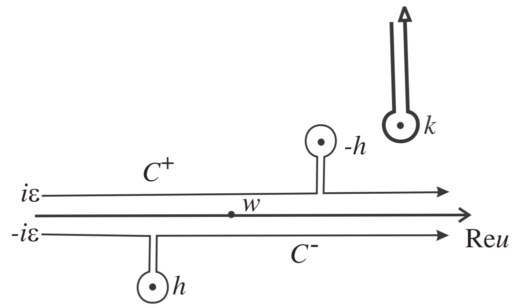

Figure 2: A plane of a complex variable w (or u).

3. CHARACTERISTIC EQUATION FOR STRIPS

The short-wave asymptotic behavior is achieved by means of the etalon integral

$$I(w) \equiv \mathbf{I}(w,u)\cdot 1,$$

using the stationary phase method. Here, the integration path in (10) is deformed up to the edge of the cut C_1, (Fig. 2) to get the contour of steepest descent that is a line parallel to imaginary axis upwards from the branch point. The result is

$$\begin{aligned} B^+(w) &\cong \mathbf{I}(-w,u)A_1(-u) + B^-(-k)I(-w), \\ B^-(w) &\cong \mathbf{I}(w,u)A_2(u) + B^+(k)I(w). \end{aligned} \tag{11}$$

The functions in (11) are found by solving the system of linear algebraic equations:

$$B^+(k) = \left(1 - I^2(-k)\right)^{-1}\left(\mathbf{I}(-k,u)A_1(-u) + I(-k)\mathbf{I}(-k,u)A_2(u)\right), \tag{12}$$

$$B^-(-k) = \left(1 - I^2(-k)\right)^{-1}\left(\mathbf{I}(-k,u)A_2(u) + I(-k)\mathbf{I}(-k,u)A_1(-u)\right), \tag{13}$$

Thus, the above-stated expressions give the dominant contribution to the solution of (10). Zeroing a denominator in (12), (13) gives a characteristic equation in the first approximation:

$$\det{}_1 = 1 - I^2(-k) = 0, \tag{14}$$

which determines the complex resonance frequencies of a strip, i.e., frequencies of self oscillations in absence of external incident waves. Here

$$I(-k) = \frac{1}{2i}\mathrm{H}_0^{(1)}(2ka) - 2ak\left(\mathrm{H}_0^{(1)}(2ka) - i\mathrm{H}_1^{(1)}(2ka)\right).$$

In order to take a account of the corrections of higher order it is necessary to expand the required functions B^+ and B^- in (10) in a Taylor series in the neighborhood of the point $u = k$ with the result:

$$\mathbf{I}(-w,u)B^-(-u) = e^{i2ak}\sum_{n=0}^{N}\frac{(i)^n}{n!}B^{-(n)}(-k)\frac{\partial^n}{\partial(2a)^n}\left(e^{-i2ak}I(-w)\right), \tag{15}$$

$$\mathbf{I}(w,u)B^+(u) = e^{i2ak}\sum_{n=0}^{N}\frac{(-i)^n}{n!}B^{+(n)}(k)\frac{\partial^n}{\partial(2a)^n}\left(e^{-i2ak}I(w)\right). \tag{16}$$

Now we will consider the characteristic equation in the second approximation for the estimation of the precision of the basic contribution of the integration by the saddle point method taking into account only the first derivatives in (15), (16) ($N = 1$):

$$\begin{aligned} \mathbf{I}(-w,u)B^-(-u) &\simeq B^-(-k)I(-w) + B^{-(1)}(-k)\left(kI(-w) + iI_a(-w)\right), \\ \mathbf{I}(w,u)B^+(u) &\simeq B^+(k)I(w) - B^{+(1)}(k)\left(kI(-w) + iI_a(-w)\right), \end{aligned}$$

where the following notation for the derivatives in the indicated points with respect to w and the parameter $b = 2a$ is introduced:

$$I_a(-w)\equiv\frac{\partial}{\partial b}I(-w),\quad B^{-(1)}(-k)\equiv\frac{\partial}{\partial w}B^-(w)\big|_{w=-k},\quad I_a\equiv I_a(-k),\quad B^{+(1)}(k)\equiv\frac{\partial}{\partial w}B^+(w)\big|_{w=k}.$$

Substituting these expressions in (10) we find the matrix

$$\begin{pmatrix} 1 & 0 & -I(-k) & -kI(-k)-iI_a \\ 0 & 1 & -I_w & -kI_w-iI_{wa} \\ -I(-k) & kI(-k)+iI_a & 1 & 0 \\ -I_w & kI_w+iI_{wa} & 0 & 1 \end{pmatrix}$$

for the system of algebraic equations, where the following notation is introduced:

$$I_w\equiv\frac{\partial}{\partial w}I(w)\big|_{w=-k},\quad I_{wa}\equiv\frac{\partial}{\partial b}I_w.$$

We get the characteristic equation in the second approximation by equating the determinant of the matrix to zero:

$$\begin{aligned}\det{}_2 \;=\; & 1-\frac{H_0^4}{16}+\frac{ka}{12}\big(8H_0H_1(1+H_0^2)+iH_0^2(24+9H_0^2+H_1^2)\big)+\frac{(ka)^2}{36}\big(H_1^2(H_1^2-112)\\ & +2H_0^2(312-47H_1^2)+129H_0^4-i16H_0H_1(H_1^2+15H_0^2+42)\big)+\frac{8(ka)^3}{9}(H_1+iH_0)\big(H_0(7H_1^2\\ & -9H_0^2-48)+iH_1(H_1^2+17H_0^2+32)\big)-\frac{64(ka)^4}{9}(H_1+iH_0)^2\big((H_1+iH_0)^2-4\big)=0,\end{aligned} \tag{17}$$

where for brevity the values of the Hankel functions are designated as

$$H_0\equiv \mathrm{H}_0^{(1)}(2ak),\quad H_1\equiv \mathrm{H}_1^{(1)}(2ak).$$

The following values of the derivatives of I have been used in the characteristic equation:

$$\begin{aligned} I_w(-k) &= a(H_0-\frac{i}{3}H_1)-\frac{8}{3}ika^2(H_0-iH_1),\quad I_a=-\frac{1}{2}kH_0+\frac{i}{2}kH_1-ikI(-k),\\ I_{wa} &= \frac{1}{2}H_0-3ak(iH_0+\frac{7}{9}H_1)-\frac{8}{3}(ak)^2(H_0-iH_1).\end{aligned}$$

The variation of $\det_1$ and $\det_2$ with real k is presented in Fig. 3 using the basic contribution of the solution (10) with the saddle point method.

The behavior of the real part of the characteristic function for real values of k is maintained if $2ak$ is larger than about 0.04 in the first and in the second approach (Fig. 3). The behavior of the imaginary part of the characteristic functions for real values k are basically identical both in the first and in the second approximation.

Hence, on solving similar diffraction problems with the saddle point method and an etalon integral, the basic contribution of the integration should be restricted to the frequency band $2ka > 0.4$.

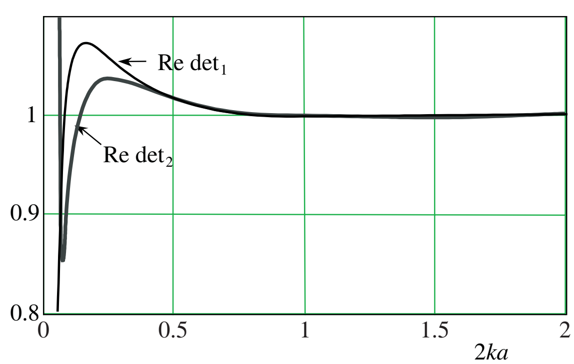

Figure 3: Real part of $\det_1$, $\det_2$ ($\mathrm{Im}ka = 0$).

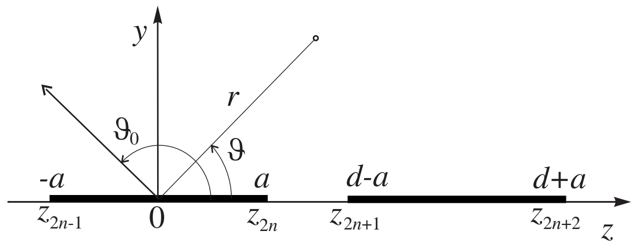

Figure 4: Diffraction of a plane wave on a grating.

The imaginary and real parts of the roots of the characteristic Equation (14) approach asymptotically

$$\mathrm{Im}(2ak) = -2.305 - 1.5\ln \mathrm{Re}(2ak), \quad \mathrm{Re}(2ak) \simeq \pi(n+1/4)\ (n = 1, 2, \ldots). \tag{18}$$

at Re$k \gg 1$. The obtained asymptotic formulas coincide with the result of [16]:

$$2ak \simeq (n+1/4)\pi - i1.5\ln\left(2\sqrt[3]{4\pi}(n+1/4)\pi\right).$$

4. CHARACTERISTIC EQUATION FOR GRATINGS

By analogy previous problem the boundary problem of a plane electromagnetic wave incident on an ideally conducting grating is reduced to a system of linear algebraic equations by means of integrating the system by the saddle point method and using etalon integrals. Then the solution of the system can be presented in the form

$$\begin{cases} B^+_{2n+1}(w) = I(-w)\left(A_{2n+2}(-k) + B^+_{2n+2}(-k) + B^-_{2n+2}(-k)\right), \\ B^-_{2n+1}(w) = J(w)\left(A_{2n}(k) + B^+_{2n}(k) + B^-_{2n}(k)\right), \\ \quad B^+_{2n}(w) = J(-w)\left(A_{2n+1}(-k) + B^+_{2n+1}(-k) + B^-_{2n+1}(-k)\right), \\ B^-_{2n+2}(w) = I(w)\left(A_{2n+1}(k) + B^+_{2n+1}(k) + B^-_{2n+1}(k)\right), \end{cases} \tag{19}$$

where

$$\mathbf{J}(w,u) = \frac{1}{2\pi i}\int_{C^+} du \frac{\exp(i(d-2a)u)}{u-w}\sqrt{\frac{k+u}{k-u}}, \quad J(w) = \mathbf{J}(w,u)\cdot 1,$$

the coefficients are defined from the following system of algebraic equations:

$$\begin{pmatrix} e^{-ihd} & 0 & 0 & 0 & 0 & -I_- & 0 & -I_- \\ 0 & e^{-ihd} & 0 & 0 & 0 & -I_+ & 0 & -I_+ \\ 0 & 0 & 1 & 0 & -J_+ & 0 & -J_+ & 0 \\ 0 & 0 & 0 & 1 & -J_- & 0 & -J_- & 0 \\ 0 & -J_- & 0 & -J_- & 1 & 0 & 0 & 0 \\ 0 & -J_+ & 0 & -J_+ & 0 & 1 & 0 & 0 \\ -I_+ & 0 & -I_+ & 0 & 0 & 0 & e^{ihd} & 0 \\ -I_- & 0 & -I_- & 0 & 0 & 0 & 0 & e^{ihd} \end{pmatrix} \begin{pmatrix} B^+_{2n+1}(k) \\ B^+_{2n+1}(-k) \\ B^-_{2n+1}(k) \\ B^-_{2n+1}(-k) \\ B^+_{2n}(k) \\ B^+_{2n}(-k) \\ B^-_{2n}(k) \\ B^-_{2n}(-k) \end{pmatrix} = \begin{pmatrix} A_{2n}(-k)I_- \\ A_{2n}(-k)I_+ \\ A_{2n}(k)J_+ \\ A_{2n}(k)J_- \\ A_{2n+1}(-k)J_- \\ A_{2n+1}(-k)J_+ \\ A_{2n+1}(k)I_+ \\ A_{2n+1}(k)I_- \end{pmatrix}. \tag{20}$$

The following notations are introduced here for brevity:

$$I_- \equiv I(-k),\ I_+ \equiv I(k) = \frac{1}{2i}\mathrm{H}_0^{(1)}(2ak),\ J_- \equiv J(-k) = -\frac{1}{2i}\mathrm{H}_0^{(1)}(k(d-2a)),\ J_+ \equiv J(k) = e^{i(d-2a)k}.$$

Further, the periodic conditions

$$\begin{aligned} &B^-_{2n+2}(w) = e^{ihd}B^-_{2n}(w), \quad B^+_{2n+2}(w) = e^{ihd}B^+_{2n}(w), \\ &B^\pm_{2n-1}(w) = \mathrm{e}^{-ihd}B^\pm_{2n+1}(w), \quad B^\pm_{2n-2}(w) = \mathrm{e}^{-ihd}B^\pm_{2n}(w) \end{aligned} \tag{21}$$

follows from the substitutions $n \to n+1$ and $z \to z+d$, remembering to use $K \to \mathrm{e}^{-ihd}K$ for the amplitude of the incident wave.

As the amplitude of a plane wave is equal to zero for self oscillations, it is necessary to note that in periodic conditions $h = 0$. Setting the denominator equal to zero, we find a characteristic equation that follows immediately from the solution (19) and (20):

$$\det = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & -I_- & 0 & -I_- \\ 0 & 1 & 0 & 0 & 0 & -I_+ & 0 & -I_+ \\ 0 & 0 & 1 & 0 & -J_+ & 0 & -J_+ & 0 \\ 0 & 0 & 0 & 1 & -J_- & 0 & -J_- & 0 \\ 0 & -J_- & 0 & -J_- & 1 & 0 & 0 & 0 \\ 0 & -J_+ & 0 & -J_+ & 0 & 1 & 0 & 0 \\ -I_+ & 0 & -I_+ & 0 & 0 & 0 & 1 & 0 \\ -I_- & 0 & -I_- & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = (1+I_-J_- - I_+J_+)^2 - (I_- + J_-)^2 = 0. \quad (22)$$

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