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The (2+1) dimensional metric $f(R)$ gravity non-minimally coupled with fermion field

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Abstract. In this article, we examine a gravitational theory including a fermion field that is non-minimally coupled to metric $f(R)$ gravity in (2+1) dimensions. We give the field equations for fermion fields and Friedmann equations. In this context, we study cosmological solutions of the field equations using these forms obtained by the existent of Noether symmetry.

1. Introduction

Modified theories of gravity have received increased attention lately due to combined motivation coming from astrophysics, cosmology and high-energy physics [1, 2, 3]. Among numerous alternatives to Einstein's gravity theory, theories which include higher-order curvature invariants, and specifically the particular class of $f(R)$ theories. In the last years, there has been a new stimulus for study $f(R)$ gravity, leading to a number of interesting results [4].

Fermionic fields are known as gravitational sources of the late-time acceleration of the universe as well as in addition to playing the role of early-time acceleration [5, 6, 7, 8]. Exact solutions of the Dirac equation in curved space-time have considerable importance in both cosmology and astrophysics. Studies of this solutions in (3+1) dimensions date back to works of Schrodinger, Fock and Tetrad [9, 10, 11]. Note that the exact solutions of the Dirac equation have been studied in various curved spacetime with different coordinate frames like (2+1) dimensions. The (2+1) dimensional quantum electrodynamics has attracted great attention due to the existence of particles with fractional spin and exotic statistics and the nontrivial topological properties. The exact solutions of the Dirac equation in the (2+1) dimensional theory have been recently found for various potentials in the flat and curved space-time.

The (2+1) dimensional gravity theories have also gained considerable importance [12, 13]. The (2+1) dimensional gravity has similar properties as the (3+1) dimensional theories of gravity. The (2+1) dimensional gravity more simple than the (3+1) dimensional gravity theories, because the Riemann tensor is reduced to the Ricci tensor. Moreover, the (2+1) Dirac equation is less complicated than the (3+1) Dirac equation [14, 15], because the Dirac matrices are reduced to Pauli matrices.

Symmetry is a well-known significant aspect of theoretical physics as well as cosmology. In this regard, Noether symmetry approach helps to find exact solutions of the defined point-like Lagrangian. It is an interesting approach that suggests a correlation between symmetry



generators of a dynamical system as well as conserved quantities [16]. Noether symmetries extensively studied in different gravity theories [17, 18, 19].

The structure of this article is the following. In Sect. 2, the field equations are derived from a point-like Lagrangian for (2+1) Friedmann-Robertson-Walker (FRW) spacetime, which is obtained from an action including the fermionic field non-minimally coupled to the $f(R)$ gravity. In Sect. 3, we search for the Noether symmetry of the Lagrangian of the theory and in Sect. 4, we obtain exact solutions of the field equations.

2. Action and field equations

We will start with action for (2+1) dimensional $f(R)$ gravity non-minimally coupled with fermion fields

$$\mathcal{S} = \int d^3x \sqrt{-g} \left\{ h(u) f(R) + \frac{i}{2} \left[(\bar{\psi} \bar{\sigma}^\mu(x) (\partial_\mu - \Omega_\mu(x)) \psi - \bar{\psi} (\overleftarrow{\partial}_\mu + \Omega_\mu(x)) \bar{\sigma}^\mu(x) \psi \right] - V(u) \right\}, \quad (1)$$

where $h(u)$ and $V(u)$ generic functions, representing the coupling with $f(R)$ gravity, where R is the Ricci scalar and the self-interaction potential of the fermion field respectively, and they depend on only functions of the bilinear $u = \bar{\psi}\psi$; g is the determinant of the metric tensor $g_{\mu\nu}$; ψ is fermion field; $\bar{\psi}$ is adjoint of the ψ and $\bar{\psi} = \psi^\dagger \sigma^3$. In this action, $\Omega_\mu(x)$ are spin connection

$$\Omega_\mu(x) = \frac{1}{4} g_{\lambda\alpha} (e_{\nu,\mu}^i e_i^\alpha - \Gamma_{\nu\mu}^\alpha) s^{\lambda\nu}(x), \quad (2)$$

where $\Gamma_{\nu\mu}^\alpha$ is Christoffel symbol, and $g_{\mu\nu}$ is given in term of triads $e_\mu^{(i)}(x)$ as follows

$$g_{\mu\nu}(x) = e_\mu^i(x) e_\nu^j(x) \eta_{ij}, \quad (3)$$

where μ and ν are curved spacetime indices running from 0 to 2. i and j are flat spacetime indices running from 0 to 2 and η_{ij} is the (2+1) dimensional Minkowskian metric with signature (1,-1,-1). The $s^{\lambda\nu}(x)$ spin operators are given by

$$s^{\lambda\nu}(x) = \frac{1}{2} [\bar{\sigma}^\lambda(x), \bar{\sigma}^\nu(x)], \quad (4)$$

where $\bar{\sigma}^\mu(x)$ are the spacetime dependent Dirac matrices in the (2+1) dimensional. Thanks to triads, $e_{(i)}^\mu(x)$, $\bar{\sigma}^\mu(x)$ are related to the flat spacetime Dirac matrices, $\bar{\sigma}^i$, as follows

$$\bar{\sigma}^\mu(x) = e_{(i)}^\mu(x) \bar{\sigma}^i, \quad (5)$$

where $\bar{\sigma}^i$ are

$$\bar{\sigma}^0 = \sigma^3, \quad \bar{\sigma}^1 = i\sigma^1, \quad \bar{\sigma}^2 = i\sigma^2. \quad (6)$$

σ^1 , σ^2 and σ^3 are Pauli matrices. To analyse the expansion of the universe, we will consider the spatially flat spacetime background in (2+1) dimensional which is analogous of the (3+1) dimensional FRW metric as follows

$$ds^2 = dt^2 - a^2(t)[dx^2 + dy^2], \quad (7)$$

where $a(t)$ is the scale factor of the Universe. The scalar curvature corresponding to the FRW metric (7) takes the form

$$R = -2 \left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) \quad (8)$$

where the dot represents differentiation with respect to cosmic time t . From the background in Eq.(7), it is possible to obtain the point-like Lagrangian

$$L = a^2 h f - a^2 h f_R R + 4h' \dot{u} f_R a \dot{a} + 4h f_{RR} \dot{R} a \dot{a} + 2h f_R \dot{a}^2 + \frac{ia^2}{2} (\bar{\psi} \sigma^3 \dot{\psi} - \dot{\bar{\psi}} \sigma^3 \psi) - a^2 V. \quad (9)$$

Here the prime denotes the derivative with respect to the bilinear u . Because of homogeneity and isotropy of the metric, it is assumed that the spinor field only depends on time t , i.e. $\psi = \psi(t)$. By using Euler-Lagrange equations for ψ and $\bar{\psi}$, we obtain Dirac's equations for the fermion field ψ and its adjoint $\bar{\psi}$ as

$$\dot{\psi} + H\psi + iV'\sigma^3\psi - ih'\sigma^3\psi(f - f_R R) + 2ih'f_R(2\dot{H} + 3H^2)\sigma^3\psi = 0, \quad (10)$$

$$\dot{\bar{\psi}} + H\bar{\psi} - iV'\bar{\psi}\sigma^3 + ih'\bar{\psi}\sigma^3(f - f_R R) - 2ih'f_R(2\dot{H} + 3H^2)\bar{\psi}\sigma^3 = 0, \quad (11)$$

where $H = \dot{a}/a$ denotes the Hubble parameter. On the other hand, from the point-like Lagrangian (9) and by considering the Dirac's equations, we find first Friedmann equation i.e. the acceleration equation for a

$$\frac{\ddot{a}}{a} = -\frac{p_f}{2h}. \quad (12)$$

Finally, we also consider the Hamiltonian constraint equation ($E_L = 0$) associated with the Lagrangian (9) which yields the second Friedmann equation as follows

$$H^2 = \frac{\rho_f}{2hf_R}. \quad (13)$$

In Eqs. (12) and (13), ρ_f and p_f are the effective energy density and pressure of the fermion field, respectively, so that they have the following expressions

$$\rho_f = -(4hf_{RR}\dot{R}H + 4h'f_RH\dot{u} - hf + hf_R R + V) \quad (14)$$

$$p_f = 2h'(\ddot{u}f_R + Hf_R\dot{u} + 2\dot{u}f_{RR}\dot{R}) + 2h''f_R\dot{u}^2 - \left[2h'f_R(2\dot{H} + 3H^2) + V'\right]u + V + (h'u - h)(f - f_R R) + 2h(f_{RRR}\dot{R}^2 + f_{RR}\ddot{R} + f_{RR}\dot{R}H) \quad (15)$$

In order to solve the field equations, we have to choose a form for the coupling function and for the potential density. To do this, in the following section we will use the Noether symmetry approach.

3. Noether symmetry

Thanks to Pauli matrices, in terms of the components of the spinor field $\psi = (\psi_1, \psi_2)^T$ and its adjoint $\bar{\psi} = (\psi_1^\dagger, -\psi_2^\dagger)$, the Lagrangian (9) can be rewritten as follows

$$L = a^2 h f - a^2 h f_R R + 4h f_{RR} \dot{R} a \dot{a} + 2h f_R \dot{a}^2 + 4h' f_R a \dot{a} \sum_{i=1}^2 \epsilon_i (\psi_i^\dagger \dot{\psi}_i + \dot{\psi}_i^\dagger \psi_i) + \frac{ia^2}{2} \left[\sum_i^2 (\psi_i^\dagger \dot{\psi}_i - \dot{\psi}_i^\dagger \psi_i) \right] - a^2 V. \quad (16)$$

where $\epsilon_i = \begin{cases} 1 & \text{for } i = 1 \\ -1 & \text{for } i = 2 \end{cases}$.

A vector field \mathbf{X} for the point Lagrangian (16) is

$$\mathbf{X} = \tau \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial a} + \beta \frac{\partial}{\partial R} + \sum_{i=1}^2 \left(\nu_i \frac{\partial}{\partial \psi_i} + \delta_i \frac{\partial}{\partial \psi_i^\dagger} \right), \quad (17)$$

where α, β_j and γ_j are depend on t, a, ψ_j and ψ_j^\dagger and they are determined from the Noether gauge symmetry condition. The first prolongation of \mathbf{X} is given by

$$\mathbf{X}^{[1]} = \mathbf{X} + \alpha_t \frac{\partial}{\partial \dot{a}} + \beta_t \frac{\partial}{\partial \dot{R}} + \sum_{i=1}^2 \left(\nu_{it} \frac{\partial}{\partial \dot{\psi}_i} + \delta_{it} \frac{\partial}{\partial \dot{\psi}_i^\dagger} \right), \quad (18)$$

where

$$\begin{aligned} \alpha_t &= D_t \alpha - \dot{a} D_t \tau, \\ \beta_t &= D_t \beta - \dot{R} D_t \tau, \\ \nu_{jt} &= D_t \nu_j - \dot{\psi}_j D_t \tau, \\ \delta_{jt} &= D_t \delta_j - \dot{\psi}_j^\dagger D_t \tau. \end{aligned} \quad (19)$$

and D_t is the operator of total differentiation with respect to t

$$D_t = \frac{\partial}{\partial t} + \dot{a} \frac{\partial}{\partial a} + \dot{R} \frac{\partial}{\partial R} + \sum_{i=1}^2 \left(\dot{\psi}_i \frac{\partial}{\partial \psi_i} + \dot{\psi}_i^\dagger \frac{\partial}{\partial \psi_i^\dagger} \right). \quad (20)$$

The significance of Noether gauge symmetry is clearly comes from the fact that if the vector field \mathbf{X} is the Noether gauge symmetry corresponding to the Lagrangian $L(t, a, \psi_j, \psi_j^\dagger, \dot{a}, \dot{\psi}_j, \dot{\psi}_j^\dagger)$,

$$\mathbf{X}^{[1]} L + L D_t(\tau) = D_t B, \quad (21)$$

Hence the Noether gauge symmetry condition (21) for the Lagrangian (16) leads to the following the over-determined system of differential equations

$$\begin{aligned} & \frac{4h}{a} \frac{\partial f}{\partial R} \frac{\partial \alpha}{\partial t} + 4h \frac{\partial^2 f}{\partial R^2} \frac{\partial \beta}{\partial t} + 4h' \frac{\partial f}{\partial R} \sum_{i=1}^2 \epsilon_i \left(\frac{\partial \nu_i}{\partial t} \psi_i^\dagger + \frac{\partial \delta_i}{\partial t} \psi_i \right) - aV \frac{\partial \tau}{\partial a} \\ & + ah \left(f - \frac{\partial f}{\partial R} R \right) \frac{\partial \tau}{\partial a} + \frac{ia}{2} \sum_{i=1}^2 \epsilon_i \left(\frac{\partial \nu_i}{\partial a} \psi_i^\dagger - \frac{\partial \delta_i}{\partial a} \psi_i \right) - \frac{1}{a} \frac{\partial B}{\partial a} = 0, \end{aligned} \quad (22)$$

$$4h \frac{\partial^2 f}{\partial R^2} \frac{\partial \alpha}{\partial t} + ah \left(f - \frac{\partial f}{\partial R} R \right) \frac{\partial \tau}{\partial R} + \frac{ia}{2} \sum_{i=1}^2 \epsilon_i \left(\frac{\partial \nu_i}{\partial R} \psi_i^\dagger - \frac{\partial \delta_i}{\partial R} \psi_i \right) - \frac{1}{a} \frac{\partial B}{\partial R} = 0, \quad (23)$$

$$\begin{aligned} & 4h' \epsilon_j \psi_j^\dagger \frac{\partial f}{\partial R} \frac{\partial \alpha}{\partial t} + i \alpha \epsilon_j \psi_j^\dagger + \frac{ia}{2} \delta + ah \left(F - \frac{\partial f}{\partial R} R \right) \frac{\partial \tau}{\partial \psi_j} \\ & - aV \frac{\partial \tau}{\partial \psi_j} + \frac{ia}{2} \sum_{i=1}^2 \epsilon_i \left(\frac{\partial \nu_i}{\partial \psi_j} \psi_i^\dagger - \frac{\partial \delta_i}{\partial \psi_j} \psi_i \right) - \frac{1}{a} \frac{\partial B}{\partial \psi_j} = 0, \end{aligned} \quad (24)$$

$$4h'\epsilon_j\psi_j\frac{\partial f}{\partial R}\frac{\partial\alpha}{\partial t} + i\alpha\epsilon_j\psi_j + \frac{ia}{2}\nu + ah\left(f - \frac{\partial f}{\partial R}R\right)\frac{\partial\tau}{\partial\psi_j^\dagger} - aV\frac{\partial\tau}{\partial\psi_j^\dagger} + \frac{ia}{2}\sum_{i=1}^2\epsilon_i\left(\frac{\partial\nu_i}{\partial\psi_j^\dagger}\psi_i^\dagger - \frac{\partial\delta_i}{\partial\psi_j^\dagger}\psi_i\right) - \frac{1}{a}\frac{\partial B}{\partial\psi_j^\dagger} = 0, \quad (25)$$

$$h'\frac{\partial f}{\partial R}\left[\sum_{i=1}^2\epsilon_i\left(\nu_i\psi_i^\dagger + \delta_i\psi_i\right) + 2a\sum_{i=1}^2\epsilon_i\left(\frac{\partial\nu_i}{\partial a}\psi_i^\dagger + \frac{\partial\delta_i}{\partial a}\psi_i\right)\right] + h\frac{\partial^2 f}{\partial R^2}\left(\beta + 2a\frac{\partial\beta}{\partial a}\right) + h\frac{\partial f}{\partial R}\left(2\frac{\partial\alpha}{\partial a} - \frac{\partial\tau}{\partial t}\right) = 0, \quad (26)$$

$$ah'\left[\frac{\partial^2 f}{\partial R^2}\sum_{i=1}^2\epsilon_i\left(\nu_i\psi_i^\dagger + \delta_i\psi_i\right) + \frac{\partial f}{\partial R}\sum_{i=1}^2\epsilon_i\left(\frac{\partial\nu_i}{\partial R}\psi_i^\dagger + \frac{\partial\delta_i}{\partial R}\psi_i\right)\right] + h\frac{\partial f}{\partial R}\frac{\partial\alpha}{\partial R} + h\frac{\partial^2 f}{\partial R^2}\left[\left(\alpha + a\frac{\partial\alpha}{\partial a}\right) + a\left(\frac{\partial\beta}{\partial R} - \frac{\partial\tau}{\partial t}\right)\right] + ah\beta\frac{\partial^3 f}{\partial R^3} = 0, \quad (27)$$

$$h''\frac{\partial f}{\partial R}\sum_{i=1}^2\epsilon_i\left(\nu_i(\psi_i^\dagger)^2 + \delta_i\psi_i^\dagger\psi_i\right) + h'\frac{\partial f}{\partial R}\sum_{i=1}^2\epsilon_i\left(\delta_i + \frac{\partial\nu_i}{\partial\psi_i^\dagger}\psi_i^\dagger + \frac{\partial\delta_i}{\partial\psi_i^\dagger}\psi_i\right) + h\frac{\partial f}{\partial R}\frac{\partial\alpha}{\partial\psi_j} + h'\epsilon_j\psi_j^\dagger\frac{\partial f}{\partial R}\left[\left(\frac{\alpha}{a} + \frac{\partial\alpha}{\partial a}\right) - \frac{\partial\tau}{\partial t}\right] + \epsilon_j\psi_j^\dagger\frac{\partial^2 f}{\partial R^2}\left(h'\beta + h\frac{\partial\beta}{\partial\psi_j}\right) = 0, \quad (28)$$

$$h''\frac{\partial f}{\partial R}\sum_{i=1}^2\epsilon_i\left(\nu_i\psi_i^\dagger\psi_i + \delta_i(\psi_i)^2\right) + h'\frac{\partial f}{\partial R}\sum_{i=1}^2\epsilon_i\left(\nu_i + \frac{\partial\nu_i}{\partial\psi_i^\dagger}\psi_i^\dagger + \frac{\partial\delta_i}{\partial\psi_i^\dagger}\psi_i\right) + h\frac{\partial f}{\partial R}\frac{\partial\alpha}{\partial\psi_j^\dagger} + h'\epsilon_j\psi_j\frac{\partial f}{\partial R}\left[\left(\frac{\alpha}{a} + \frac{\partial\alpha}{\partial a}\right) - \frac{\partial\tau}{\partial t}\right] + \epsilon_j\psi_j\frac{\partial^2 f}{\partial R^2}\left(h'\beta + h\frac{\partial\beta}{\partial\psi_j^\dagger}\right) = 0, \quad (29)$$

$$h\frac{\partial^2 f}{\partial R^2}\frac{\partial\alpha}{\partial R} = 0, \quad h'\frac{\partial f}{\partial R}\frac{\partial\alpha}{\partial\psi_j} = 0, \quad h'\frac{\partial f}{\partial R}\frac{\partial\alpha}{\partial\psi_j^\dagger} = 0, \quad (30)$$

$$h'\psi_j^\dagger\frac{\partial f}{\partial R}\frac{\partial\tau}{\partial\psi_j} = 0, \quad h'\psi_j\frac{\partial f}{\partial R}\frac{\partial\tau}{\partial\psi_j^\dagger} = 0, \quad h\frac{\partial f}{\partial R}\frac{\partial\tau}{\partial a} = 0, \quad h\frac{\partial^2 f}{\partial R^2}\frac{\partial\tau}{\partial R} = 0 \quad (31)$$

$$\left(2\alpha + a\frac{\partial\tau}{\partial t}\right)V + \frac{1}{a}\frac{\partial B}{\partial t} + aV'\sum_{i=1}^2\epsilon_i\left(\nu_i\psi_i^\dagger + \delta_i\psi_i\right) + 2h\alpha\left(\frac{\partial f}{\partial R}R - f\right) + ah'\sum_{i=1}^2\epsilon_i\left(\nu_i\psi_i^\dagger + \delta_i\psi_i\right)\left(\frac{\partial f}{\partial R}R - f\right) - \frac{ia}{2}\sum_{i=1}^2\left(\psi_i^\dagger\frac{\partial\nu_i}{\partial t} - \psi_i\frac{\partial\delta_i}{\partial t}\right) = 0. \quad (32)$$

This system of equations given by Eqs. (22)-(32) are obtained by imposing the fact that the coefficients of $\dot{a}^2, \dot{a}, \dot{R}, \dot{R}^2, \dot{\psi}_j, \dot{\psi}_j^\dagger, \dot{a}\dot{\psi}_j, \dot{\psi}_j\dot{\psi}_l$ and so on, vanish. From the rest Noether gauge symmetry equations, the complete solution is obtained as follows

$$\begin{aligned}\alpha &= -\frac{c_1}{2(n-1)}a, \\ \beta &= \frac{c_1}{2(n-1)}R, \\ \nu_j &= \frac{c_1}{2(n-1)}\psi_j - \epsilon_j c_2 \psi_j, \\ \delta_j &= \frac{c_1}{2(n-1)}\psi_j^\dagger + \epsilon_j c_2 \psi_j^\dagger, \\ \tau &= c_1 t + c_3, \quad B = c_4,\end{aligned}\tag{33}$$

and the coupling and the potential function are power law forms of the function of the bilinear u

$$h(u) = h_0 u^n,\tag{34}$$

$$V(u) = \lambda u^{2-n},\tag{35}$$

and

$$f(R) = f_0 R^{2n-1},\tag{36}$$

where c_l, λ, f_0, h_0 and n constants of integration.

4. Cosmological solutions

Since the coupling function h depends on the bilinear function u , from Dirac's equations (10) and (11) one gets

$$\dot{u} + 2\frac{\dot{a}}{a}u = 0,\tag{37}$$

which integrates to give

$$u = \frac{u_0}{a^2},\tag{38}$$

where u_0 is a constant of integration. Using Friedmann equation (13) with (14) we obtain

$$\frac{\dot{a}}{a} - \sqrt{\frac{\lambda}{6f_0 h_0}} = 0,\tag{39}$$

which has the solution

$$a(t) = a_0 e^{H_0 t}, \quad \text{where} \quad H_0 = \sqrt{\frac{\lambda}{6f_0 h_0}}\tag{40}$$

5. Conclusions

In the present article, we have studied fermion field non-minimally coupled with (2+1) dimensional $f(R)$ gravity. We have derived full set of field equations for our model. By using Noether symmetry approach with gauge term we get cosmological solutions.

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