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DETERMINATION THE CONNECTION BETWEEN THE METRIC AND POTENTIAL IN THE EQUATIONS OF ASSOCIATIVITY

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Introduction. The WDVV equations, in general, have the following form [1, 2, 3]:

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^q \partial t^k \partial t^r} = \frac{\partial^3 F}{\partial t^j \partial t^k \partial t^p}\eta^{pq}\frac{\partial^3 F}{\partial t^i \partial t^q \partial t^r}, \quad \forall i, j, k, r \in \{1,...,n\},$$

where F is a prepotential, η is a metric, n is the dimension of a manifold. This is an overdetermined system of partial differential equations for a single scalar function F known as the prepotential. It appeared in the work of Witten and Dijkgraaf, E.Verlinde, H.Verlinde [1, 2, 3, 4] in context of Topological Quantum Field Theory (TQFT). A geometric interpretation of these equations was given by Dubrovin in [1], which led to the formal definition of a Frobenius manifold. This work of Dubrovin further strengthened the link between the WDVV equations and TQFT, and therefore of integrable systems. The WDVV equations themselves are also integrable since they can be written as a zero curvature equation for a so-called deformed, or Dubrovin, connection.

The solution F must meet the following two conditions:

1) (the normalization condition):

$$\frac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \delta_{\alpha+\beta,n+1}$$

2) the homogeneity equation:

$$L_E F = (m+3)F + \sum_{\alpha\beta} A_{\alpha\beta} t^\alpha t^\beta + \sum_{\alpha} B_\alpha t^\alpha + C, \quad A_{\alpha\beta}, B_\alpha, C = const,$$

where E is Euler vector field.

Let us consider a function of n independent variables $F(t^1,...,t^n)$ satisfying the following two conditions [5, 6]:

1. The matrix $\eta_{\alpha\beta} := \frac{\partial^3 F(t)}{\partial t^1 \partial t^\alpha \partial t^\beta}$ is constant and nondegenerate. Note that the matrix $\eta_{\alpha\beta}$ completely determines dependence of the function F on the fixed variable t^1.

2. For all t = $(t^1,..., t^n)$ the functions $c^p_{\alpha\beta} = \eta^{pq} \frac{\partial^3 F(t)}{\partial t^q \partial t^\alpha \partial t^\beta}$ are structural constants of an associative algebra A(t) in n-dimensional space with a basis e_1, ..., e_n.

The conditions 1 and 2 impose a complicated overdetermined system of nonlinear partial differential equations of the third order on the function F. This system is known in two-dimensional topological field theory as the equations of associativity or the Witten-Dijkgraaf-H.Verlinde-E.Verlinde (WDVV) system.

Determination the connection between the metric and potential in the equations of associativity. We consider various cases [7] of the metric of the associativity equation. In this case we have that $\eta_{11} = 0$. The metric for n = 2 cases is as follows

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Since the metric is antidiagonal we have the condition

$$\frac{\partial^3 F}{\partial x \partial x \partial x} = 0$$

from this condition we obtain the following form of F :

$$F = k_1(y)\frac{x^2}{2} + xk_2(y) + k_3(y). \tag{1}$$

Now we need to find $k_1(y)$ and $k_2(y)$. Since the metric is antidiagonal we have the conditions

$$\frac{\partial^3 F}{\partial x \partial x \partial y} = 1, \quad \frac{\partial^3 F}{\partial x \partial y \partial y} = 0$$

from which it follows that $k_1(y) = y$ and $k_2(y) = 0$. Hence we have the following form of solutions

$$F = \frac{x^2 y}{2} + f(y). \tag{2}$$

Now we consider case when $\eta_{11} \neq 0$. The metric for n = 2 cases is as follows

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the above metric we have the condition

$$\frac{\partial^3 F}{\partial x \partial x \partial x} = 1$$

from this condition, we obtain the following form of F :

$$F = \frac{x^3}{6} + k_1(y)\frac{x^2}{2} + xk_2(y) + k_3(y). \tag{3}$$

Now we need to find $k_1(y)$ and $k_2(y)$. Since the above metric we have the conditions

$$\frac{\partial^3 F}{\partial x \partial x \partial y} = 0, \quad \frac{\partial^3 F}{\partial x \partial y \partial y} = 1$$

from which it follows that $k_1(y) = 0$ and $k_2(y) = \frac{y^2}{2}$. Hence we have the following form of solutions

$$F = \frac{x^3}{6} + \frac{xy^2}{2} + f(y). \tag{4}$$

Now we consider case when $\eta_{11} \neq 0$. The metric for n = 3 cases is as follows

$$\eta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Since the above metric we have the condition

$$\frac{\partial^3 F}{\partial x \partial x \partial x} = 1$$

from this condition, we obtain the following form of F :

$$F = \frac{x^3}{6} + k_1(y,z)\frac{x^2}{2} + k_2(y,z)x + k_3(y,z). \tag{5}$$

Now we need to find $k_1(y,z)$ and $k_2(y,z)$. Since the above metric we have the conditions

$$\frac{\partial^3 F}{\partial x \partial x \partial y} = 0, \quad \frac{\partial^3 F}{\partial x \partial y \partial z} = 1$$

from which it follows that $k_1(y,z) = 0$ and $k_2(y,z) = yz$. Hence we have the following form of solutions

$$F = \frac{x^3}{6} + xyz + f(y,z). \quad (6)$$

In this case we have that $\eta_{11} = 0$. The metric for n = 3 cases is as follows

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Since the metric is antidiagonal we have the condition

$$\frac{\partial^3 F}{\partial x \partial x \partial x} = 0$$

from this condition, we obtain the following form of F:

$$F = k_1(y,z)\frac{x^2}{2} + k_2(y,z)x + k_3(y,z). \quad (7)$$

Now we need to find $k_1(y,z)$ and $k_2(y,z)$. Since the metric is antidiagonal we have the conditions

$$\frac{\partial^3 F}{\partial x \partial x \partial z} = 1, \quad \frac{\partial^3 F}{\partial x \partial y \partial y} = 1$$

from which it follows that $k_1(y,z) = z$ and $k_2(y,z) = \frac{y^2}{2}$. Hence we have the following form of solutions

$$F = \frac{x^2 z}{2} + \frac{xy^2}{2} + f(y,z). \quad (8)$$

Now we consider case when $\eta_{11} = 0$. The metric for n = 4 cases is as follows

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Since the metric is antidiagonal we have the condition

$$\frac{\partial^3 F}{\partial x \partial x \partial x} = 0$$

from this condition, we obtain the following form of F:

$$F = k_1(y,z,t)\frac{x^2}{2} + k_2(y,z,t)x + k_3(y,z,t). \qquad (9)$$

Now we need to find $k_1(y,z,t)$ and $k_2(y,z,t)$. Since the metric is antidiagonal we have the conditions

$$\frac{\partial^3 F}{\partial x \partial x \partial t} = 1, \quad \frac{\partial^3 F}{\partial x \partial y \partial z} = 1$$

from which it follows that $k_1(y,z,t) = t$ and $k_2(y,z,t) = yz$. Hence we have the following form of solutions

$$F = \frac{x^2 t}{2} + xyz + f(y,z,t). \qquad (10)$$

Now we consider case when the metric for n = 4 is as follows

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Since the above metric we have the condition

$$\frac{\partial^3 F}{\partial x \partial x \partial x} = 0$$

from this condition, we obtain the following form of F:

$$F = k_1(y,z,t)\frac{x^2}{2} + k_2(y,z,t)x + k_3(y,z,t). \qquad (11)$$

Now we need to find $k_1(y,z,t)$ and $k_2(y,z,t)$. Since the above metric we have the conditions

$$\frac{\partial^3 F}{\partial x \partial x \partial t} = 1, \quad \frac{\partial^3 F}{\partial x \partial y \partial y} = 1, \quad \frac{\partial^3 F}{\partial x \partial y \partial t} = 0, \quad \frac{\partial^3 F}{\partial x \partial z \partial z} = 1,$$

from which it follows that $k_1(y,z,t) = t$ and $k_2(y,z,t) = \frac{y^2}{2} + \frac{z^2}{2}$. Hence we have the following form of solutions

$$F = \frac{x^2 t}{2} + \frac{xy^2}{2} + \frac{xz^2}{2} + f_3(y,z,t). \qquad (12)$$

Conclusion. In this paper, we consider determination the connection between the metric and potential in the equations of associativity. We consider various cases of the metric of the associativity equations. We obtain the conditions arising from the metric. We consider the different cases of the metric for n=2, 3, 4. We obtain the different form of solution for potential F in the equations (2, 4, 6, 8, 10, 12).

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